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PERSISTENT HOMOLOGY AND DUALITY

Doctoral thesis

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VZTRAJNA HOMOLOGIJA IN DUALNOST

Doktorska disertacija

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Abstract

An important problem with sensor networks is that they do not provide information about the regions that are not covered by their sensors. If the sensors in a network are static, then the Alexander Duality Theorem from classic algebraic topology is sufficient to determine the coverage of a network. However, in many networks the nodes change position with time. In the case of dynamic sensor networks, we consider the covered and uncovered regions as parametrized spaces with respect to time. Parametrized homology is a variant of zigzag persistent homology that measures how the homology of the levelsets of the space changes as we vary the parameter. We will present a few theorems that extend different versions of classical Alexander Duality theorem to the setting of parametrized homology theories. This approach sheds light on the practical problem of 'wandering' loss of coverage within dynamic sensor networks.

Math. Subj. Class. (2000): $55N35$, $55U30$

Keywords: Alexander duality, persistent homology, zigzag persistence, levelset zigzag persistence, parametrized homology.

Povzetek

Eden izmed večjih problemov pri preučevanju senzorskih omrežij je, da nudijo le informacijo o področju, ki ga senzorji pokrivajo. V statičnih senzorskih omrežjih klasična Aleksandrova dualnost zadošča kot kriterij za pokritost, ampak v mnogo omrežjih se položaj senzorjev spreminja s časom in ta izrek ni dovolj. V primeru dinamičnih senzorskih omrežij sta območji pokritosti in nepokritosti parametrizirana prostora glede na čas. Parametrizirana homologija je različica cikcak vztrajne homologije, ki meri, kako se homologija nivojnic prostora spreminja, če spreminjamo parameter. V tej disertaciji bom predstavila parametrizirane ekvivalente nekaj različic klasične Aleksandrove dualnosti. Parametrizirana Aleksandrova dualnost nam tudi pomaga pri razumevanju 'problema vsiljivca'.

Math. Subj. Class. (2000): $55N35$, $55U30$

Ključne besede: Aleksandrova dualnost, vztrajna homologija, cikcak vztrajnost, cikcak vztrajnost za nivojnice, parametrizirana homologija.

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I thank my boyfriend Peter J. Verovšek for being closest to me while I was writing this dissertation and for helping me with my English.

Vin de Silva almost single-handedly invented all the concepts that I needed to state and prove my theorems. Without his insights during our many discussions, I would not have been able to complete this dissertation.

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Introduction

The rapid development of information technology in the last few decades has produced data at an unprecedented rate. Unfortunately, we are unable to take full advantage of this deluge of information. The quantity of data is typically too large for a complete analysis and in many cases is not necessarily relevant to the questions we are trying to answer. One of the key problems in sorting through this mass of data is determining which datasets should be grouped together. For example, if we have two images taken by a security camera at different times of day, each showing a face from a different angle and distance, we need to determine whether it is the same face. In this situation we are looking for features to distinguish between the different possibilities as well as for a fast and reliable algorithm to measure them.

Algebraic topology has provided a number of important tools for the analysis of large data [31, 19]. This is due to the nature of the data. Large datasets are often given in the form of very long vectors or arrays (for instance, DNA sequences or pixel arrays). To codify data in the form of a vector, we have to choose coordinates that are not necessarily natural (intrinsic) to our problem. Since topology studies the intrinsic properties of geometric objects that do not depend on a particular choice of coordinates, it allows us to analyze data merely by establishing how 'close' two datasets are to each other. Invariants of topological spaces (for example, homology groups) are also useful in this context, because they are insensitive to small deformations.

Despite their advantages, one of the drawbacks of the invariants of algebraic topology is their sensitivity to 'noise' in the data. In practice, real-valued parameters cannot be measured exactly, and even very small differences can affect invariants. Edelsbrunner, Letscher and Zomorodian [22] solved this problem by introducing persistent homology. Zomorodian and Carlsson then gave this idea a firm theoretical footing [13].

Developments in persistent homology have allowed these methods to be applied in many different settings. Among the first was the invention of a computational test of coverage in sensor networks [26]. Persistent homology is also popular as a basis for techniques of visualization [18, 23]. The application of persistent homology to shape analysis led to the introduction of the barcode [11, 12, 10] and of the persistence diagram [16]. The ability to represent persistent homology through the persistence diagram contributed to the proof of the stability of persistent homology [16].

The development of zigzag persistence [7] is particularly important. It is the most general version of one-dimensional persistence, because it admits a classification theorem similar to that of persistent homology [13]. This approach is based on Gabriel's Representation Theorem. When studying a filtration $\left\{X_r = p_X^{-1}(r)\right\}$ assigned to a function $p_X \colon X \to \mathbb{R}$, we do not wish to restrict ourselves to considering only inclusions, but also want to allow sequences of zigzag inclusions $X_r \subset p_X^{-1}[r,s] \supset X_s$. This is where zigzag persistence comes into play.

Zigzag persistence is commonly described either as a multiset of intervals (a barcode) or as a multiset of points in the half plane (a persistence diagram). However, these descriptions do not distinguish between different types of intervals (for example, $[p,q]$, $[p,q)$, $(p,q]$ and (p,q)). In order to capture the homology of spaces parametrized over $\mathbb{R}$, Chazal et al. [14] introduced decorated real numbers. They also developed a new approach for expressing persistence. The intuition is that if we know how many points of the diagram are contained in each rectangle in the half plane, then we know the diagram itself. Counting the points in the rectangles leads to the introduction of r-measures. The Equivalence Theorem formally establishes the correspondence between r-measures and 'decorated' persistence diagrams.

A parametrized space is a pair (X, p_X), where $p_X \colon X \to \mathbb{R}$ is a continuous map on a topological space X. This function defines levelsets $X_r = p_X^{-1}(r)$ and slices $X_a^b = p_X^{-1}([a,b])$ for different intervals $[a,b] \subset \mathbb{R}$. Carlsson, de Silva and Morozov recently introduced levelset zigzag persistence [9] to capture the topology of a parametrized space. This requires the imposition of reasonable restrictions on the space X and the function p_X (such as assuming a function to be Morse and a space to be compact).

Carlsson et al. use levelset zigzag persistence in combination with measure theory to define 'parametrized homology' [8]. Let H_j be a singular homology functor. For $a < b < c < d$ we are interested in determining homological features that persist on X_b^c, but are absent outside of X_a^d. More precisely, we want to count the number of indecomposable summands in the zigzag persistent homology of the diagram

$$\begin{array}{ccccccc} & \mathrm{H}_j(X_a^b) & & \mathrm{H}_j(X_b^c) & & \mathrm{H}_j(X_c^d) & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ \mathrm{H}_j(X_a) & & \mathrm{H}_j(X_b) & & \mathrm{H}_j(X_c) & & \mathrm{H}_j(X_d), \end{array}$$

which persist over the closed interval $[b,c]$, but not over the open interval (a,d). There are four types of such indecomposable summands. By counting them we can define four different 'decorated' persistence diagrams for X. Parametrized singular homology of X is a collection of these diagrams over all j. We define other parametrized homology theories using a similar approach.

In this work we extend different versions of Alexander Duality to the setting of parametrized homology theories. If X is a subspace of the Cartesian product $\mathbb{R}^n \times \mathbb{R}$, and $p_X \colon X \to \mathbb{R}$ is the projection onto the last coordinate, we can define a parametrized space $Y = \mathbb{R}^n \times \mathbb{R} \setminus X$ (together with projection onto the last coordinate). If the levelsets are compact and locally contractible, an Alexander Duality isomorphism exists: $\widetilde{\mathrm{H}}_{n-j-1}(Y_r; \mathbf{k}) \cong \mathrm{H}^j(X_r; \mathbf{k})$ for each r and each field $\mathbf{k}$. Here H denotes singular cohomology and $\widetilde{\mathrm{H}}$ reduced singular homology.

Unlike previous attempts to expand Alexander Duality, our approach allows us to generalize Alexander Duality directly to an appropriate parametrized version. Edelsbrunner and Kerber operate within extended persistence diagrams [20]. They consider Alexander Duality to be a statement about two complementary subsets of the sphere that intersect in an n-manifold. By contrast, we consider it to be a statement about a subset of a Euclidean space with compact levelsets and slices, and its complement. An advantage of our approach is that it covers situations that their theorem leaves out.

One practical application of this research is in sensor networks [29]. The classic Alexander Duality can only be used to find gaps in static networks. However, in many networks the sensors move continuously through time. An 'evasion path' may exist within dynamic networks, as an intruder could avoid detection by moving around the domain. Since the covered and uncovered regions are parametrized spaces with respect to time, the 'evasion problem' calls for a parametrized version of Alexander Duality. This approach allows us to present a test for coverage in dynamic sensor networks [1].

Contents

Chapter 1

Zigzag Persistence

This chapter reviews the theory of Zigzag persistence. In Section 1.1, we formulate a few basic definitions from the theory of quiver representations and state Gabriel's theorem (for a more complete exposition of this material, see Brion [5]). Section 1.2 is devoted to zigzag modules, a special case of quiver representations [7]. In Section 1.3, we state and prove the Diamond Principle [7], which is one of the ingredients needed to prove parametrized versions of Alexander duality. In Section 1.4 we present one application of the Diamond Principle, called the Mayer-Vietoris Diamond Principle [7].

Conventions Throughout this chapter, we consider vector spaces and linear maps over a fixed field $\mathbf{k}$.

1.1 Quivers

Definition 1.1. *A* quiver *is a quadruple*

$$Q = (Q_0, Q_1, h, t),$$

where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows between them and

$$h, t\colon Q_1 \to Q_0$$

are maps assigning the head and tail to each respective arrow.

A quiver is merely a directed graph. Forgetting the orientations of the arrows yields the underlying undirected graph of a quiver.

Definition 1.2. *A* representation *V of a quiver Q consists of a collection of finite dimensional vector spaces V_i indexed by the vertices $i \in Q_0$, together with a collection of linear maps $V_a\colon V_{t(a)} \to V_{h(a)}$ indexed by the arrows $a \in Q_1$.*

The *total dimension* of a quiver representation V is $\dim_{\mathbf{k}} V = \sum_{i \in Q_0} \dim V_i$.

Every quiver has a zero representation, which assigns to each vertex the zero space (and consequently to each arrow the zero map). We can also attach a one-dimensional representation S_i to each $i \in Q_0$, such that $(S_i)_j = 0$ for $i \neq j \in Q_0$, $(S_i)_i = \mathbf{k}$, and $(S_i)_a = 0$ for all $a \in Q_1$.

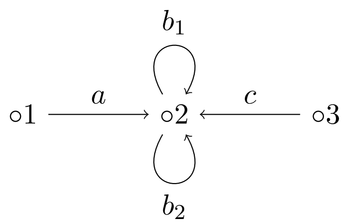


Figure 1.1: The quiver with vertices $1, 2, 3$ and arrows $a\colon 1 \to 2$, $b_1\colon 2 \to 2$, $b_2\colon 2 \to 2$, $c\colon 3 \to 2$ is depicted. Vertex 2 is the head and vertex 1 the tail of arrow a.

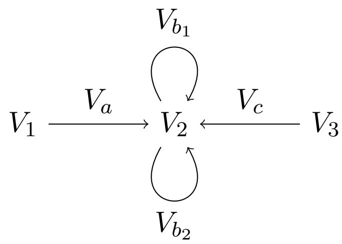
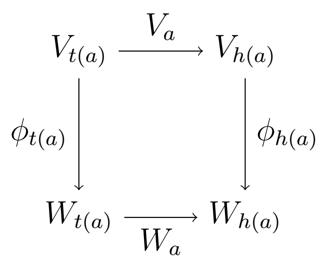


Figure 1.2: A representation of the quiver in Figure 1.1 is depicted. It is a collection of three vector spaces V_1, V_2 and V_3, together with four linear maps $V_a\colon V_1 \to V_2$, $V_c\colon V_3 \to V_2$, $V_{b_1}\colon V_2 \to V_2$ and $V_{b_2}\colon V_2 \to V_2$.

Definition 1.3. *Given representations $V = (V_i, V_a)$ and $W = (W_i, W_a)$ of a quiver Q, a* morphism *$\phi\colon V \to W$ is a collection of linear maps $(\phi_i\colon V_i \to W_i \mid i \in Q_0)$ such that the diagram*



commutes for every arrow $a \in Q_1$. That is, $W_a \circ \phi_{t(a)} = \phi_{h(a)} \circ V_a$ for all $a \in Q_1$.

For any two morphisms $\phi\colon V \to W$ and $\psi\colon W \to U$, the collection of compositions $(\psi_i \circ \phi_i)_{i \in Q_0}$ is a morphism $\psi \circ \phi\colon V \to U$. This operation is associative and $(\mathrm{Id}_{V_i})_{i \in Q_0}$ is the identity element. Therefore, for each quiver Q and field $\mathbf{k}$, we can form a category whose objects are representations of Q with morphisms as defined above. The zero object is the zero representation. We denote this category by $\mathrm{Rep}(Q, \mathbf{k})$.

Definition 1.4. *Let V and W be two representations of the quiver Q. The* direct sum $V \oplus W$ *of V and W is a collection of vector spaces*

$$(V \oplus W)_i := V_i \oplus W_i$$

for $i \in Q_0$ and linear maps

$$(V \oplus W)_a := \begin{bmatrix} V_a & 0 \\ 0 & W_a \end{bmatrix} : V_{t(a)} \oplus W_{t(a)} \to V_{h(a)} \oplus W_{h(a)}$$

for $a \in Q_1$.

If V is isomorphic to a direct sum of nonzero representations W and Z, then V is called decomposable. Otherwise V is called indecomposable. For example, the one-dimensional representation S_i is indecomposable for each $i \in Q_0$.

Any non-zero quiver representation can be decomposed into a finite direct sum of indecomposable representations. In other words, for any $V \neq 0$ there exist indecomposable representations W_j such that $V \cong W_1 \oplus \ldots \oplus W_n$. This follows by induction on the total dimension $\dim_{\mathbf{k}} V$. These decompositions, known as Remak decompositions, are not unique. However, the Krull-Schmidt theorem tells us that the summands in such a decomposition are unique up to reordering:

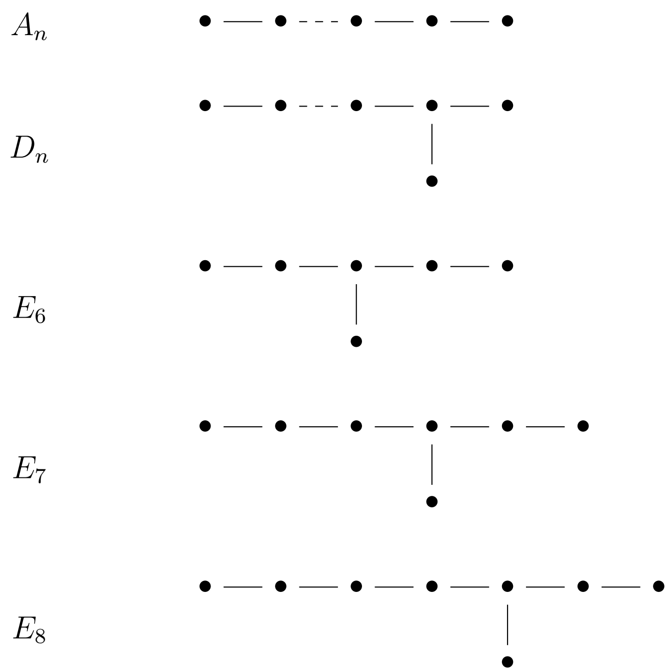
Theorem 1.5 (Krull-Remak-Schmidt)**.** *Let V be a non-zero quiver representation. In this case there exists a decomposition $V \cong a_1V_1 \oplus \ldots \oplus a_nV_n$ with the V_i pairwise non-isomorphic indecomposable representations and each $a_i \geq 1$. If $V \cong b_1W_1 \oplus \ldots \oplus b_mW_m$ is another such decomposition, then $n = m$, and a permutation π of $1, \ldots, n$ exists such that $V_i \cong W_{\pi(i)}$ for any $1 \leq i \leq n$, and $a_i = b_{\pi(i)}$.*

Proof. See Barot [2]. □

A quiver that has only finitely many isomorphism classes of indecomposable representations is called a quiver of *finite type*.

Gabriel's Theorem yields a complete description of this type of quivers:

Theorem 1.6 (Gabriel's Theorem)**.** *A quiver is of finite type if and only if the underlying undirected graph is a union of Dynkin graphs of type shown below:*



The number of isomorphism classes of indecomposable representations for the different Dynkin types is as follows:

A_n	D_n	E_6	E_7	E_8
$\frac{1}{2}n(n+1)$	$n(n-1)$	*36*	*69*	*120.*

Proof. See Gabriel [25] or Brion [5]. □

1.2 Zigzag Persistence

Applied topologists are interested in quiver representations whose underlying graph is A_n. Following Carlsson and de Silva [7] we will refer to these as *zigzag modules.*

Definition 1.7. *A zigzag module V is a sequence of finite dimensional vector spaces and maps*

$$V_1 \overset{p_1}{\longleftrightarrow} V_2 \overset{p_2}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n.$$

Each $\overset{p_i}{\longleftrightarrow}$ *represents either a forward map* $\overset{f_i}{\longrightarrow}$ *or a backward map* $\overset{g_i}{\longleftarrow}$ *.*

The sequence of f's and g's is called the type τ of a zigzag module. For example, a zigzag module of type $\tau = ffgg$ looks like this:

$$V_1 \longrightarrow V_2 \longrightarrow V_3 \longleftarrow V_4 \longleftarrow V_5.$$

The length of a zigzag module is the number of vertices in its underlying graph. For example, the zigzag module depicted above is of length 5. We call the zigzag modules of a fixed type τ and length n τ-modules. The category of τ-modules is denoted τ-Mod. By reversing all the arrows in a τ-module we get a zigzag module of type τ^{op}.

Remark 1.8. *Persistence modules (see Edelsbrunner et al. [21]) are zigzag modules of type $\tau = ff\dots f$.*

Definition 1.9. *Let τ be of length n. The* interval τ-module $\mathbb{I}_\tau(b,d)$ *with birth time b and death time d for $1 \le b \le d \le n$ is a collection of vector spaces*

$$(\mathbb{I}_\tau(b,d))_i = \begin{cases} \mathbf{k} & \text{if } b \le i \le d \\ 0 & \text{otherwise} \end{cases} .$$

It has identity maps between adjacent copies of $\mathbf{k}$ and zero maps otherwise. When τ is implicit, we usually suppress it and write $\mathbb{I}(b,d)$.

Example 1.10. *If $\tau = ffgg$, then $\mathbb{I}(2,3)$ is*

$$0 \xrightarrow{0} \mathbf{k} \xrightarrow{1} \mathbf{k} \xleftarrow{0} 0 \xleftarrow{0} 0$$

Proposition 1.11. *Interval τ-modules are indecomposable.*

Proof. Interval modules $\mathbb{I}(b,b)$ are clearly indecomposable. Now let $b < d$. Suppose $\mathbb{I}(b,d) = V \oplus W$. For $b \le i < d$ consider adjacent terms $\mathbf{k}$ at positions i and $i+1$ connected by an identity map. Since $\mathbf{k} = V_i \oplus W_i$, one of V_i, W_i is zero. Without loss of generality we assume that $V_i = 0$ and $W_i = \mathbf{k}$. It follows that $V_i \longleftrightarrow V_{i+1}$ is the zero map. It follows that $W_i \leftrightarrow W_{i+1}$ is nonzero (otherwise their direct sum would be a zero map and not the identity map as required) and consequently $W_{i+1} = \mathbf{k}$ and $V_{i+1} = 0$. Repeating this argument we get that V is 0. So interval τ-modules are indecomposable. □

We have the following important consequence of Gabriel's Theorem, which is the cornerstone of the theory of zigzag persistence.

Theorem 1.12. *The indecomposable τ-modules of length n are precisely the interval τ-modules $\mathbb{I}(b,d)$, where $1 \le b \le d \le n$. Every τ-module can be written as a direct sum of the interval τ-modules.*

Proof. By Gabriel's Theorem there are $\frac{1}{2}n(n+1)$ indecomposable τ-modules of length n, so indecomposable τ-modules are precisely the intervals $\mathbb{I}_\tau(b,d)$, where $1 \le b \le d \le n$. The second part of the theorem follows since the underlying graph of a τ-module of length n is A_n. □

Example 1.13. *Let $\tau = gf$. The underlying graph is of type A_3, so there are 6 indecomposable representations:*

$$0 \xleftarrow{0} 0 \xrightarrow{0} \mathbf{k}\ , \quad 0 \xleftarrow{0} \mathbf{k} \xrightarrow{0} 0\ , \quad \mathbf{k} \xleftarrow{0} 0 \xrightarrow{0} 0\ ,$$

$$0 \xleftarrow{0} \mathbf{k} \xrightarrow{1} \mathbf{k}\ , \quad \mathbf{k} \xleftarrow{1} \mathbf{k} \xrightarrow{0} 0\ , \quad \mathbf{k} \xleftarrow{1} \mathbf{k} \xrightarrow{1} \mathbf{k}\ .$$

Now consider the τ-module V defined as follows:

$$\mathbf{k} \xleftarrow{g_1} \mathbf{k}^2 \xrightarrow{f_2} \mathbf{k}$$

$$x \longleftarrow\!\shortmid (x, y) \longmapsto y$$

The interval decomposition of V is $\mathbb{I}(1,2) \oplus \mathbb{I}(2,3)$. The summands are $\mathbf{k} \longleftarrow \mathbf{k} \longrightarrow 0$ *and* $0 \longleftarrow \mathbf{k} \longrightarrow \mathbf{k}$ *.*

By Theorem 1.12 any τ-module V is a direct sum of interval τ-modules. By the Krull-Remak-Schmidt Theorem the summands in this decomposition are unique up to reordering. Thus the multiset of interval τ-modules that appear in the decomposition of V is an isomorphism invariant. We call it zigzag persistence of V. Recall that a multiset is a pair $A = (S, m)$, where S is a set and $m : S \to \{1, 2, 3, \ldots\} \cup \{\infty\}$ is the multiplicity function, which tells us how many times each element of S occurs in A.

Definition 1.14. *Let V be a zigzag module of type τ. By Gabriel's Theorem there exist interval τ-modules such that $V \cong \mathbb{I}(b_1, d_1) \oplus \ldots \oplus \mathbb{I}(b_n, d_n)$. The zigzag persistence of V is the multiset*

$$\operatorname{Pers}(V) = \{[b_j, d_j] \subset \{1, \ldots, n\} \mid j = 1, \ldots, n\}.$$

An interval $[b_j, d_j]$ represents a homological feature that is born at time b_j and dies at time $d_j + 1$.

We can represent $\operatorname{Pers}(V)$ graphically as a *barcode* or as a *persistence diagram.*

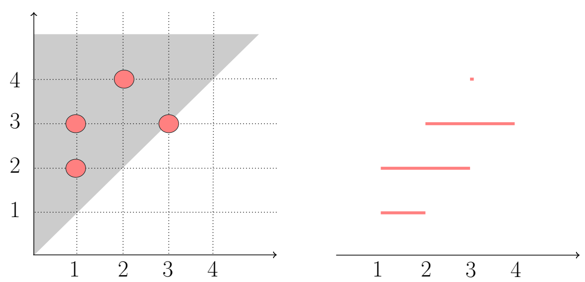


Figure 1.3: Persistence diagram (left) and barcode (right) representations of the zigzag persistence $\{[1,2], [1,3], [3,3], [2,4]\}$ of a zigzag module of length 4.

A barcode is a collection of horizontal line segments $[b_j, d_j] \subseteq [0, \infty)$ measured against a horizontal axis with labels $\{1, \ldots, n\}$. A persistence diagram is a multiset of points in $\mathbb{R}^2$ lying on or above the diagonal. We represent each interval $[b_j, d_j]$ as a point (b_j, d_j) in the plane.

1.3 The Diamond Principle

Consider the following diagram of vector spaces and linear maps between them.

$$
\begin{array}{ccccccccccc}
 & & & & & W_k & & & & & \\
 & & & & {}^{f_{k-1}}\nearrow & & \nwarrow^{g_k} & & & & \\
V_1 \overset{p_1}{\longleftrightarrow} \cdots \overset{p_{k-3}}{\longleftrightarrow} & V_{k-2} & \overset{p_{k-2}}{\longleftrightarrow} & V_{k-1} & & & & V_{k+1} & \overset{p_{k+1}}{\longleftrightarrow} & V_{k+2} & \overset{p_{k+2}}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n \\
 & & & & {}_{g_{k-1}}\nwarrow & & \nearrow_{f_k} & & & & \\
 & & & & & U_k & & & & &
\end{array}
$$

We say that the diamond in the center is exact if $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$ in the following sequence

$$U_k \xrightarrow{D_1} V_{k-1} \oplus V_{k+1} \xrightarrow{D_2} W_k,$$

where $D_1(u) = g_{k-1}(u) \oplus f_k(u)$, and $D_2(v \oplus v') = f_{k-1}(v) - g_k(v')$.

Let V^+ and V^- denote the upper and lower zigzag modules.

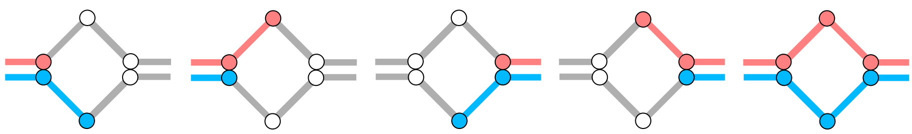
$$V^+ = \quad V_1 \overset{p_1}{\longleftrightarrow} \cdots \overset{p_{k-2}}{\longleftrightarrow} V_{k-1} \xrightarrow{f_{k-1}} W_k \xleftarrow{g_k} V_{k+1} \overset{p_{k+1}}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n,$$

$$V^- = \quad V_1 \overset{p_1}{\longleftrightarrow} \cdots \overset{p_{k-2}}{\longleftrightarrow} V_{k-1} \xleftarrow{g_{k-1}} U_k \xrightarrow{f_k} V_{k+1} \overset{p_{k+1}}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n.$$

We have the following relation between zigzag persistence of V^+ and that of V^-.

Theorem 1.15 (The Diamond Principle). *Given V^+ and V^- as above, suppose that the middle diamond is exact. There is a partial bijection between* $\mathrm{Pers}(V^+)$ *and* $\mathrm{Pers}(V^-)$. *Intervals are matched according to the following rules:*

- *Intervals of type $[k, k]$ are unmatched,*
- *Type $[b, k]$ is matched with type $[b, k-1]$ and vice versa, for $b \leq k-1$,*
- *Type $[k, d]$ is matched with type $[k+1, d]$ and vice versa, for $d \geq k+1$,*
- *Type $[b, d]$ is matched with type $[b, d]$ in all other cases.*

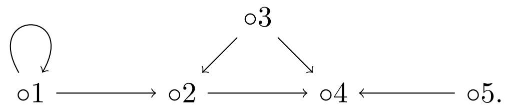


Carlsson and de Silva prove the Diamond principle in [7]. We present a different proof more suitable for this exposition involving the Bernstein, Gelfand and Ponomarev reflection functor (BGP reflection functor) [4].

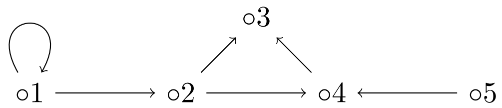
1.3.1 Reflection Functors

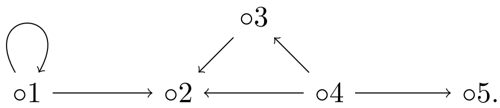
For each vertex $k \in Q_0$, we denote the quiver obtained from Q by reversing the direction of all the arrows a satisfying $t(a) = k$ or $h(a) = k$ by $\sigma_k Q$.

Example 1.16. *Let Q be the following quiver*



Then $\sigma_3 Q$ and $\sigma_4 Q$ are as follows:





To define BGP reflection functors, we need the concepts of a sink and a source in the quiver Q. A vertex k of Q is called a sink if there is no arrow in Q starting at k. The same vertex is called a source if there is no arrow ending at k.

For example, the quiver Q in Example 1.16 has two sources 3 and 5, and one sink 4. If $k \in Q_0$ is a sink, then it is a source of the quiver $\sigma_k Q$ and vice versa.

Let k be a sink in Q. For each representation $V = (V_i, V_a) \in \mathrm{Rep}(Q, \mathbf{k})$, we define a representation $\mathscr{S}_k^+(V) = (W_i, W_a) \in \mathrm{Rep}(\sigma_k Q, \mathbf{k})$ as follows. For all $i \neq k$, we set $W_i = V_i$. We define W_k to be the kernel of the map

$$\xi_k \colon \bigoplus_{a \in Q_1, h(a) = k} V_{t(a)} \longrightarrow V_k, \quad (x_{t(a)})_a \longmapsto \sum_a V_a(x_{t(a)}).$$

For an arrow a in $\sigma_k Q$, let $W_a = V_a$ if $t(a) \neq k$. If $t(a) = k$, then W_a is a composition

$$W_k = \mathrm{Ker}\, \xi_k \hookrightarrow \bigoplus\nolimits_{a \in Q_1, h(a) = k} V_{t(a)} \xrightarrow{\pi_{h(a)}} W_{h(a)},$$

where $\pi_{h(a)}$ is the canonical projection.

With each morphism $\phi = (\phi_i) \colon V \to V'$ in $\mathrm{Rep}(Q, \mathbf{k})$, we associate a morphism $\mathscr{S}_k^+(\phi) = \psi = (\psi_i) \colon \mathscr{S}_k^+(V) \to \mathscr{S}_k^+(V')$ in $\mathrm{Rep}(\sigma_k Q, \mathbf{k})$ by setting $\psi_i = \phi_i$, for $i \neq k$. We define ψ_k as a restriction map by the following natural commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ker}\, \xi_k & \hookrightarrow & \bigoplus_{a \in Q_1, h(a) = k} V_{t(a)} & \xrightarrow{\xi_k} & V_k \\
{\scriptstyle \psi_k} \downarrow & & \downarrow {\scriptstyle \bigoplus_{a \in Q_1, h(a) = k} \phi_{t(a)}} & & \downarrow {\scriptstyle \phi_k} \\
\mathrm{Ker}\, \xi'_k & \hookrightarrow & \bigoplus_{a \in Q_1, h(a) = k} V'_{t(a)} & \xrightarrow{\xi'_k} & V'_k
\end{array}$$

In this way, we obtain a functor $\mathscr{S}_k^+ \colon \mathrm{Rep}(Q, \mathbf{k}) \to \mathrm{Rep}(\sigma_k Q, \mathbf{k})$.

Example 1.17. *Consider the interval τ-module below*

$$\mathbf{k} \xrightarrow{1} \mathbf{k} \xrightarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} 0.$$

Vertex 3 in the underlying graph is a sink. Applying the $\mathscr{S}_3^+$ functor we get

$$\mathbf{k} \xrightarrow{1} \mathbf{k} \xleftarrow{1} \mathbf{k} \xrightarrow{0} 0 \xleftarrow{0} 0.$$

We construct $\mathscr{S}_k^-$ dually. Let the vertex k be a source in Q. To each representation $V = (V_i, V_a) \in \operatorname{Rep}(Q, \mathbf{k})$ we assign a representation

$$\mathscr{S}_k^-(V) = (W_i, W_a) \in \operatorname{Rep}(\sigma_k Q, \mathbf{k}).$$

For all $i \neq k$, we set $W_i = V_i$. We define W_k to be the cokernel of the map

$$\nu_k \colon V_k \longrightarrow \textstyle\bigoplus_{a\in Q_1, t(a)=k} V_{h(a)}, \qquad x \longmapsto \textstyle\bigoplus_{a\in Q_1, t(a)=k} V_a(x).$$

For an arrow a in $\sigma_k Q$, let $W_a = V_a$ if $h(a) \neq k$. If $h(a) = k$ then W_a is a composition

$$W_{t(a)} \hookrightarrow \textstyle\bigoplus_{a\in Q_1, t(a)=k} V_{h(a)} \xrightarrow{q_k} W_k$$

where q_k is the canonical quotient map.

As in the case of $\mathscr{S}_k^+$, we can assign to each morphism $\phi = (\phi_i) \colon V \to V'$ in $\operatorname{Rep}(Q, \mathbf{k})$ a morphism $\mathscr{S}_k^-(\phi) = \psi = (\psi_i) \colon \mathscr{S}_k^-(V) \to \mathscr{S}_k^-(V')$ in $\operatorname{Rep}(\sigma_k Q, \mathbf{k})$. As a result we obtain a functor $\mathscr{S}_k^- \colon \operatorname{Rep}(Q, \mathbf{k}) \to \operatorname{Rep}(\sigma_k Q, \mathbf{k})$.

Example 1.18. *Consider the following interval τ-module*

$$\mathbf{k} \xrightarrow{1} \mathbf{k} \xleftarrow{1} \mathbf{k} \xrightarrow{0} 0 \xleftarrow{0} 0.$$

Vertex 3 in the underlying graph is a source, so we can apply the $\mathscr{S}_3^-$ functor. The resulting quiver representation is

$$\mathbf{k} \xrightarrow{1} \mathbf{k} \xrightarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} 0.$$

Functors $\mathscr{S}^\pm$ respect the direct sum decomposition.

Proposition 1.19. *Let V and V' be representations of Q and k either a sink or a source. Then*

$$\mathscr{S}_k^\pm(V \oplus V') = \mathscr{S}_k^\pm(V) \oplus \mathscr{S}_k^\pm(V').$$

Proof. The statement follows from the observation that $\mathscr{S}_k^\pm$ is a functor satisfying $\mathscr{S}_k^\pm(\phi \oplus \psi) = \mathscr{S}_k^\pm(\phi) \oplus \mathscr{S}_k^\pm(\psi)$ for any pair of morphisms ϕ, ψ. □

Proposition 1.20. *Let V and V' be isomorphic representations of Q and k either a sink or a source. Then*

$$\mathscr{S}_k^{\pm}(V) \cong \mathscr{S}_k^{\pm}(V').$$

We now state and prove a theorem by Bernstein, Gelfand and Ponomarev [4].

Theorem 1.21. *Let Q be a quiver and V a representation of Q.*

1. *Let k be a sink in Q. Then there is a canonical monomorphism*

$$\phi\colon \mathscr{S}_k^- \mathscr{S}_k^+ V \to V.$$

Furthermore, ϕ splits and V is isomorphic to a direct sum of $\mathscr{S}_k^- \mathscr{S}_k^+ V$ and various copies of S_k.

2. *Let k be a source in Q. Then there is a canonical epimorphism*

$$\psi\colon V \to \mathscr{S}_k^+ \mathscr{S}_k^- V.$$

Furthermore, ψ splits and V is isomorphic to a direct sum of $\mathscr{S}_k^+ \mathscr{S}_k^- V$ and various copies of S_k.

Proof. 1. The construction gives the following diagram with the top row exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathscr{S}_k^+ V)_k & \hookrightarrow & \bigoplus_{a\in Q_1, h(a)=k} V_{t(a)} & \xrightarrow{q_k} & (\mathscr{S}_k^- \mathscr{S}_k^+ V)_k & \longrightarrow & 0 \\ & & & & \xi_k \Big\downarrow & \swarrow \phi_k & & & \\ & & & & V_k & & & & \end{array}$$

Then there is a unique linear map $\phi_k\colon (\mathscr{S}_k^- \mathscr{S}_k^+ V)_k \to V_k$ for which $\xi_k = \phi_k \circ q_k$. The map ϕ_k is injective. We can define a monomorphism $\phi\colon \mathscr{S}_k^- \mathscr{S}_k^+ V \to V$ as follows

$$\phi_i = \begin{cases} \mathrm{Id}_{V_i} & \text{for all } i \neq k \\ \phi_k & \text{otherwise.} \end{cases}$$

Moreover, this monomorphism ϕ splits and induces an isomorphism

$$V \cong \mathscr{S}_k^- \mathscr{S}_k^+ V \oplus m S_k,$$

where $m = \dim V_k - \dim \operatorname{Im} \xi_k$.

2. The proof of (2) is similar to (1). Consider the diagram

$$\begin{array}{ccccccccc} & & & & V_k & & & & \\ & & & \psi_k \swarrow & \Big\downarrow \nu_k & & & & \\ 0 & \longrightarrow & (\mathscr{S}_k^+ \mathscr{S}_k^- V)_k & \xrightarrow[\xi_k]{} & \bigoplus_{a\in Q_1, t(a)=k} V_{h(a)} & \xrightarrow[q_k]{} & (\mathscr{S}_k^- V)_k & \longrightarrow & 0. \end{array}$$

$\square$

This theorem gives rise to the following useful corollary.

Corollary 1.22. *Let k be a sink in Q. The functors $\mathscr{S}_k^+$ and $\mathscr{S}_k^-$ induce mutually inverse bijections between the isomorphism classes of indecomposable representations of Q and the isomorphism classes of indecomposable representations of $\sigma_k Q$. The exception is the representation S_k, which is annihilated by these functors.*

In the special case of zigzag modules we have:

Corollary 1.23. *Let k be a sink in the underlying graph of a τ-module V. The functors $\mathscr{S}_k^+$ and $\mathscr{S}_k^-$ induce mutually inverse bijections between the isomorphism classes of interval modules of Q and the isomorphism classes of interval modules of $\sigma_k Q$. The exception is the interval $\mathbb{I}_\tau[k,k]$, which is annihilated by these functors:*

$$\begin{array}{rcll} \mathbb{I}_\tau(k,k), \mathbb{I}_{\sigma_k\tau}(k,k) & \longrightarrow & 0 & \\ \mathbb{I}_\tau(b,k-1) & \longleftrightarrow & \mathbb{I}_{\sigma_k\tau}(b,k) & \textit{for } b \le k-1 \\ \mathbb{I}_\tau(k+1,d) & \longleftrightarrow & \mathbb{I}_{\sigma_k\tau}(k,d) & \textit{for } d \ge k+1 \\ \mathbb{I}_\tau(b,d) & \longleftrightarrow & \mathbb{I}_{\sigma_k\tau}(b,d) & \textit{in other cases.} \end{array}$$

1.3.2 Proving the Diamond Principle

To prove the Diamond Principle we will need the following lemma.

Lemma 1.24. *Let*

$$W^+ = \quad V_1 \xleftrightarrow{p_1} \cdots \xleftrightarrow{p_{k-2}} V_{k-1} \xrightarrow{f_{k-1}} W_k \xleftarrow{-g_k} V_{k+1} \xleftrightarrow{p_{k+1}} \cdots \xleftrightarrow{p_{n-1}} V_n,$$

and

$$V^- = \quad V_1 \xleftrightarrow{p_1} \cdots \xleftrightarrow{p_{k-2}} V_{k-1} \xleftarrow{g_{k-1}} U_k \xrightarrow{f_k} V_{k+1} \xleftrightarrow{p_{k+1}} \cdots \xleftrightarrow{p_{n-1}} V_n.$$

We define $D_1(u) = g_{k-1}(u) \oplus f_k(u)$ and $D_2(v \oplus v') = f_{k-1}(v) - g_k(v')$. Assume that $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$. Then $\mathscr{S}_k^+ W^+ \oplus m\mathbb{I}(k,k) \cong V^-$.

Proof. First we see that

$$\mathscr{S}_k^+ W^+ = \quad V_1 \xleftrightarrow{p_1} \cdots \xleftrightarrow{p_{k-2}} V_{k-1} \xleftarrow{\pi_{V_{k-1}}} \mathrm{Ker}(D_2) \xrightarrow{\pi_{V_{k+1}}} V_{k+1} \xleftrightarrow{p_{k+1}} \cdots \xleftrightarrow{p_{n-1}} V_n,$$

where $\pi_{V_{k-1}}$ and $\pi_{V_{k+1}}$ are canonical projections. Since $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$, we have the following isomorphism of τ-modules:

$$\begin{array}{ccccccccccccc} V_1 & \xleftrightarrow{p_1} & \cdots & \xleftrightarrow{p_{k-2}} & V_{k-1} & \xleftarrow{\pi_{V_{k-1}}} & \mathrm{Im}(D_1) & \xrightarrow{\pi_{V_{k+1}}} & V_{k+1} & \xleftrightarrow{p_{k+1}} & \cdots & \xleftrightarrow{p_{n-1}} & V_n \\ \mathrm{Id}\updownarrow & & \mathrm{Id}\updownarrow & & \mathrm{Id}\updownarrow & & \mathrm{Id}\updownarrow & & \mathrm{Id}\updownarrow & & \mathrm{Id}\updownarrow & & \mathrm{Id}\updownarrow \\ V_1 & \xleftrightarrow{p_1} & \cdots & \xleftrightarrow{p_{k-2}} & V_{k-1} & \xleftarrow{\pi_{V_{k-1}}} & \mathrm{Ker}(D_2) & \xrightarrow{\pi_{V_{k+1}}} & V_{k+1} & \xleftrightarrow{p_{k+1}} & \cdots & \xleftrightarrow{p_{n-1}} & V_n. \end{array}$$

From linear algebra we know that $U_k \cong \mathrm{Im}(D_1) \oplus \mathrm{Ker}(D_1)$. This induces an isomorphism between τ-modules

$$\begin{array}{ccccccccccccc}
V_1 & \overset{p_1}{\longleftrightarrow} & \cdots & \overset{p_{k-2}}{\longleftrightarrow} & V_{k-1} & \overset{g_{k-1}}{\longleftarrow} & U_k & \overset{f_k}{\longrightarrow} & V_{k+1} & \overset{p_{k+1}}{\longleftrightarrow} & \cdots & \overset{p_{n-1}}{\longleftrightarrow} & V_n \\
{\scriptstyle \mathrm{Id}\oplus 0}\big\downarrow & & {\scriptstyle \mathrm{Id}\oplus 0}\big\downarrow & & {\scriptstyle \mathrm{Id}\oplus 0}\big\downarrow & & {\scriptstyle \cong}\big\downarrow & & {\scriptstyle \mathrm{Id}\oplus 0}\big\downarrow & & {\scriptstyle \mathrm{Id}\oplus 0}\big\downarrow & & {\scriptstyle \mathrm{Id}\oplus 0}\big\downarrow \\
V_1 & \overset{p_1}{\longleftrightarrow} & \cdots & \overset{p_{k-2}}{\longleftrightarrow} & V_{k-1} & \overset{\pi_{V_{k-1}}}{\longleftarrow} & \mathrm{Im}(D_1) & \overset{\pi_{V_{k+1}}}{\longrightarrow} & V_{k+1} & \overset{p_{k+1}}{\longleftrightarrow} & \cdots & \overset{p_{n-1}}{\longleftrightarrow} & V_n \\
 & & & & & & \bigoplus & & & & & & \\
0 & \overset{0}{\longleftrightarrow} & \cdots & \overset{0}{\longleftrightarrow} & 0 & \overset{0}{\longleftarrow} & \mathrm{Ker}(D_1) & \overset{0}{\longrightarrow} & 0 & \overset{0}{\longleftrightarrow} & \cdots & \overset{0}{\longleftrightarrow} & 0.
\end{array}$$

$\mathbb{I}(k,k)$ span the lower quiver. It follows that $\mathscr{S}_k^+ W^+ \oplus m\mathbb{I}(k,k) \cong V^-$, where $m = \dim \mathrm{Ker}\, D_1$.

□

Proof of the Diamond Principle. The map $\phi = (\phi_i)$ where

$$\phi_i = \begin{cases} \mathrm{Id}_{V_i} & \text{for } i \neq k+1 \\ -\mathrm{Id}_{V_{k+1}} & \text{otherwise} \end{cases}$$

is an isomorphism of τ-modules V^+ and W^+. Zigzag persistence is an isomorphism invariant, so $\mathrm{Pers}(V^+) = \mathrm{Pers}(W^+)$. The quiver W^+ is decomposable by Gabriel's theorem, therefore b_i, d_i exist for $1 \leq i \leq n$ such that

$$W^+ \cong \oplus_{1\leq i\leq n} \mathbb{I}(b_i, d_i).$$

It follows from Lemma 1.24 and Proposition 1.19 that

$$\begin{array}{rcl} V^- & \cong & \mathscr{S}_k^+ W^+ \oplus m\mathbb{I}(k,k) \\ & \cong & \mathscr{S}^+(\oplus_{1\leq i\leq n}\mathbb{I}(b_i,d_i)) \oplus m\mathbb{I}(k,k) \\ & \cong & \oplus_{1\leq i\leq n}\mathscr{S}^+\mathbb{I}(b_i,d_i) \oplus m\mathbb{I}(k,k) \end{array}$$

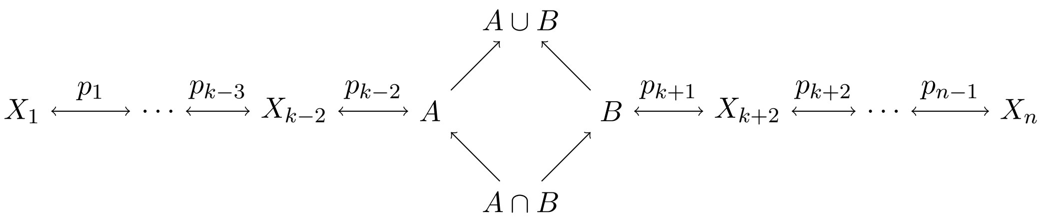
By Corollary 1.23 we have a partial bijection between $\mathrm{Pers}(V^-)$ and $\mathrm{Pers}(W^+)$ (and consequently $\mathrm{Pers}(V^+)$). The intervals change as follows:

- Intervals of type $[k,k]$ are unmatched,
- Type $[b,k]$ is matched with type $[b,k-1]$ and vice versa, for $b \leq k-1$,
- Type $[k,d]$ is matched with type $[k+1,d]$ and vice versa, for $d \geq k+1$,
- Type $[b,d]$ is matched with type $[b,d]$ in all other cases.

This finishes the proof of the Diamond Principle. □

1.4 The Mayer-Vietoris Diamond Principle

We can apply the Diamond Principle to the following diagram of topological spaces and continuous maps.



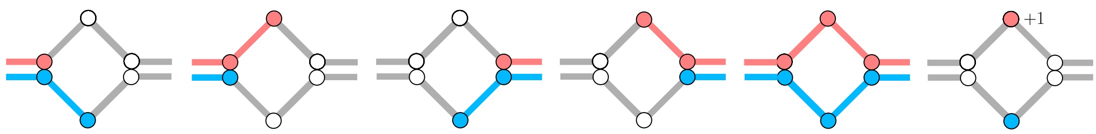
Let X^+ denote the upper and X^- the lower zigzag diagram contained in this picture. Applying the singular homology functor H_j with coefficients in $\mathbf{k}$ to X^+, we get a zigzag module that we denote by $\mathrm{H}_j(X^+)$. We denote by $\mathrm{Pers}(\mathrm{H}_*(X^+))$ the collection of $\mathrm{Pers}(\mathrm{H}_j(X^+))$ over all j. We use a similar notation for zigzag modules associated with X^-.

Theorem 1.25 (The Mayer-Vietoris Diamond Principle)**.** *Given X^+ and X^- (see above) there is a bijection between $\mathrm{Pers}(\mathrm{H}_*(X^+))$ and $\mathrm{Pers}(\mathrm{H}_*(X^-))$. Intervals are matched according to the following rules:*

- $[k, k] \in \mathrm{Pers}(\mathrm{H}_{j+1}(X^+))$ *is matched with* $[k, k] \in \mathrm{Pers}(\mathrm{H}_j(X^-))$.

In the remaining cases, the matching preserves homological dimension:

- *Type $[b, k]$ is matched with type $[b, k-1]$ and vice versa, for $b \leq k-1$,*
- *Type $[k, d]$ is matched with type $[k+1, d]$ and vice versa, for $d \geq k+1$,*
- *Type $[b, d]$ is matched with type $[b, d]$ in all other cases.*



Proof. We apply the homology functor H_j to the diagram. According to the Mayer-Vietoris theorem

$$\cdots \longrightarrow \mathrm{H}_j(A \cap B) \xrightarrow{D_1} \mathrm{H}_j(A) \oplus \mathrm{H}_j(B) \xrightarrow{D_2} \mathrm{H}_j(A \cup B) \longrightarrow \cdots$$

is an exact sequence. Therefore the diamond in the center is exact. By the Diamond Principle we have a partial bijection between $\mathrm{Pers}(\mathrm{H}_*(X^+))$ and $\mathrm{Pers}(\mathrm{H}_*(X^-))$ which accounts for all intervals except those of type $[k, k]$.

To get the bijection between intervals of type $[k,k]$, we consider the connecting homomorphism in the same Mayer-Vietoris sequence:

$$\cdots \xrightarrow{D_2} \mathrm{H}_{j+1}(A \cup B) \xrightarrow{\partial} \mathrm{H}_j(A \cap B) \xrightarrow{D_1} \cdots$$

∂ induces an isomorphism between the $\mathrm{Coker}(D_2)$ and $\mathrm{Ker}(D_1)$. Since the $[k,k]$ summands of $\mathrm{Pers}(\mathrm{H}_*(X^+))$ span $\mathrm{Coker}(D_2)$, and the $[k,k]$ summands of $\mathrm{Pers}(\mathrm{H}_*(X^-))$ span $\mathrm{Ker}(D_1)$, the statement follows. □

Chapter 2

Parametrized Spaces and Levelset Zigzag Persistence

In this chapter, we use the theory of zigzag persistence to identify homological features of some well-behaved parametrized spaces extending over a range of values of the parameter. Section 2.1 is a collection of basic definitions and statements about parametrized spaces. In Section 2.2 we show how to model a parametrized space by a zigzag diagram and present a construction called 'levelset zigzag persistence' [9].

2.1 Category of Parametrized Spaces

Definition 2.1. *A* parametrized space *is a pair* $\mathrm{X} = (X, p_X)$*, where* X *is a topological space and* $p_X \colon X \to \mathbb{R}$ *is a continuous function.*

Definition 2.2. *A* morphism of parametrized spaces $\mathrm{f} \colon \mathrm{X} \to \mathrm{Y}$ *is a continuous map* $f \colon X \to Y$ *such that* $p_Y \circ f = p_X$*.*

Proposition 2.3. *If* $f \colon X \to Y$ *is a morphism between parametrized spaces* X *and* Y*, and* $g \colon Y \to Z$ *is a morphism between parametrized spaces* Y *and* Z*, then* $g \circ f$ *is a morphism between parametrized spaces* X *and* Z*.*

The composition of parametrized morphisms is associative. Furthermore, for a parametrized space X, $\mathrm{Id}_X \colon X \to X$ is the identity morphism. Thus the class of parametrized spaces has the structure of a category.

Proposition 2.4. *Let* X *and* Y *be parametrized spaces. If* X *and* Y *are isomorphic as parametrized spaces, then* X *and* Y *are homeomorphic as topological spaces.*

Proof. Since X and Y are isomorphic as parametrized spaces, there exist continuous maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $p_Y \circ f = p_X$, $p_X \circ g = p_Y$, $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. These last two equations show that X and Y are homeomorphic. □

Definition 2.5. *Let* X *and* Y *be parametrized spaces. A parametrized embedding* $\mathrm{f} \colon \mathrm{X} \hookrightarrow \mathrm{Y}$ *is an embedding* $f \colon X \hookrightarrow Y$ *such that* $p_Y \circ f = p_X$*.*

Let (X, p_X) be a parametrized space. The function p_X defines *levelsets* $X_a = p_X^{-1}(a)$ and *slices* X_I for intervals $I \subseteq \mathbb{R}$. To make it immediately clear what interval we are referring to when building zigzags with different slices, we sometimes write $X_a^b = p_X^{-1}([a, b])$.

2.2 Levelset Zigzag Persistence

Let $\mathrm{X} = (X, p_X)$ be a parametrized space.

Given a discretization

$$a = s_0 \leq \ldots \leq s_n = b$$

of the interval $[a, b] \subseteq \mathbb{R}$ we build a zigzag diagram that models the slice X_a^b:

$$\begin{array}{ccccccccccccccccccccc} & & X_{s_0}^{s_1} & & & & X_{s_1}^{s_2} & & & & \cdots & & & & X_{s_{n-2}}^{s_{n-1}} & & & & X_{s_{n-1}}^{s_n} & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ X_{s_0} & & & & X_{s_1} & & & & X_{s_2} & & & & X_{s_{n-2}} & & & & X_{s_{n-1}} & & & & X_{s_n}. \end{array}$$

Now we apply the j-dimensional homology functor H_j to obtain:

$$\begin{array}{ccccccccccccccccccccc} & & \mathrm{H}_j(X_{s_0}^{s_1}) & & & & \mathrm{H}_j(X_{s_1}^{s_2}) & & & & \cdots & & & & \mathrm{H}_j(X_{s_{n-2}}^{s_{n-1}}) & & & & \mathrm{H}_j(X_{s_{n-1}}^{s_n}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & \\ \mathrm{H}_j(X_{s_0}) & & & & \mathrm{H}_j(X_{s_1}) & & & & \mathrm{H}_j(X_{s_2}) & & & & \mathrm{H}_j(X_{s_{n-2}}) & & & & \mathrm{H}_j(X_{s_{n-1}}) & & & & \mathrm{H}_j(X_{s_n}). \end{array}$$

We denote this diagram of vector spaces by $\mathrm{H}_j(\mathrm{X}_{\{s_0, s_1, \ldots, s_n\}})$. We want to be able to analyze how the homology changes as we vary the parameter. In order to do so, we have to restrict ourselves to a smaller class of parametrized spaces.

Definition 2.6. *A parametrized space* $\mathrm{X} = (X, p_X)$ *is of* Morse-type *if there is a finite set of real-valued indices* $a_1 < a_2 < \ldots < a_n$ *called homological critical values, such that over each open interval*

$$I \in \{(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, \infty)\}$$

the slice over I *is homeomorphic to a product of the form* $Y \times I$*, with* p_X *being the projection onto the factor* I*. Additionally, each slice* X_I *has a finitely-generated homology. Finally, each homeomorphism* $Y \times I \to X_I$ *should extend to a continuous function* $Y \times \overline{I} \to X_{\overline{I}}$*, where* $\overline{I}$ *is the closure of* I *in* $\mathbb{R}$.

Example 2.7. X *is a compact manifold and* p_X *is a Morse function.*

If we allow X to be non-compact, then in order for (X, p_X) to be of Morse-type, we need to impose some restrictions on its behavior at infinity. We say that a parametrized space is cylindrical at infinity if an $a > 0$ exists such that X_a^∞ is isomorphic to $X_a \times [a, \infty)$ and $X_{-\infty}^{-a}$ to $X_{-a} \times (-\infty, -a]$ in the category of parametrized spaces.

Example 2.8. *X is a manifold without boundary which is cylindrical at infinity, and p_X is a proper Morse function with finitely many critical points.*

Now assume that X is of Morse-type. We select a set of indices s_i which satisfy

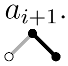
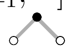
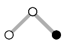
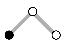
$$-\infty < s_0 < a_1 < s_1 < a_2 < \ldots < s_{n-1} < a_n < s_n < \infty.$$

Since all slices X_I have a finitely-generated homology, $\mathrm{H}_j(\mathrm{X}_{\{s_0,s_1,\ldots,s_n\}})$ is a zigzag module.

Definition 2.9. *Let* X *be of Morse-type and s_i for $i = 0, 1, \ldots, n$ as above.* The levelset zigzag persistence *of* X *in dimension j is the zigzag persistence of* $\mathrm{H}_j(\mathrm{X}_{\{s_0,s_1,\ldots,s_n\}})$. *The collection of these over all j is the levelset zigzag persistence of* X.

This definition is independent of the choice of intermediate values s_i, because of the product structure between critical values. We now need to determine how the intervals in the levelset zigzag persistence of X relate to the homological features of X. Consider, for example,

$$\begin{array}{ccc} & \mathrm{H}_j(X_{s_i}^{s_{i+1}}) & \\ \nearrow & & \nwarrow \\ \mathrm{H}_j(X_{s_i}) & & \mathrm{H}_j(X_{s_{i+1}}). \end{array}$$

Suppose an interval in the levelset zigzag persistence of X restricted to these three indices is . This means that we have a j-cycle that persists over the interval (a_i, a_{i+2}). If it is , this means that there is a j-cycle that does not exist beyond a_{i+1}. This cycle corresponds to an interval of the form $[-, a_{i+1}]$. The restriction  indicates a j-cycle that is born at a_{i+1} and corresponds to an interval $[a_{i+1}, -]$. Lastly, we consider intervals whose restriction to the three indices is a point ,  or . The first interval comes from a j-cycle that is only present at a_{i+1}, while the second appears only after a_{i+1}. The corresponding interval in this case is $(a_{i+1}, -)$. In the third case the corresponding j-cycle is only present up to a_{i+1} and the interval is $(-, a_{i+1})$.

We translate between the notation of intervals that appear in the levelset zigzag persistence of X and critical values as follows:

$$\begin{array}{lcll} [2i+2,\ 2j+2] & \leftrightarrow & [a_i, a_j] & \text{for } 1 \le i \le j \le n, \\ [2i+2,\ 2j+1] & \leftrightarrow & [a_i, a_j) & \text{for } 1 \le i < j \le n+1, \\ [2i+3,\ 2j+2] & \leftrightarrow & (a_i, a_j] & \text{for } 1 \le i \le j \le n, \\ [2i+3,\ 2j+1] & \leftrightarrow & (a_i, a_j) & \text{for } 1 \le i < j \le n+1. \end{array}$$

Here $a_0 = -\infty$ and $a_{n+1} = \infty$.

We have seen that each interval, of any of the four types, may be labelled by a corresponding point $(a_i, a_j) \in \mathbb{R}^2$. The multiset of these points labelled by homological dimension is called the levelset zigzag persistence diagram.

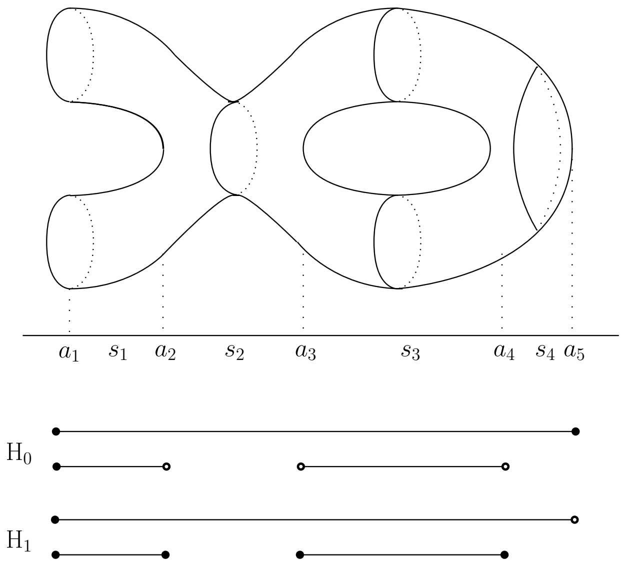


Figure 2.1: A Morse function on a 2-manifold with boundary.

Example 2.10. *Consider the surface X in Figure 2.1. Let p_X be a projection onto the horizontal axis. The parametrized space (X, p_X) is of Morse-type. The levelset zigzag persistence intervals in* H_0 *and* H_1 *expressed in critical value notation are depicted below the picture of the surface.*

Chapter 3

Decorated Persistence Diagrams and Measures

Persistence is commonly described either as a multiset of intervals (a barcode) or as a multiset of points in the half plane (a persistence diagram). Since the latter does not distinguish between different types of intervals (for example, $[p, q]$, $[p, q)$, $(p, q]$ and (p, q)), we introduce decorated points (Section 3.1). In Section 3.2 we establish a correspondence between finite rectangle measures and the decorated persistence diagrams in the half plane (the Equivalence Theorem)[14]. In Section 3.3 we extend this theorem to reach points at infinity.

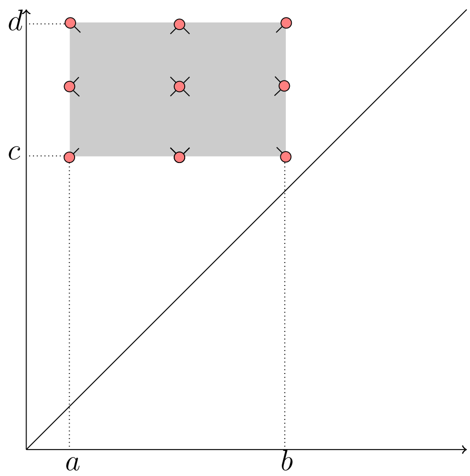
3.1 Decorated Persistence Diagrams

Chazal et al. [14] introduced decorated real numbers to keep track of all possible interval topologies (open, closed, half-open). They are written as ordinary real numbers but with a superscript $^+$ (plus) or $^-$ (minus). We can represent every decorated point as a point in the half plane with a tick specifying the decoration: We adopt the following notation:

$[p, q)$	is written	(p^-, q^-)	and drawn	
$[p, q]$	is written	(p^-, q^+)	and drawn	
(p, q)	is written	(p^+, q^-)	and drawn	
$(p, q]$	is written	(p^+, q^+)	and drawn	

We require $p < q$, except for the one point interval $[p, p]$. The notation for an arbitrary interval is (p^*, q^*).

Let $R = [a, b] \times [c, d]$, where $a < b < c < d$, be a rectangle. Let (p^*, q^*) be a decorated point. Then $(p^*, q^*) \in R$ if $[b, c] \subset (p^*, q^*) \subset (a, d)$. This happens exactly when the point (p, q) and its decoration tick are contained in the closed rectangle R:



Definition 3.1. *A decorated persistence diagram is a locally finite multiset of decorated points in the half plane.*

3.2 The Equivalence Theorem

Chazal et al. [14] introduce a new approach for expressing persistence that is especially well-suited for a continuous parameter. The basic idea is that if we know how many points of the diagram are contained in each rectangle in the half plane, then we know the diagram itself. R-measures are the result of counting the points in the rectangles.

Let $\mathscr{H} = \{(p,q) \in \mathbb{R}^2 \mid p < q\}$ be the open half plane. The set of rectangles in $\mathscr{H}$ is

$$\operatorname{Rect}(\mathscr{H}) = \{[a,b] \times [c,d] \subset \mathscr{H} \mid a < b < c < d\}.$$

Definition 3.2. *A rectangle measure or r-measure on $\mathscr{H}$ is a function*

$$\mu\colon \operatorname{Rect}(\mathscr{H}) \to \{0,1,2,3,\ldots\} \cup \{\infty\}$$

that is additive under vertical and horizontal splitting. This means that $\mu(R) = \mu(R_1) + \mu(R_2)$, whenever

$$\boxed{R} = \boxed{R_1}\boxed{R_2} \text{ or } \boxed{R} = \begin{array}{|c|}\hline R_1 \\ \hline R_2 \\ \hline \end{array}\,.$$

A rectangle measure is not a true measure in the classical sense. Instead, we combine and decompose rectangles in the sense of tiling theory.

Proposition 3.3. *Let μ be an r-measure on $\mathscr{H}$.*

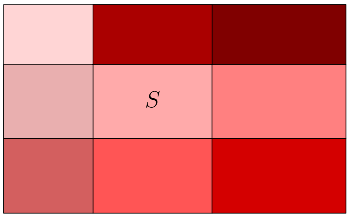
1. *μ is finitely additive. More precisely, if $R \in \operatorname{Rect}(\mathscr{H})$ can be written as a union $R = R_1 \cup R_2 \cup \ldots R_k$ of rectangles with disjoint interiors, then*

$$\mu(R) = \mu(R_1) + \ldots \mu(R_k).$$

2. *μ is monotone. If $S \subseteq R$, then $\mu(S) \subseteq \mu(R)$.*

Proof. We can prove (1) inductively using the vertical and horizontal splitting properties of μ.

For (2) consider the decomposition of R depicted below:



The claim follows since μ is nonnegative and finitely additive. □

Finite r-measures correspond exactly to decorated persistence diagrams [14].

Theorem 3.4 (The Equivalence Theorem in $\mathscr{H}$)**.** *There is a bijective correspondence between:*

- *Finite r-measures μ on $\mathscr{H}$. Here 'finite' means that $\mu(R) < \infty$ for every $R \in \mathrm{Rect}(\mathscr{H})$.*
- *Locally finite multisets A of decorated points in $\mathscr{H}$. Here 'locally finite' means that $\mathrm{card}(A|_R) < \infty$ for every $R \in \mathrm{Rect}(\mathscr{H})$.*

The measure μ corresponding to a multiset A satisfies the formula

$$\mu(R) = \mathrm{card}(A|_R)$$

for every $R \in \mathrm{Rect}(\mathscr{H})$.

Proof of the Equivalence Theorem ($\Uparrow$). Let A be a locally finite multiset of decorated points in $\mathscr{H}$. We define $\mu\colon \mathrm{Rect}(\mathscr{H}) \to \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$ by

$$\mu(R) = \mathrm{card}(A|_R).$$

This μ is finite since A is a locally finite multiset. To show that μ is additive we assume that R splits horizontally into R_1 and R_2. Each decorated point in R belongs either to R_1 or R_2, but it cannot belong to both. It follows that

$$\mu(R) = \mathrm{card}(A|_R) = \mathrm{card}(A|_{R_1}) + \mathrm{card}(A|_{R_2}) = \mu(R_1) + \mu(R_2).$$

We can use a similar proof when R splits vertically. □

Proving the other direction of the Equivalence Theorem ($\Downarrow$) is more difficult. We construct the locally finite multiset A determined by the measure μ by computing multiplicities. The multiplicity of (p^*, q^*) with respect to μ is

$$\mathrm{m}_\mu(p^*, q^*) = \min\{\mu(R) \mid (p^*, q^*) \in R, R \in \mathrm{Rect}(\mathscr{H})\}.$$

Alternatively, we can pick a nested sequence $R_1 \supset R_2 \supset R_3 \supset \ldots$ of closed rectangles that contain (p^*, q^*) such that $\cap_n R_n = (p, q)$ (see Figure 3.1). Then

$$\mathrm{m}_\mu(p^*, q^*) = \lim_{n\to\infty} \mu(R_n).$$

To see that this equation holds, first observe that for every rectangle R that contains (p^*, q^*) a large enough i exists such that $R_i \subset R$. The sequence $\{\mu(R_i)\}$ is non-increasing, because μ is monotone. Since it is also bounded below by 0, the minimum is its limit. So

$$\mathrm{m}_\mu(p^*, q^*) \leq \min \mu(R_i) = \lim_{i\to\infty} \mu(R_i) \leq \mu(R)$$

for any R containing (p^*, q^*). Taking the minimum over all R, the right side becomes $\mathrm{m}_\mu(p^*, q^*)$. By the Sandwich Theorem $\lim_{i\to\infty} \mu(R_i) = m_\mu(p^*, q^*)$.

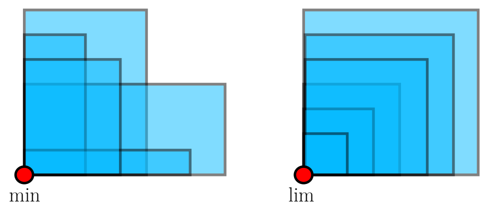


Figure 3.1: Computing multiplicities.

Proof of the Equivalence Theorem (⇓). Let μ be a finite r-measure. For each decorated point $(p^*, q^*) \in \mathscr{H}$, we compute its multiplicity. If this multiplicity is greater or equal to 1, we include $((p^*, q^*), \mathrm{m}_\mu(p^*, q^*))$ in the multiset A. We claim that this A is a unique locally finite multiset that satisfies

$$\mu(R) = \mathrm{card}(A|_R). \tag{3.1}$$

- Local finiteness follows since μ is a finite r-measure.
- Next we show that A satisfies Equation (3.1). We set

  $$\nu(R) = \mathrm{card}(A|_R) = \sum_{(p^*,q^*)\in R} \mathrm{m}_\mu(p^*, q^*).$$

  The second equation follows by the definition of multiplicity. We must show that $\nu = \mu$. We prove this by induction on $k = \mu(R)$.

  Base case

  Let $\mu(R) = 0$. For every $(p^*, q^*) \in R$ we have

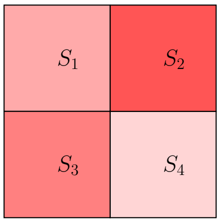
  $$0 \leq \mathrm{m}_\mu(p^*, q^*) \leq \mu(R) = 0.$$

  Consequently, $\nu(R) = 0$.

  Inductive step

  Suppose that $\mu(R) = \nu(R)$ for every rectangle R with $\mu(R) < k$. Let R_0 be a rectangle with $\mu(R_0) = k$. We must show that $\nu(R_0) = k$.

  We split R_0 into four equal quadrants S_1, S_2, S_3, S_4 (see picture).



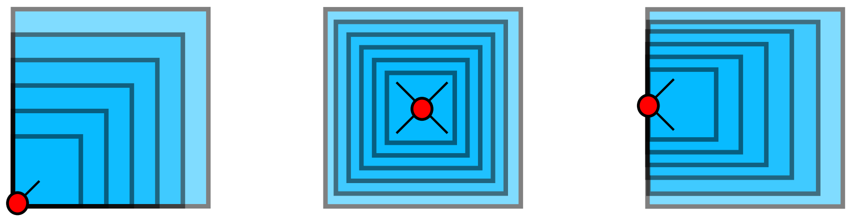
By finite additivity of measures μ and ν

$$\begin{aligned}\mu(R_0) &= \mu(S_1) + \mu(S_2) + \mu(S_3) + \mu(S_4),\\ \nu(R_0) &= \nu(S_1) + \nu(S_2) + \nu(S_3) + \nu(S_4).\end{aligned}$$

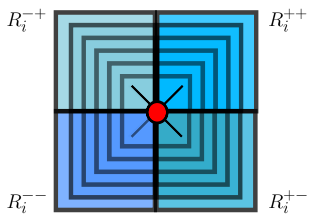
If every quadrant satisfies $\mu(S_i) < k$, then $\mu(R_0) = \nu(R_0)$ by induction. If an i exists such that $\mu(S_i) = k$, then $\mu(S_j) = 0$ for $j \neq i$ and consequently $\nu(S_j) = 0$. We set $R_1 = S_i$. It suffices to show that $\nu(R_1) = k$. We repeat the argument and subdivide R_i into four equal quadrants. The proof is complete if all four quadrants satisfy the inductive hypothesis. If not, we find a quadrant R_i with $\mu(R_i) = k$. We have to show that $\nu(R_i) = k$.

If this iteration does not terminate, we obtain a sequence of closed rectangles $R_0 \supseteq R_1 \supseteq R_2 \ldots$ with $\mu(R_i) = k$ for all $i = 1, 2, \ldots$. Since the diameters of the rectangles tend to zero, their intersection is a single point (r, s).

We now compute multiplicities of all points $(p^*, q^*) \in R_0$. Suppose that $(p, q) \neq (r, s)$. Since $\cap_i R_i = (r, s)$, an R_i must exist such that $(p^*, q^*) \notin R_i$. Consequently $(p^*, q^*) \notin R_j$ for $j \geq i$. The multiplicity of such a point is 0. So the only points with potentially nonzero multiplicity are (r^*, s^*). Their number depends on how the nested sequence of rectangles converges to its limit.



Let us assume that (r, s) lies in the interior of every rectangle R_i. In this case (r^+, s^+), (r^+, s^-), (r^-, s^+), and (r^-, s^-) all belong to every R_i. We divide each R_i into 4 rectangles R_i^{++}, R_i^{+-}, R_i^{-+}, and R_i^{--} (see picture below).



Taking limits over these four decreasing families of rectangles, we get

$$\begin{array}{rclrcl} \mathrm{m}_\mu(r^+,s^+) &=& \lim_{i\to\infty}\mu(R_i^{++}), & \mathrm{m}_\mu(r^-,s^+) &=& \lim_{i\to\infty}\mu(R_i^{-+}),\\ \mathrm{m}_\mu(r^+,s^-) &=& \lim_{i\to\infty}\mu(R_i^{+-}), & \mathrm{m}_\mu(r^-,s^-) &=& \lim_{i\to\infty}\mu(R_i^{--}). \end{array}$$

Since these are decreasing values of nonnegative integers, they eventually stabilize. Therefore, for large i

$$\begin{array}{rcl} \nu(R_0) &=& \mathrm{m}_\mu(r^+,s^+)+\mathrm{m}_\mu(r^+,s^-)+\mathrm{m}_\mu(r^-,s^+)+\mathrm{m}_\mu(r^-,s^-)\\ &=& \mu(R_i^{++})+\mu(R_i^{+-})+\mu(R_i^{-+})+\mu(R_i^{--})\\ &=& \mu(R_i)\\ &=& k. \end{array}$$

The proof is similar in the cases where only 1 or 2 of the decorated points belong to every R_i.

- Lastly, we show uniqueness. Suppose $\mathrm{m'}_\mu$ is some other multiplicity function on $\mathscr{H}$ for which

  $$\mu(R)=\nu'(R)=\sum_{(p^*,q^*)\in R}\mathrm{m'}_\mu(p^*,q^*).$$

  We need to show that $\mathrm{m}_\mu = \mathrm{m'}_\mu$.

  Let $(p^*,q^*)\in\mathscr{H}$. We pick a nested family of rectangles R_i of which each contains (p^*,q^*) at its corner. Since μ is a finite measure, (p^*,q^*) is the only decorated point with positive multiplicity m_μ or $\mathrm{m'}_\mu$ in R_i for sufficiently large i.

  Then

  $$\mathrm{m}_\mu(p^*,q^*)=\nu(R)=\mu(R)=\nu'(R)=\mathrm{m'}_\mu(p^*,q^*)$$

  and consequently $\mathrm{m}_\mu = \mathrm{m'}_\mu$.

□

Corollary 3.5. *If μ and ν are finite r-measures in the half plane that satisfy*

$$\mu(R)=\nu(R)$$

for all $R\in Rect(\mathscr{H})$, then the associated decorated persistence diagrams are the same.

3.3 The Diagram at Infinity

The decorated point notation extends to infinite intervals. Since real intervals are open at infinity, the symbols ∞ and $-\infty$ implicitly carry the superscripts ∞^+ and $-\infty^-$ even though they are usually omitted. For instance, $(-\infty^-, q)$ means $(-\infty, q)$.

To represent an infinite interval as a decorated point, we work in the extended half-plane

$$\overline{\mathscr{H}} = \mathscr{H} \cup \{-\infty\} \times \mathbb{R} \cup \mathbb{R} \times \{\infty\} \cup \{(-\infty, \infty)\},$$

which we can drawn schematically as a triangle.

To access the full persistence diagram that includes the points at infinity, we define μ on infinite rectangles. We replace $\operatorname{Rect}(\mathscr{H})$ with a larger class $\operatorname{Rect}(\overline{\mathscr{H}})$ of:

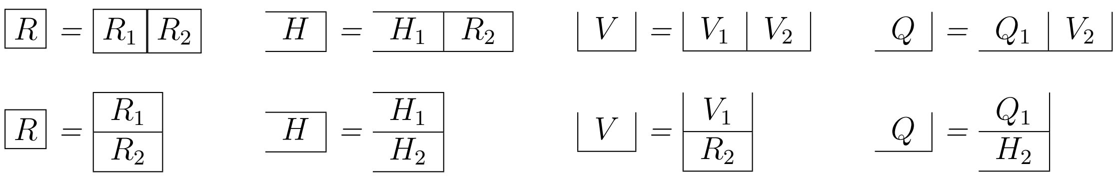
$$\begin{array}{ll} \text{finite rectangles} & R = [a,b] \times [c,d], \\ \text{horizontal strips} & H = (-\infty, b] \times [c,d], \\ \text{vertical strips} & V = [a,b] \times [c,\infty), \\ \text{quadrants} & Q = (-\infty, b] \times [c, \infty), \end{array}$$

with $a < b < c < d$.

Definition 3.6. *A rectangle measure or r-measure on* $\overline{\mathscr{H}}$ *is a function*

$$\mu_\infty \colon \operatorname{Rect}(\overline{\mathscr{H}}) \to \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$$

that is additive under vertical and horizontal splitting.



Theorem 3.7 (The Equivalence Theorem in $\overline{\mathscr{H}}$)**.** *There is a bijective correspondence between:*

- *Finite r-measures* μ_∞ *on* $\overline{\mathscr{H}}$*. Here 'finite' means that* $\mu_\infty(R) < \infty$ *for every* $R \in \operatorname{Rect}(\overline{\mathscr{H}})$*.*
- *Locally finite multisets* A *of decorated points in* $\overline{\mathscr{H}}$*. Here 'locally finite' means that* $\operatorname{card}(A|_R) < \infty$ *for every* $R \in \operatorname{Rect}(\overline{\mathscr{H}})$*.*

The measure μ_∞ *corresponding to a multiset* A *satisfies the formula*

$$\mu_\infty(R) = \operatorname{card}(A|_R)$$

for every $R \in \operatorname{Rect}(\overline{\mathscr{H}})$*.*

Proof. The following transformation of the plane

$$x' = \arctan x, \quad y' = \arctan y$$

identifies $\overline{\mathbb{R}^2}$ with the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. This claim follows since the proof of the Equivalence Theorem in $\mathscr{H}$ is invariant under this reparametrization of the plane. $\square$

Corollary 3.8. *If μ_∞ and ν_∞ are finite r-measures in the extended half plane that satisfy*

$$\mu_\infty(R) = \nu_\infty(R)$$

for all $R \in Rect(\overline{\mathscr{H}})$, then the associated decorated persistence diagrams are the same.

We compute the multiplicities of points at infinity with respect to a finite r-measure μ_∞, using the following formulas:

$$\begin{array}{lcl} m_{\mu_\infty}(-\infty, q^-) & = & \min \mu_\infty((-\infty, b] \times [c, q]), \\ m_{\mu_\infty}(-\infty, q^+) & = & \min \mu_\infty((-\infty, b] \times [q, d]), \\ m_{\mu_\infty}(p^-, \infty) & = & \min \mu_\infty([a, p] \times [c, \infty)), \\ m_{\mu_\infty}(p^+, \infty) & = & \min \mu_\infty([p, b] \times [c, \infty)), \\ m_{\mu_\infty}(-\infty, \infty) & = & \min \mu_\infty((-\infty, b] \times [c, \infty)). \end{array}$$

Chapter 4

Parametrized Homology Theories

In Chapter 2 we make use of 'levelset zigzag persistence' to determine how the homology of a parametrized space X changes as we vary the parameter. However, there we restrict ourselves to parametrized spaces for which the topology changes at only a finite set of 'critical values.' While this strategy may be appropriate when working with real-world data, it excludes some common theoretical situations. We now present a more flexible approach using measure theory [8]. In Section 4.1 we develop a number of parametrized homology theories (for example, parametrized singular homology, parametrized cohomology, etc). In Section 4.2 we identify sufficient conditions for a parametrized space to have a well-defined parametrized homology theory. The chapter finishes with a few examples.

4.1 Parametrized Homology Theories

4.1.1 Parametrized Singular Homology

Given a rectangle $R = [a, b] \times [c, d]$ with $-\infty < a < b < c < d < \infty$, our goal is to count the homological features of X that persist over the closed $[b, c]$, but not over the open interval (a, d). We assume that all slices and levelsets have finite-dimensional homology groups taken with coefficients in a field $\mathbf{k}$.

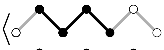
Consider the following diagram of spaces and inclusion maps.

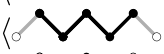
$$\begin{array}{ccccccc} & X_a^b & & X_b^c & & X_c^d & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ X_a & & X_b & & X_c & & X_d \end{array}$$

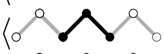
We denote it by $\mathrm{X}_{\{a,b,c,d\}}$. We apply j-dimensional homology to obtain:

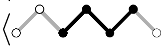
$$\begin{array}{ccccccc} & \mathrm{H}_j(X_a^b) & & \mathrm{H}_j(X_b^c) & & \mathrm{H}_j(X_c^d) & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ \mathrm{H}_j(X_a) & & \mathrm{H}_j(X_b) & & \mathrm{H}_j(X_c) & & \mathrm{H}_j(X_d). \end{array}$$

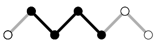
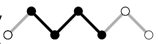
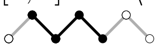
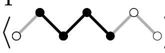
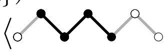
This is a representation of a quiver of type A_7. We denote it by $\mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})$. It is decomposable by Gabriel's Theorem. There are four types of indecomposable summands that meet b and c, but not a and d. By counting each of these summands, we get four quantities presented in the notation introduced by Chazal et al. [14]:

$${}_j\mu^{\backslash\backslash}_{\mathrm{X}}(R) = \langle$$  $$\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle,$$

$${}_j\mu^{\vee}_{\mathrm{X}}(R) = \langle$$  $$\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle,$$

$${}_j\mu^{\wedge}_{\mathrm{X}}(R) = \langle$$  $$\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle,$$

$${}_j\mu^{//}_{\mathrm{X}}(R) = \langle$$  $$\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle.$$

Here  represents $\mathbb{I}[2,5]$ and $\langle$$\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle$ denotes the number of times the summand  appears in the interval decomposition of $\mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})$. When the zigzag module is clear from the context, we write $\langle$$\rangle$ instead of $\langle$$\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle$.

These four quantities are functions on $\mathrm{Rect}(\mathscr{H})$ and we refer to them as pre-measures.

Definition 4.1. *A parametrized space* X *has a well-defined parametrized homology if the four pre-measures above are finite r-measures. By the Equivalence Theorem each determines a decorated persistence diagram. Let* $Dgm^*_j(\mathrm{X})$ *be the diagram determined by* ${}_j\mu^*_{\mathrm{X}}$. *The* parametrized homology *of* X, $\mathrm{Par}_{\mathrm{H}_*}(\mathrm{X})$, *is the collection of* $Dgm^{\backslash\backslash}_j(\mathrm{X})$, $Dgm^{\vee}_j(\mathrm{X})$, $Dgm^{\wedge}_j(\mathrm{X})$, *and* $Dgm^{//}_j(\mathrm{X})$ *over all* j.

The four diagrams in the definition above demonstrate how homological features perish (whether j-dimensional cycles are *killed* in homology by $(j+1)$-dimensional chains or whether they *cease to exist*):

- $\mathrm{Dgm}^{\backslash\backslash}_j(\mathrm{X})$ contains decorated points (p^*, q^*) corresponding to homology j-cycles that cease to exist beyond p, and are killed at q;
- $\mathrm{Dgm}^{\vee}_j(\mathrm{X})$ contains decorated points (p^*, q^*) corresponding to homology j-cycles that cease to exist beyond both endpoints;
- $\mathrm{Dgm}^{\wedge}_j(\mathrm{X})$ contains decorated points (p^*, q^*) corresponding to homology j-cycles that are killed at both endpoints;
- $\mathrm{Dgm}^{//}_j(\mathrm{X})$ contains decorated points (p^*, q^*) corresponding to homology j-cycles that are killed p and cease to exist beyond q.

Figure 4.1 shows examples of each type of homological feature discussed above.

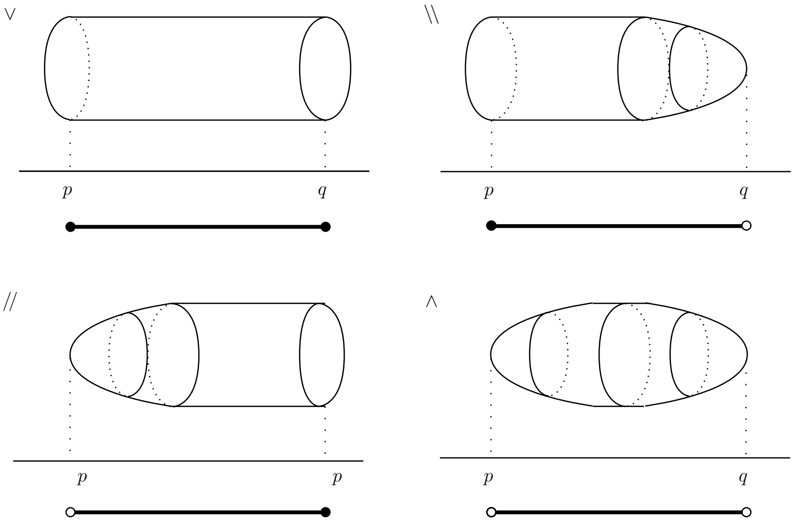


Figure 4.1: The 1-dimensional cycle on the upper left ceases to exist beyond both endpoints, whereas that on the upper right ceases to exist beyond p and and is killed by a disc at q.

The Diagram at Infinity

Let X be a parametrized space. We assume that all slices and levelsets of X have finite dimensional homology groups taken with coefficients in **k**. To access the full persistence diagram, we must extend the four pre-measures defined on $\mathrm{Rect}(\mathscr{H})$ to functions on $\mathrm{Rect}(\overline{\mathscr{H}})$. For each of the three types of infinite rectangles (horizontal strips, vertical strips and quadrants), we consider an appropriate diagram of spaces and maps:

$$H = (-\infty, b] \times [c, d] \qquad \emptyset \nearrow X^b_{-\infty} \nwarrow X_b \nearrow X^c_b \nwarrow X_c \nearrow X^d_c \nwarrow X_d,$$

$$V = [a, b] \times [c, \infty) \qquad X_a \nearrow X^b_a \nwarrow X_b \nearrow X^c_b \nwarrow X_c \nearrow X^\infty_c \nwarrow \emptyset,$$

$$Q = (-\infty, b] \times [c, \infty) \qquad \emptyset \nearrow X^b_\infty \nwarrow X_b \nearrow X^c_b \nwarrow X_c \nearrow X^\infty_c \nwarrow \emptyset.$$

Applying homology with coefficients in **k** we get a quiver of type A_7 for each infinite rectangle. We can now proceed as in the case of finite rectangles.

Observe that $(-\infty, \infty)$ is contained in $\mathrm{Dgm}^\vee_j(\mathrm{X})$ for some j, $(-\infty, q^*)$ in $\mathrm{Dgm}^\vee_j$ or $\mathrm{Dgm}^{\backslash\backslash}_j$, and (p^*, ∞) either in $\mathrm{Dgm}^\vee_j$ or $\mathrm{Dgm}^{//}_j$.

Definition 4.2. *A parametrized space* X *has a well-defined parametrized homology in the extended plane* $\overline{\mathscr{H}}$ *if the four pre-measures are finite r-measures on* $\mathrm{Rect}(\overline{\mathscr{H}})$*. By the Equivalence Theorem in* $\overline{\mathscr{H}}$ *each determines a decorated persistence diagram. Let* ${}_\infty Dgm_j^*(\mathrm{X})$ *be the diagram determined by* ${}_j\mu_\infty^*$*. The* parametrized homology *in* $\overline{\mathscr{H}}$ *of* X*,* ${}_\infty\,\mathrm{Par}_{\mathrm{H}_*}(\mathrm{X})$*, is the collection of* ${}_\infty Dgm_j^{\backslash\backslash}(\mathrm{X})$*,* ${}_\infty Dgm_j^{\vee}(\mathrm{X})$*,* ${}_\infty Dgm_j^{\wedge}(\mathrm{X})$*, and* ${}_\infty Dgm_j^{//}(\mathrm{X})$ *over all* j*.*

4.1.2 Other Parametrized (Co)homology Theories

Let X be a parametrized space. In the previous section, we applied singular homology to $\mathrm{X}_{\{a,b,c,d\}}$ for each rectangle $[a,b]\times[c,d]$ in the half plane. We can apply the same procedure to any other homology theory functor. We always take coefficients in a field **k**.

As in the standard version of Alexander Duality, reduced homology groups also appear in the parametrized version. We denote the four measures with respect to $\widetilde{\mathrm{H}}_j$ by ${}_j\widetilde{\mu}_{\mathrm{X}}^{\backslash\backslash}$, ${}_j\widetilde{\mu}_{\mathrm{X}}^{\vee}$, ${}_j\widetilde{\mu}_{\mathrm{X}}^{\wedge}$, and ${}_j\widetilde{\mu}_{\mathrm{X}}^{//}$. The corresponding diagrams are $\widetilde{\mathrm{D}}\mathrm{gm}_j^{\backslash\backslash}(\mathrm{X})$, $\widetilde{\mathrm{D}}\mathrm{gm}_j^{\vee}(\mathrm{X})$, $\widetilde{\mathrm{D}}\mathrm{gm}_j^{\wedge}(\mathrm{X})$, and $\widetilde{\mathrm{D}}\mathrm{gm}_j^{//}(\mathrm{X})$. The *reduced parametrized homology*, $\mathrm{Par}_{\widetilde{\mathrm{H}}_*}(\mathrm{X})$, is the collection of these diagrams over all j.

We draw on two parametrized cohomology theories in order to formulate the parametrized Alexander Duality Theorem. Firstly, singular cohomology yields *parametrized cohomology* of X ($\mathrm{Par}_{\mathrm{H}^*}(\mathrm{X})$). Secondly, Čech cohomology yields *parametrized Čech cohomology* of X ($\mathrm{Par}_{\check{\mathrm{H}}^*}(\mathrm{X})$).

We treat the points at infinity as in the case of parametrized singular homology.

4.2 Pre-measures into Measures

For the four pre-measures defined at the beginning of Section 4.1 to be r-measures, they must be finite and additive with respect to horizontal and vertical splitting. In this section, we discuss the necessary requirements for pre-measures on parametrized spaces to be r-measures. For parametrized homology we summarize the proofs that appear in Carlsson et al. [8]. Since one version of Alexander Duality requires a parametrized version of Čech cohomology, we discuss additivity and finiteness of pre-measures associated to Čech cohomology. To prove finiteness of pre-measures for Čech cohomology we have to define a dual of well groups [3]. Uninspiringly, we call them *dual well groups*.

4.2.1 Additivity

Parametrized Singular Homology

Let $\mathrm{X}=(X,p_X)$ be a parametrized space and let H_j be a singular homology functor with coefficients in **k**. The proof of additivity requires the Mayer-Vietoris Diamond

Principle for diagrams of the form

$$\begin{array}{ccccc} & & \mathrm{H}_j(X_p^r) & & \\ & \nearrow & & \nwarrow & \\ \mathrm{H}_j(X_p^q) & & & & \mathrm{H}_j(X_q^r) \\ & \nwarrow & & \nearrow & \\ & & \mathrm{H}_j(X_q) & & \end{array}$$

where $p < q < r$. Since this principle follows from the Mayer-Vietoris Theorem, we have to make sure that it applies. For singular homology, the theorem assumes that the triad (X, A, B) is excisive with respect to singular homology. This is not true in our case, because $\mathrm{Int}(X_p^q) \cup \mathrm{Int}(X_q^r)$ does not always include X_q.

This means that we must restrict X to those parametrized spaces whose level sets are embedded in a certain way. For example, this holds when X_q is a neighborhood deformation retract of X_q^r or of X_p^q. We can thicken slightly on that side of X_q to satisfy the interior covering condition without changing the homotopy types in the diagram. This is satisfied if X is a locally finite simplicial complex and p_X a proper piecewise-linear map. Similarly, we can apply the Mayer-Vietoris Theorem if X is a manifold and p_X is a proper Morse function.

Another option would be to use a homology theory that satisfies the 'strong excision' axiom, for which the Mayer-Vietoris Theorem holds without any restrictions on how A and B cover X. Carlsson et al. explore this option in [8].

Theorem 4.3. *Let (X, p_X) be a parametrized space. If X_q is a neighborhood deformation retract of X_q^r or of X_p^q for all $p < q < r$, then each of the four pre-measures ${}_j\mu_{\mathrm{X}}^{\backslash\backslash}$, ${}_j\mu_{\mathrm{X}}^{\vee}$, ${}_j\mu_{\mathrm{X}}^{\wedge}$ and ${}_j\mu_{\mathrm{X}}^{//}$ is additive.*

Proof. First we prove that ${}_j\mu_{\mathrm{X}}^{\backslash\backslash}$ is additive with to respect to horizontal splitting. Let

$$R = [a, b] \times [c, d] \quad R_1 = [a, b] \times [c, q] \quad R_2 = [a, b] \times [q, d]$$

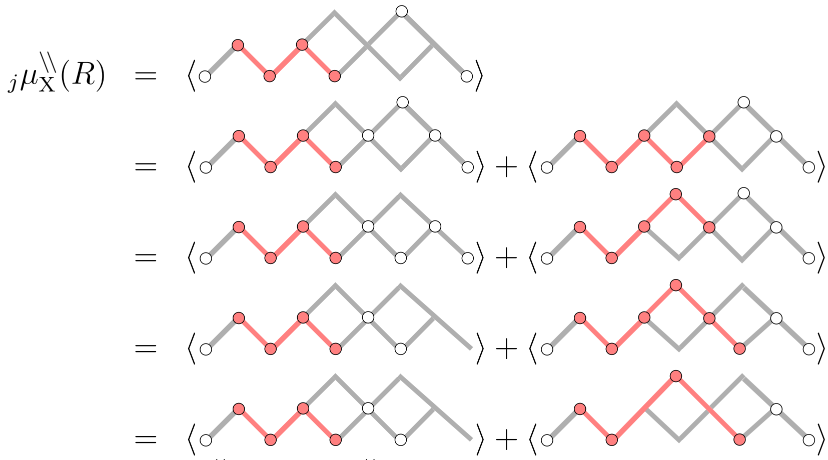
for $a < b < c < q < d$.

Consider the following diagram of spaces and maps:

$$\begin{array}{ccccccccccccccccc} &&&&&&&& X_b^q &&&& X_c^d &&&& \\ &&&&&&& \nearrow && \nwarrow && \nearrow && \nwarrow &&& \\ && X_a^b &&&& X_b^c &&&& X_c^q &&&& X_q^d && \\ & \nearrow && \nwarrow && \nearrow && \nwarrow && \nearrow && \nwarrow && \nearrow && \nwarrow & \\ X_a &&&& X_b &&&& X_c &&&& X_q &&&& X_d \end{array}$$

We apply homology functor H_j to this diagram. The diagram that we get contains the defining zigzags for all three terms. Moreover, both diamonds are exact.

As a result, we can use the Mayer-Vietoris Diamond Principle.



$$= {}_j\mu^{\backslash\backslash}_{\mathrm{X}}(R_1) + {}_j\mu^{\backslash\backslash}_{\mathrm{X}}(R_2).$$

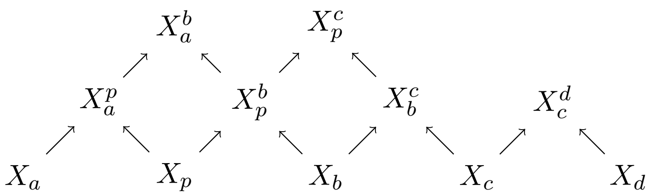
In the first step we refine the 7-term zigzag in the first line to the 9-term zigzag in the second line, by including nodes $\mathrm{H}_j(X_c^q)$ and $\mathrm{H}_j(X_q^d)$. Every interval decomposition of the 9-term zigzag induces an interval decomposition of the 7-term zigzag. So every interval in the 7-term zigzag arises from an interval in the 9-term zigzag that restricts to it. This gives us the two terms in the second line. In the following steps, we combine this restriction principle with the Mayer-Vietoris Diamond Principle to finish the proof of additivity with respect to the horizontal splitting.

To show additivity with respect to vertical splitting, let

$$R = [a, b] \times [c, d] \quad R_1 = [a, p] \times [c, d] \quad R_2 = [p, b] \times [c, d]$$

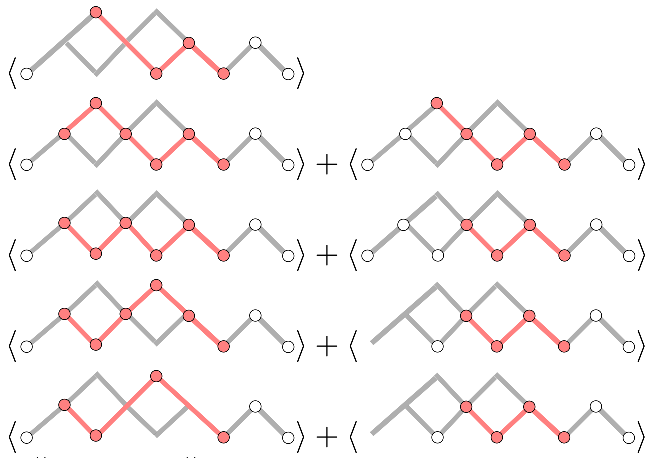
for $a < b < c < q < d$.

Consider the following diagram of spaces and maps:



We apply the homology functor H_j to this diagram. We use the same procedure to calculate:

$${}_j\mu_{\mathrm{X}}^{\backslash\backslash}(R) = \ldots$$



$$= {}_j\mu_{\mathrm{X}}^{\backslash\backslash}(R_1) + {}_j\mu_{\mathrm{X}}^{\backslash\backslash}(R_2).$$

The proof for the other three pre-measure follows the same pattern. □

Čech Cohomology

The cohomology theory we are interested in is Čech cohomology. Let (X, p_X) be a parametrized space. We assume throughout this section that X is Hausdorff and paracompact. Since Čech cohomology satisfies the 'strong excision' axiom, Mayer-Vietoris holds for closed pairs without any restrictions. The diamonds that come into play in this setting are

$$\begin{array}{ccc} & \check{\mathrm{H}}^j(X_p^r) & \\ \swarrow & & \searrow \\ \check{\mathrm{H}}^j(X_p^q) & & \check{\mathrm{H}}^j(X_q^r) \\ \searrow & & \swarrow \\ & \check{\mathrm{H}}^j(X_q). & \end{array}$$

Theorem 4.4. *Let (X, p_X) be a parametrized space. Each of the four pre-measures ${}^j\check{\mu}_{\mathrm{X}}^{\backslash\backslash}$, ${}^j\check{\mu}_{\mathrm{X}}^{\vee}$, ${}^j\check{\mu}_{\mathrm{X}}^{\wedge}$ and ${}^j\check{\mu}_{\mathrm{X}}^{//}$ is additive.*

Proof. See the proof of Theorem 4.3. All the diamonds are exact because Čech cohomology has 'strong excision' property. □

4.2.2 Finiteness

Parametrized Singular Homology

Let $\mathrm{X} = (X, p_X)$ be a parametrized space. Recall that a measure μ is finite if $\mu(R) < \infty$ for all rectangles in the half plane $R = [a, b] \times [c, d]$.

Bendich et al. [3] define well groups to capture the notion of a 'stable topological feature'.

Definition 4.5. *Let* H_* *denote singular homology. The well group of* (X, p_X) *is*

$$WX_{m-\epsilon}^{m+\epsilon} = \cap_Y \operatorname{Im}(\mathrm{H}_j(Y) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon}))$$

the intersection being taken over all possible $Y = g^{-1}(m)$*, where* $g\colon X \to \mathbb{R}$ *satisfies* $||g - p_X|| \leq \epsilon$*.*

Bendich et al. [3] also prove the following.

Theorem 4.6. *Let* X *be a parametrized space. We assume that for each triple* $p < q < r$ X_q *is a neighborhood deformation retract of* X_q^r *or of* X_p^q*. Then*

$$WX_{m-\epsilon}^{m+\epsilon} = \operatorname{Im}(\mathrm{H}_j(X_{m-\epsilon}) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon})) \cap \operatorname{Im}(\mathrm{H}_j(X_{m+\epsilon}) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon}))$$

for all m *and* ϵ*.*

In other words, the intersection in the definition of the well group is cut out by the terms $X_{m-\epsilon}$ and $X_{m+\epsilon}$.

Proof. The containment $\subseteq$ follows by definition of the intersection. We must prove $\supseteq$. It suffices to show that

$$\operatorname{Im}(\mathrm{H}_j(Y) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon})) \supseteq \operatorname{Im}(\mathrm{H}_j(X_{m-\epsilon}) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon})) \cap \operatorname{Im}(\mathrm{H}_j(X_{m+\epsilon}) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon}))$$

for $Y = g^{-1}(m)$ where $||g - p_X|| \leq \epsilon$.

Now we fix a function $g\colon X \to \mathbb{R}$ for which $||g - p_X|| \leq \epsilon$. Let $Y = g^{-1}(m)$. We write $X_{m-\epsilon}^{m+\epsilon} = A \cup B$ where

$$A = \{x \in X \mid g(x) \leq m\} \cap X_{m-\epsilon}^{m+\epsilon} \quad \text{and} \quad B = \{x \in X \mid g(x) \geq m\} \cap X_{m-\epsilon}^{m+\epsilon}.$$

The intersection of A and B is Y. Since $||g - p_X|| \leq \epsilon$, we also have $X_{m+\epsilon} \subset B$ and $X_{m-\epsilon} \subset A$.

Consider the following diagram of spaces and maps:

$$\begin{array}{ccccccc} & & & \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon}) & & & \\ & & \nearrow & & \nwarrow & & \\ \mathrm{H}_j(X_{m-\epsilon}) & \longrightarrow & \mathrm{H}_j(A) & & \mathrm{H}_j(B) & \longleftarrow & \mathrm{H}_j(X_{m+\epsilon}) \\ & & \nwarrow & & \nearrow & & \\ & & & \mathrm{H}_j(Y) & & & \end{array}$$

Suppose $c_{X_{m-\epsilon}^{m+\epsilon}} \in \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon})$ is a homology cycle that belongs to

$$\operatorname{Im}(\mathrm{H}_j(X_{m-\epsilon}) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon})) \cap \operatorname{Im}(\mathrm{H}_j(X_{m+\epsilon}) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon})).$$

By definition $c_{X_{m-\epsilon}} \in \mathrm{H}_j(X_{m-\epsilon})$ and $c_{X_{m+\epsilon}} \in \mathrm{H}_j(X_{m+\epsilon})$ exist that map to $c_{X_{m-\epsilon}^{m+\epsilon}}$ under corresponding maps induced by inclusions. We can push $c_{X_{m-\epsilon}}, c_{X_{m+\epsilon}}$ to cycles $c_A \in \mathrm{H}_j(A)$ and $c_B \in \mathrm{H}_j(B)$ that both map to $c_{X_{m-\epsilon}^{m+\epsilon}}$. Since the diamond is exact there exists a class c_Y that gets mapped both to c_A and c_B. Therefore $c_{X_{m-\epsilon}^{m+\epsilon}}$ being the image of c_Y belongs to $\operatorname{Im}(\mathrm{H}_j(Y) \to \mathrm{H}_j(X_{m-\epsilon}^{m+\epsilon}))$. $\square$

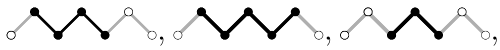
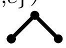
Proposition 4.7. *Let* X *be as in Theorem 4.6. For* $R = [a,b] \times [c,d]$ *with* $a < b < c < d$ *and* $* = \backslash\backslash, \wedge, \vee, //$*, we have*

$$_j\mu^*_X(R) \leq \dim WX^c_b.$$

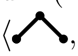
Proof. By Theorem 4.6 $WX^c_b = \mathrm{Im}(\mathrm{H}_j(X_b) \to \mathrm{H}_j(X^c_b)) \cap \mathrm{Im}(\mathrm{H}_j(X_c) \to \mathrm{H}_j(X^c_b))$ when we take $m - \epsilon = b$ and $m + \epsilon = c$.

The dimension of the well group is the number of summands of type  of the diagram

$$\begin{array}{ccc} & \mathrm{H}_j(X^c_b) & \\ \nearrow & & \nwarrow \\ \mathrm{H}_j(X_b) & & \mathrm{H}_j(X_c) \end{array}$$

We denote it by $\mathrm{H}_j(\mathrm{X}_{\{b,c\}})$. Each of the four types   restricts to  when supported only over these three indices. Therefore

$$\dim WX^c_b = \dim(\mathrm{Im}(\mathrm{H}_j(X_b) \to \mathrm{H}_j(X^c_b)) \cap \mathrm{Im}(\mathrm{H}_j(X_c) \to \mathrm{H}_j(X^c_b)))$$

$= \langle$  $, \mathrm{H}_j(\mathrm{X}_{\{b,c\}})\rangle$

$$\geq {}_j\mu^*_X(R).$$

□

Theorem 4.8. *Let* $\mathrm{X} = (X, p_X)$ *where* X *is a locally compact triangulable space and* p_X *a proper continuous map. Then* $_j\mu^*_X$ *is finite for* $* = \backslash\backslash, \wedge, \vee, //$.

Proof. First we show that every $WX^{m+\epsilon}_{m-\epsilon}$ is finite dimensional. We fix a piecewise-linear structure on X, and approximate $p_X : X \to \mathbb{R}$ with a piecewise-linear map $g : X \to \mathbb{R}$ for which $||g - p_X|| \leq \epsilon$. Since g is also proper, $Y = g^{-1}(m)$ is triangulable as a finite simplicial complex. By Theorem 4.6 it follows that

$$\begin{aligned} \dim WX^{m+\epsilon}_{m-\epsilon} &\leq \dim \mathrm{Im}(\mathrm{H}_j(Y) \to \mathrm{H}_j(X^{m+\epsilon}_{m-\epsilon})) \\ &\leq \dim \mathrm{H}_j(Y) \\ &< \infty. \end{aligned}$$

We can apply the theorem since Y is a neighbourhood retract (being a piecewise-linear subspace of a piecewise linear space). The claim now follows by Proposition 4.7. □

We can deduce the following from Theorem 4.3 and Theorem 4.8.

Theorem 4.9. *A parametrized space* $\mathrm{X} = (X, p_X)$ *has a well-defined parametrized homology if:*

- X *is a compact manifold with a boundary and* p_X *is Morse;*
- X *is a finite simplicial complex and* p_X *is a piecewise-linear map.*

Remark 4.10. *In the case when* (X, p_X) *has a well-defined parametrized homology and levelset zigzag persistence diagram, leaving out the decorations on the points in the parametrized homology yields the levelset zigzag persistence diagram.*

Remark 4.11. *When* X *is a compact manifold and* p *is Morse, the four decorations correspond exactly with how features perish at endpoints [8]:*

This is not always the case as we see in Example 5.24.

Čech Cohomology

To observe finiteness for parametrized Čech cohomology, we need a different approach. First we need a concept dual to well groups for singular homology.

Definition 4.12. *Let* $\check{\mathrm{H}}^j$ *be the Čech cohomology functor. The* dual well group *of* (X, p_X) *is*

$$W^*X_{m-\epsilon}^{m+\epsilon} = \sum_Y \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(Y)),$$

where the internal sum is taken over all $Y = g^{-1}(m)$*, where* $g\colon X \to \mathbb{R}$ *satisfies* $||g - p_X|| \leq \epsilon$*.*

The following equivalent of Theorem 4.6 holds.

Theorem 4.13. *Let* X *be a parametrized space. Then*

$$W^*X_{m-\epsilon}^{m+\epsilon} = \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m-\epsilon})) + \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m+\epsilon}))$$

for all m *and* ϵ*.*

Proof. The containment $\supseteq$ follows by definition of the sum (take $g = p_X + \epsilon$ and $g = p_X - \epsilon$).

We fix a function $g\colon X \to \mathbb{R}$ for which $||g - p_X|| \leq \epsilon$. Let $Y = g^{-1}(m)$. To prove $\subseteq$ it suffices to show that

$$\mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(Y)) \subseteq \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m-\epsilon})) + \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m+\epsilon})).$$

We write $X_{m-\epsilon}^{m+\epsilon} = A \cup B$ where

$$A = \{x \in X \mid g(x) \leq m\} \cap X_{m-\epsilon}^{m+\epsilon} \quad \text{and} \quad B = \{x \in X \mid g(x) \geq m\} \cap X_{m-\epsilon}^{m+\epsilon}.$$

The intersection of A and B is Y. Since $||g - p_X|| \leq \epsilon$, we also have $X_{m+\epsilon} \subset B$ and $X_{m-\epsilon} \subset A$.

Consider the following diagram of spaces and maps:

$$\begin{array}{ccccccc} & & & \check{\mathrm{H}}^j(Y) & & & \\ & & \nearrow & & \nwarrow & & \\ \check{\mathrm{H}}^j(X_{m-\epsilon}) & \longleftarrow & \check{\mathrm{H}}^j(A) & & \check{\mathrm{H}}^j(B) & \longrightarrow & \check{\mathrm{H}}^j(X_{m+\epsilon}) \\ & & \nwarrow & & \nearrow & & \\ & & & \check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) & & & \end{array}$$

Suppose $c_{X_{m-\epsilon}^{m+\epsilon}} \in \check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon})$ belongs to $\mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(Y))$. Under maps induced by inclusions, $c_{X_{m-\epsilon}^{m+\epsilon}}$ gets mapped to $c_A \in \check{\mathrm{H}}^j(A)$ and $c_B \in \check{\mathrm{H}}^j(B)$. Both c_A and c_B are mapped to 0 in $\check{\mathrm{H}}^j(Y)$ with corresponding maps induced by inclusions. Čech cohomology has strong excision property, so the diamond is exact. By exactness, $c_\alpha \in \check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon})$ and $c_\beta \in \check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon})$ exist such that

$$c_\alpha \mapsto c_A \oplus 0 \quad \text{and} \quad c_\beta \mapsto 0 \oplus c_B.$$

Next we observe that

$$c_{X_{m-\epsilon}^{m+\epsilon}} - c_\beta \in \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m+\epsilon})), \quad c_{X_{m-\epsilon}^{m+\epsilon}} - c_\alpha \in \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m-\epsilon})),$$

and

$$c_{X_{m-\epsilon}^{m+\epsilon}} - c_\beta - c_\alpha \in \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m-\epsilon})) \cap \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m+\epsilon})).$$

Since we can write

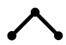
$$c_{X_{m-\epsilon}^{m+\epsilon}} = (c_{X_{m-\epsilon}^{m+\epsilon}} - c_\beta) + (c_{X_{m-\epsilon}^{m+\epsilon}} - c_\alpha) - (c_{X_{m-\epsilon}^{m+\epsilon}} - c_\beta - c_\alpha),$$

$c_{X_{m-\epsilon}^{m+\epsilon}}$ is contained in $\mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m-\epsilon})) + \mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(X_{m+\epsilon}))$. □

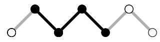
Proposition 4.14. *Let* X *be a parametrized space. For* $R = [a,b] \times [c,d]$ *with* $a < b < c < d$ *and* $* = \backslash\backslash, \wedge, \vee, //$, *we have*

$${}^j\check{\mu}_X^*(R) \leq \dim(\check{\mathrm{H}}^j(X_b^c)/W^* X_b^c).$$

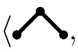
Proof. By Theorem 4.13 $W^* X_b^c = \mathrm{Ker}(\check{\mathrm{H}}^j(X_b^c) \to \check{\mathrm{H}}^j(X_b)) + \mathrm{Ker}(\check{\mathrm{H}}^j(X_b^c) \to \check{\mathrm{H}}^j(X_c))$ when we take $m - \epsilon = b$ and $m + \epsilon = c$.

The dimension of $\check{\mathrm{H}}^j(X_b^c)/W^* X_b^c$ is the number of summands of type  of the diagram

$$\begin{array}{ccc} & \check{\mathrm{H}}^j(X_b^c) & \\ \swarrow & & \searrow \\ \check{\mathrm{H}}^j(X_b) & & \check{\mathrm{H}}^j(X_c). \end{array}$$

We denote it by $\check{\mathrm{H}}^j(\mathrm{X}_{\{b,c\}})$. Each of the four types , , ,  restricts to  when supported only over these three indices. Therefore

$$\begin{aligned} \dim \check{\mathrm{H}}^j(X_b^c)/W^*X_b^c &= \dim \check{\mathrm{H}}^j(X_b^c)/(\mathrm{Ker}(\check{\mathrm{H}}^j(X_b^c) \to \check{\mathrm{H}}^j(X_b)) + \mathrm{Ker}(\check{\mathrm{H}}^j(X_b^c) \to \check{\mathrm{H}}^j(X_c))) \\ &= \langle \ \cdot \ , \check{\mathrm{H}}^j(\mathrm{X}_{\{b,c\}})\rangle \\ &\geq {}^j\check{\mu}_X^*(R). \end{aligned}$$



□

Theorem 4.15. *Let* $\mathrm{X} = (X, p_X)$ *where* X *is a locally compact triangulable space and* p_X *a proper continuous map. Then* ${}^j\check{\mu}_X^*$ *is finite for* $* = \backslash\backslash, \wedge, \vee, //$.

Proof. First we show that every $\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon})/W^*X_{m-\epsilon}^{m+\epsilon}$ is finite dimensional. We fix a piecewise-linear structure on X, and approximate $p_X : X \to \mathbb{R}$ with a piecewise-linear map $g : X \to \mathbb{R}$ for which $||g - p_X|| \leq \epsilon$. Since g is also proper, $Y = g^{-1}(m)$ is triangulable as a finite simplicial complex. Being a piecewise-linear subspace of a piecewise-linear space it is a neighborhood retract, so

$$\begin{aligned} \dim \check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon})/W^*X_{m-\epsilon}^{m+\epsilon} &\leq \dim \check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon})/\,\mathrm{Ker}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(Y)) \\ &= \dim \mathrm{Im}(\check{\mathrm{H}}^j(X_{m-\epsilon}^{m+\epsilon}) \to \check{\mathrm{H}}^j(Y)) \\ &\leq \dim \check{\mathrm{H}}^j(Y) \\ &< \infty. \end{aligned}$$

The claim now follows by Proposition 4.14. □

We can deduce the following from Theorem 4.4 and Theorem 4.15.

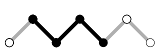
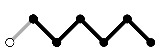
Theorem 4.16. *A parametrized space* $\mathrm{X} = (X, p_X)$ *has a well-defined parametrized Čech cohomology if* X *is a locally compact triangulable space and* p_X *is a proper continuous map.*

4.3 Examples

First we revisit Example 2.10. This parametrized space of Morse-type has a well-defined parametrized homology whose diagram lies in the half plane.

Example 4.17. *Consider the surface* X *in Figure 4.2. Since the projection* p_X *onto the horizontal axis is Morse,* X *has a well-defined parametrized homology.*

To determine the diagrams belonging to each of the four measures, we compute the multiplicities of the decorated points. When p *or* q *is a regular point, the multiplicity of* (p^*, q^*) *is* 0 *for the four measures defined above. The only situations we have left to compute are when* p *and* q *are critical points. For example, we can now calculate the multiplicities of* (a_1^-, a_2^-) *with respect to the four measures. Pick* $\epsilon > 0$ *such that* $a_1 - \epsilon < a_1 < a_2 - \epsilon < a_2$*. We have*

$$\mathrm{H}_0(\mathrm{X}_{a_1-\epsilon, a_1, a_2-\epsilon, a_2}) \cong$$  $\oplus$ .

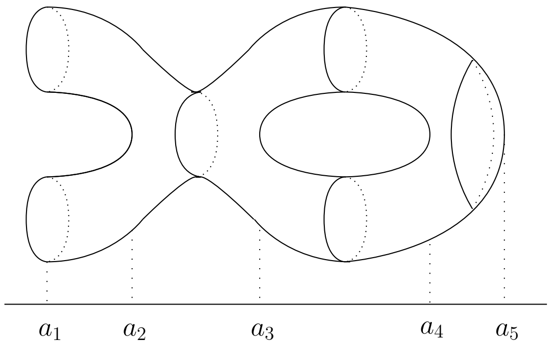


Figure 4.2: Morse function on a 2-manifold with boundary.

The summand on the right is not registered by any of the measures, whereas the one on the left is detected by ${}_0\mu_X^{\setminus\setminus}$. *Since these values are the same for all* $0 < \epsilon < a_2 - a_1$, *we have*

$$\begin{aligned}
m_{{}_0\mu_X^{\setminus\setminus}}(a_1^-, a_2^-) &= \lim_{\epsilon \to 0} {}_0\mu_X^{\setminus\setminus}([a_1 - \epsilon, a_1] \times [a_2 - \epsilon, a_2]) &= 1, \\
m_{{}_0\mu_X^{\vee}}(a_1^-, a_2^-) &= \lim_{\epsilon \to 0} {}_0\mu_X^{\vee}([a_1 - \epsilon, a_1] \times [a_2 - \epsilon, a_2]) &= 0, \\
m_{{}_0\mu_X^{\wedge}}(a_1^-, a_2^-) &= \lim_{\epsilon \to 0} {}_0\mu_X^{\wedge}([a_1 - \epsilon, a_1] \times [a_2 - \epsilon, a_2]) &= 0, \\
m_{{}_0\mu_X^{//}}(a_1^-, a_2^-) &= \lim_{\epsilon \to 0} {}_0\mu_X^{//}([a_1 - \epsilon, a_1] \times [a_2 - \epsilon, a_2]) &= 0.
\end{aligned}$$

This means that (a_1^-, a_2^-) *is a point in the decorated persistence diagram belonging to* ${}_0\mu_X^{\setminus\setminus}$ *with a multiplicity of 1. The corresponding 0-homology cycle ceases to exist beyond* a_1, *and is killed at* a_2. *We repeat this procedure to compute the other multiplicities. The parametrized homology of* X *is represented in Figure 4.3.*

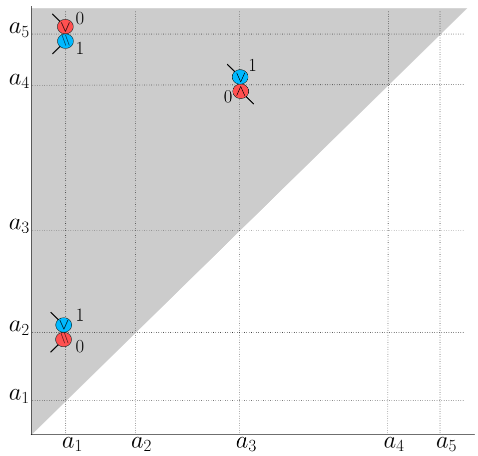


Figure 4.3: The parametrized homology of X. The color of the point indicates the dimension, while the symbol designates to which of the diagrams it belongs.

Now we give an example of a Morse-type parametrized space whose persistence diagram lies not in the half plane as in the previous example, but in the extended half plane.

Example 4.18. *Consider the surface X in Figure 4.4. Since the projection p_X onto the horizontal axis is Morse,* X *has a well-defined parametrized homology.*

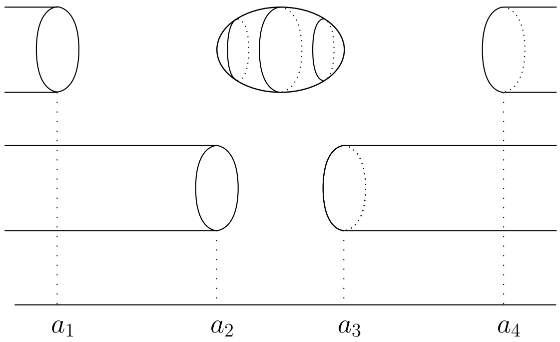
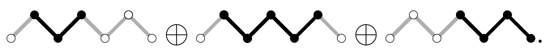


Figure 4.4: A parametrized space whose persistence is not captured in the half plane.

As before we determine the diagrams by computing the multiplicities of the decorated points. We repeat the argument from the previous example for finite rectangles. We now calculate the multiplicity of $(-\infty, a_2^+)$ with respect to the four measures in dimension 0. For $0 < \epsilon < a_3 - a_2$, $b < a_1$ we compute the four measures on $H_b^\epsilon = (-\infty, b] \times [a_2, a_2 + \epsilon]$:

$$\mathrm{H}_0(\mathrm{X}_{-\infty,b,a_2,a_2+\epsilon}) \cong$$



It follows that $m_{{}_0\mu_{\mathrm{X}}^{\backslash\backslash}}(-\infty, a_2^+) = m_{{}_0\mu_{\mathrm{X}}^{\wedge}}(-\infty, a_2^+) = m_{{}_0\mu_{\mathrm{X}}^{//}}(-\infty, a_2^+) = 0$. Recall that

$$m_{\mu_{\mathrm{X}}^{\vee}}(-\infty, a_2^+) = \min\{\mu_{\mathrm{X}^\vee}((-\infty, b] \times [a_2, d])\}.$$

Therefore we have $m_{\mu_{\mathrm{X}}^{\vee}}(-\infty, a_2^+) \leq 1$. Since any horizontal strip of the form $(-\infty, b] \times [a_2, d]$ contains $H_{b'}^\epsilon$ for some b' and ϵ and by monotonicity of r-measures, $m_{\mu_{\mathrm{X}}^{\vee}}(-\infty, a_2^+) = 1$. This means that $(-\infty, a_2^+)$ is a point in the decorated persistence diagram in the extended plane belonging to ${}_0\mu_{\mathrm{X}}^{\vee}$ with a multiplicity of 1. We repeat this procedure to compute the other multiplicities. The parametrized homology of X *is represented in Figure 4.5.*

Lastly, we give an example of a parametrized space that is not of Morse-type, but nevertheless has a well-defined persistence diagram.

Example 4.19. *Consider the surface X in Figure 4.6. For every integer k, we have an object appearing at $2k + 1$ that persists until $2k + 2$.*

We determine the diagrams by computing the multiplicities of the decorated points as before. The parametrized homology of X *is represented in Figure 4.7.*

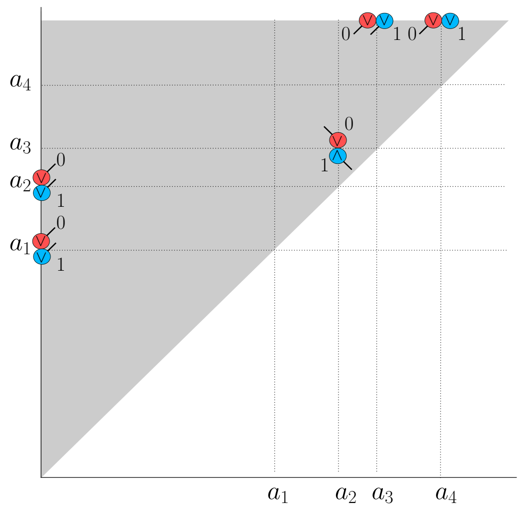


Figure 4.5: The parametrized homology of X.

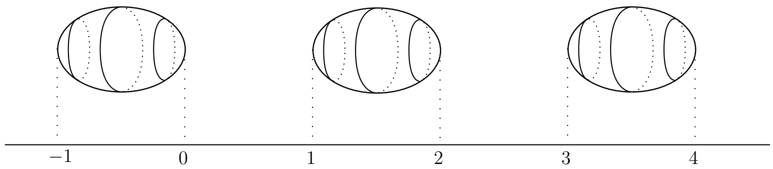


Figure 4.6: This space has infinitely many critical points and is therefore not of Morse-type.

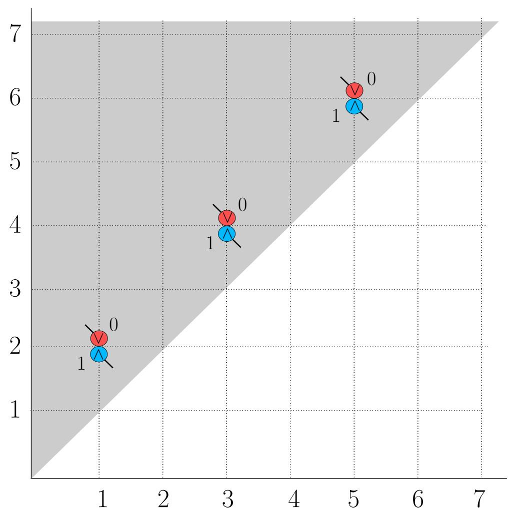


Figure 4.7: The parametrized homology of X.

Chapter 5

Parametrized Alexander Duality

In Section 5.1 we state and prove the classical Alexander Duality. We assume that the reader is familiar with basic constructions from cohomology theory (for example, cup and cap products) as well as Poincaré Duality (see Hatcher [27] and Spanier [30]). In Section 5.2 we prove that any two embeddings of a parametrized compact space into parametrized Euclidean space are 'stably equivalent.' Consequently, the complement does not depend on the embedding. This indicates that some kind of Alexander duality theorem might exist in a parametrized space setting. In Section 5.3 we state and prove several parametrized Alexander Duality Theorems [28].

5.1 Classical Alexander Duality

There are many dualities in algebraic topology. There is the duality between homology and cohomology, a duality between cup products and cap products, as well as between suspension and looping.

When J.W. Alexander was working on the Jordan-Brouwer separation theorem in 1922, he gave a new proof and generalized the result to what we call Alexander Duality. The original theorem claimed that $\dim \widetilde{\mathrm{H}}^j(X;\mathbb{Z}_2) = \dim \widetilde{\mathrm{H}}_{n-j-1}(S^n \setminus X;\mathbb{Z}_2)$ when X is a subcomplex of S^n viewed as a polyhedron.

The theorem took its final form in the 1960s (see for example Spanier [30]). In its more general form, it is a statement about the relationship between the cohomology groups of a locally contractible, compact subset of $\mathbb{R}^n$ and the homology groups of the complement.

Theorem 5.1 (Alexander Duality). *If X is a locally contractible, compact subset of $\mathbb{R}^n$, then for all $j = 0, \dots, n-1$,*

$$\widetilde{\mathrm{H}}_{n-j-1}(\mathbb{R}^n \setminus X) \cong \mathrm{H}^j(X).$$

Lemma 5.2. *Let X be a locally contractible, compact subset of $\mathbb{R}^n$. Then for all $j = 0, \dots, n-1$*

$$\operatorname{colim} \mathrm{H}^j(U) \cong \mathrm{H}^j(X)$$

where the direct limit is taken with respect to open neighborhoods U of X.

Proof. Hatcher's Theorem A.7 [27] states that a locally contractible compact set is a retract of some neighborhood U_0 in $\mathbb{R}^n$. In computing the direct limit we can restrict our attention to the open sets $U \subset U_0$, which all retract to X. Each neighborhood $U \subset U_0$ contains a neighborhood U' of X, for which the inclusion $U' \hookrightarrow U$ is homotopic to the retraction $U' \to X \subset U$. To show this, observe that the linear homotopy $H\colon U \times [0,1] \to \mathbb{R}^n$ from the identity to the retraction $U \to X$ takes $X \times [0,1]$ to X. For each $(x,t) \in X \times [0,1]$, pick open sets $x \in U_{\{x,t\}} \subset U$ and $t \in V_{\{x,t\}} \subset [0,1]$ such that

$$H(U_{\{x,t\}} \times V_{\{x,t\}}) \subset U.$$

This is possible since H is continuous and U is open. Now for every $t \in [0,1]$, the set $X \times \{t\}$ is compact, hence there is a finite integer N_t and points $x_{\{t,1\}}$, $x_{\{t,2\}}$, ..., $x_{\{t,N_t\}}$ such that $X \times \{t\}$ is covered with the sets

$$U_{\{x_{\{t,j\}},t\}} \times V_{\{x_{\{t,j\}},t\}}, \quad j = 1, \ldots, N_t. \tag{5.1}$$

Define the open sets

$$U_t = \bigcup_{j=1}^{N_y} U_{\{x_{\{t,j\}},t\}}, \quad V_t = \bigcap_{j=1}^{N_t} V_{\{x_{\{t,j\}},t\}}.$$

Then $U_t \times V_t \supset X \times \{t\}$, and $U_t \times V_t$ is contained in the union of all the sets in (5.1), thus $H(U_t \times V_t) \subset U$. The compact set $K \times [0,1]$ is covered by the open sets $U_t \times V_t$, where $t \in [0,1]$. By compactness, only finitely many such sets suffice, say $U_{\{t_1\}} \times V_{\{t_1\}}, \ldots, U_{\{t_N\}} \times V_{\{t_N\}}$. Set

$$U' = \bigcap_{i=1}^{N} U_{t_i}, \quad V' = \bigcup_{i=1}^{N} V_{t_i}.$$

Then $U' \supset X$ and $V' = [0,1]$ are open, and $H(U' \times [0,1]) \subset \bigcup_{i=1}^{N} H(U_{t_i} \times V_{t_i}) \subset U$, as desired.

The restriction map $\operatorname{colim} \mathrm{H}^j(U) \to \mathrm{H}^j(X)$ is surjective since we can pull back each element of $\mathrm{H}^j(X)$ to the direct limit via the retractions $U \to X$.

To show that $\operatorname{colim} \mathrm{H}^j(U) \to \mathrm{H}^j(X)$ is injective, take an element in $\operatorname{colim} \mathrm{H}^j(U)$ that gets mapped to 0. By definition, this element comes from $H_j(U)$ for some U. Given this U, we take U' as defined above. Since $U' \hookrightarrow U$ factors as $U' \hookrightarrow X \subset U$, the restriction $\mathrm{H}^j(U) \to \mathrm{H}^j(U')$ factors through $\mathrm{H}^j(X)$. Therefore, if an element in $\mathrm{H}^j(U)$ restricts to zero in $\mathrm{H}^j(X)$, it also restricts to zero in $\mathrm{H}^j(U')$. It follows that $\operatorname{colim} \mathrm{H}^j(U) \to \mathrm{H}^j(X)$ is injective. □

Proof of Alexander Duality. The Alexander Duality isomorphism is a composition of the following isomorphisms:

$$\begin{aligned}
\mathrm{H}_{n-j-1}(\mathbb{R}^n \setminus X) &\cong \mathrm{H}_c^{j+1}(\mathbb{R}^n \setminus X) \\
&\cong \operatorname{colim} \mathrm{H}^{j+1}(\mathbb{R}^n \setminus X, \mathbb{R}^n \setminus (U_i^C \cap B_i)) \\
&= \operatorname{colim} \mathrm{H}^{j+1}(\mathbb{R}^n \setminus X, U_i \setminus X \cup B_i^C) \\
&\cong \operatorname{colim} \mathrm{H}^{j+1}(\mathbb{R}^n, U_i \cup B_i^C) \\
&\cong \operatorname{colim} \widetilde{\mathrm{H}}^j(U_i \cup B_i^C) \\
&\cong \operatorname{colim} \mathrm{H}^j(U_i) \oplus \widetilde{\mathrm{H}}^j(B_i^C) \\
&\cong \mathrm{H}^j(X) \oplus \widetilde{\mathrm{H}}^j(S^{n-1}).
\end{aligned}$$

The direct limits are taken with respect to $\{U_i^C \cap B_i\}_i$. Here $\{U_i\}$ is a nested sequence of neighborhoods of X that retract to X, such that $\cap_i U_i = X$. $\{B_i\}$ is an increasing sequence of closed balls containing X centered at the origin, such that $\cup_i B_i = \mathbb{R}^n$. By Theorem A.7 of [27], such a sequence $\{U_i\}$ exists since X is compact and locally contractible. To compute $\mathrm{H}_c^{j+1}(\mathbb{R}^n \setminus X)$ it is sufficient to let the compact sets range over $\{U_i^C \cap B_i\}_i$, since any compact set in $\mathbb{R}^n \setminus X$ is contained in $U_i^C \cap B_i$ for some i.

There are four isomorphisms that are important for our purposes. The first is the Poincaré Duality isomorphism. The second is the definition of cohomology with compact supports along with the observation in the previous paragraph. The third is excision. The fourth comes from the long exact sequences of the pairs $(\mathbb{R}^n, U_i \cup B_i^C)$ for reduced homology.

By Lemma 5.2 $\operatorname{colim} \mathrm{H}^j(U_i) \cong \mathrm{H}^j(X)$. To compute $\operatorname{colim} \widetilde{\mathrm{H}}^j(B_i^C)$ observe that $\widetilde{\mathrm{H}}^j(B_i^C) \cong \widetilde{\mathrm{H}}^j(S^{n-1})$ and that all $\widetilde{\mathrm{H}}^j(B_i^C) \to \widetilde{\mathrm{H}}^j(B_{i+1}^C)$ are isomorphisms.

If $j \neq n-1$, then $\operatorname{colim} \widetilde{\mathrm{H}}^j(S^{n-1}) = 0$. It follows that $\widetilde{\mathrm{H}}_{n-j-1}(\mathbb{R}^n \setminus X) \cong \mathrm{H}^j(X)$ for $j \neq n-1$.

If $j = n-1$, then $\operatorname{colim} \widetilde{\mathrm{H}}^j(S^{n-1}) = \mathbf{k}$. This chain of isomorphisms yields $\mathrm{H}_0(\mathbb{R}^n \setminus X) \cong \mathrm{H}^{n-1}(X) \oplus \mathbf{k}$. As a result, $\widetilde{\mathrm{H}}_0(\mathbb{R}^n \setminus X) \cong \mathrm{H}^{n-1}(X)$. □

If the space is not locally contractible, it is necessary to use Čech cohomology on the left side of this isomorphism. Consider the Warsaw circle X depicted in Figure 5.1. The singular cohomology of X is isomorphic to that of a point. Since X separates $\mathbb{R}^2$, Alexander Duality predicts $\mathrm{H}^1(X) \cong \mathbf{k}$. However, this prediction fails in singular cohomology.

In general, we construct Čech cohomology groups as follows. To each open cover $U = \{U_\alpha\}_{\alpha \in A}$ of a given space X we can associate a simplicial complex $\mathscr{N}(U)$ called the nerve of U. Its vertex set is A and a family $\{\alpha_0, \ldots, \alpha_k\}$ spans a k-simplex if and only if $U_{\alpha_0} \cap \ldots \cap U_{\alpha_k} = \emptyset$. Another cover $V = \{V_\beta\}$ is a refinement of U if each V_β is contained in some U_α. In this case these inclusions induce a simplicial map $\mathscr{N}(V) \to \mathscr{N}(U)$ that is well-defined up to homotopy. We define Čech cohomology by the formula

$$\check{\mathrm{H}}^j(X) = \operatorname{colim} \mathrm{H}^j(\mathscr{N}(U)),$$

where we take the limit with respect to ever more refined open covers U.

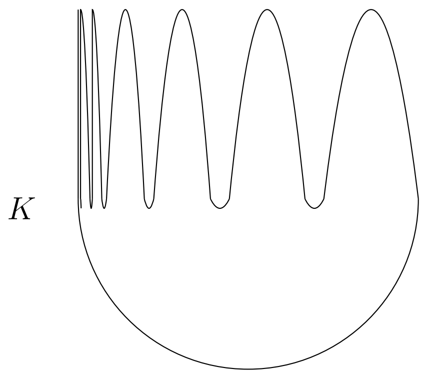


Figure 5.1: Warsaw circle is not locally contractible.

If X is embedded in a manifold M, we can compute Čech cohomology as the direct limit over neighborhoods in M containing X. We take this to be the definition of Čech cohomology, since this is always the case in the situations we treat.

Definition 5.3. *Let M be a manifold and X a closed subset of M. The Čech cohomology group of the embedding $X \subset M$ is the colimit*

$$\check{\mathrm{H}}^j(X) = \operatorname{colim} \mathrm{H}^j(U),$$

taken with respect to open neighborhoods U of X.

Theorem 5.4 (Alexander Duality for Čech cohomology)**.** *If X is a compact subset of $\mathbb{R}^n$, then for all $j = 0, \dots, n-1$,*

$$\widetilde{\mathrm{H}}_{n-j-1}(\mathbb{R}^n \setminus X) \cong \check{\mathrm{H}}^j(X).$$

Proof. We use the same argument as in the proof of Theorem 5.1. However, since $\check{\mathrm{H}}^j(X) = \operatorname{colim} \mathrm{H}^j(U_i)$ by cofinality, Lemma 5.2 is obsolete. □

5.2 The prerequisite for parametrized Alexander Duality

In order for Alexander Duality to exist, any two embeddings of a compact space X must be ‘stably equivalent.’ As a result, the homology of the complement does not depend on the embedding. We first show the statement for ordinary topological spaces and then extend it to the setting of parametrized spaces.

Proposition 5.5. *Let X be a compact set. Let $f\colon X \hookrightarrow \mathbb{R}^n$ and $g\colon X \hookrightarrow \mathbb{R}^m$ be embeddings. Then*

$$\mathrm{H}_{j-m}(\mathbb{R}^n - f(X)) \cong \mathrm{H}_{j-n}(\mathbb{R}^m - g(X))$$

for $\max\{n, m\} \leq j$.

We prove that any two embeddings of a compact space X are 'stably equivalent'.

Lemma 5.6. *Let X be a compact set. Let $f\colon X \hookrightarrow \mathbb{R}^n$ and $g\colon X \hookrightarrow \mathbb{R}^m$ be embeddings. Moreover, let $i_n\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, $u \hookrightarrow (u,0)$, and $i_m\colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$, $v \hookrightarrow (0,v)$, be inclusions. Then $\mathrm{Im}(i_n \circ f)$ is ambiently homeomorphic to $\mathrm{Im}(i_m \circ g)$.*

Proof. Both $i_n \circ f$ and $i_m \circ g$ are embeddings of X into $\mathbb{R}^n \times \mathbb{R}^m$. Now we define

$$G'(f(x)) = g(x) \quad \text{for } x \in X$$

and

$$F'(g(x)) = f(x) \quad \text{for } x \in X.$$

$\mathrm{Im}(f)$ is closed in $\mathbb{R}^n$. By Tietze's extension theorem $G\colon \mathbb{R}^n \to \mathbb{R}^m$ exists such that $G|_{\mathrm{Im}(f)} = G'$. Similarly, since $\mathrm{Im}(g)$ is closed in $\mathbb{R}^m$, $F\colon \mathbb{R}^m \to \mathbb{R}^n$ exists such that $F|_{\mathrm{Im}(g)} = F'$.

Next we define

$$\Phi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m \quad \text{by } (u,v) \mapsto (u, v + G(u)).$$

This Φ is a homeomorphism that maps $\mathrm{Im}(i_n \circ f)$ to $\mathrm{Im}(f,g)$.

Similarly, we define

$$\Psi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m \quad \text{by } (u,v) \mapsto (u + F(v), v).$$

This map takes $\mathrm{Im}(i_m \circ g)$ to $\mathrm{Im}(f,g)$ homeomorphically.

We finish the proof by observing that the composition $\Psi^{-1} \circ \Phi$ is a homeomorphism that maps $\mathrm{Im}(i_n \circ f)$ to $\mathrm{Im}(i_m \circ g)$. □

The following two Lemmas will come in handy to prove the parametrized version of Proposition 5.5.

Lemma 5.7. *Let $X \subseteq \mathbb{R}^n$ be a compact set. Let i_{n+1} be an embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ given by $u \mapsto (0,u)$. Then*

$$\mathrm{H}_{j+1}(\mathbb{R}^{n+1} \setminus i_{n+1}(X)) \cong \mathrm{H}_j(\mathbb{R}^n \setminus X)$$

for $j = 0, \ldots, n-1$.

Proof. Let

$$A = \{(t,u) \,|\, t > 0\} \cup \{(t,u) \,|\, -\epsilon < t \leq 0, (0,u) \notin i_{n+1}(X)\}$$

and

$$B = \{(t,u) \,|\, t < 0\} \cup \{(t,u) \,|\, 0 \leq t < \epsilon, (0,u) \notin i_{n+1}(X)\}.$$

Both A and B are contractible. Furthermore, $A \cap B = (-\epsilon, \epsilon) \times i_{n+1}(\mathbb{R}^n \setminus X)$ and $A \cup B = \mathbb{R}^{n+1} \setminus i_{n+1}(X)$. Using the Mayer-Vietoris Principle for the pair $A \cup B$, we get

$$\mathrm{H}_{j+1}(\mathbb{R}^n \setminus X) \xrightarrow{\phi_*} 0 \oplus 0 \xrightarrow{\psi_*} \mathrm{H}_{j+1}(\mathbb{R}^{n+1} \setminus i_{n+1}(X)) \xrightarrow{\partial_*} \mathrm{H}_j(\mathbb{R}^n \setminus X) \xrightarrow{\phi_*} 0 \oplus 0$$

By exactness it follows that $\mathrm{H}_{j+1}(\mathbb{R}^{n+1} \setminus i_{n+1}(X)) \cong \mathrm{H}_j(\mathbb{R}^n \setminus X)$ for $j = 0, \ldots, n-1$. □

By slightly modifying the argument we can prove Proposition 5.8.

Proposition 5.8. *Let $X \subseteq \mathbb{R}^n \times [a,b]$ be a compact set. Let i_{n+2} be an embedding of $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$ given by $u \mapsto (0,u)$. Then*

$$\mathrm{H}_j(\mathbb{R}^n \times [a,b] \setminus X) \cong \mathrm{H}_{j+1}(\mathbb{R}^{n+1} \times [a,b] \setminus i_{n+2}(X))$$

for $j = 0, \ldots, n$.

Proof of Proposition 5.5. By Lemma 5.6 $\mathbb{R}^n \times \mathbb{R}^m \setminus \mathrm{Im}(i_n \circ f)$ is homeomorphic to $\mathbb{R}^n \times \mathbb{R}^m \setminus \mathrm{Im}(i_m \circ g)$. Consequently,

$$\mathrm{H}_j(\mathbb{R}^n \times \mathbb{R}^m \setminus \mathrm{Im}(i_n \circ f)) = \mathrm{H}_j(\mathbb{R}^n \times \mathbb{R}^m \setminus \mathrm{Im}(i_m \circ g)).$$

Repeatedly using Lemma 5.7 we get $\mathrm{H}_{j-m}(\mathbb{R}^n - f(X)) \cong \mathrm{H}_{j-n}(\mathbb{R}^m - g(X))$. □

In order to prove the equivalent of Proposition 5.5 for parametrized spaces, we need a few additional theorems and lemmas.

Theorem 5.9. *Let X and Y be isomorphic in the category of parametrized spaces. If X has a well-defined parametrized homology, then Y has a well-defined parametrized homology and $\mathrm{Par}_{\mathrm{H}_*}(\mathrm{X}) = \mathrm{Par}_{\mathrm{H}_*}(\mathrm{Y})$.*

Proof. Since X and Y are isomorphic, $f\colon X \to Y$ and $g\colon Y \to X$ exist such that $p_Y \circ f = p_X$, $p_X \circ g = p_Y$, $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. These f and g induce homeomorphisms between levelsets and slices.

By Corollary 3.5 it is sufficient to show that ${}_j\mu^*_{\mathrm{X}}(R) = {}_j\mu^*_{\mathrm{Y}}(R)$ for any $R = [a,b] \times [c,d]$ with $-\infty < a < b < c < d < \infty$, $j = 1, \ldots, n-1$ and $* = \backslash\backslash, \vee, \wedge, //$.

The parametrized homeomorphism f induces the following isomorphism of zigzag modules:

$$\begin{array}{ccccc}
 & \mathrm{H}_j(X_d) & \xrightarrow{(f_d)_*} & \mathrm{H}_j(Y_d) & \\
 & \swarrow & & \swarrow & \\
\mathrm{H}_j(X_c^d) & & \xrightarrow{(f_c^d)_*} & & \mathrm{H}_j(Y_c^d) \\
 & \nwarrow & & \nwarrow & \\
 & \mathrm{H}_j(X_c) & \xrightarrow{(f_c)_*} & \mathrm{H}_j(Y_c) & \\
 & \swarrow & & \swarrow & \\
\mathrm{H}_j(X_b^c) & & \xrightarrow{(f_b^c)_*} & & \mathrm{H}_j(Y_b^c) \\
 & \nwarrow & & \nwarrow & \\
 & \mathrm{H}_j(X_b) & \xrightarrow{(f_b)_*} & \mathrm{H}_j(Y_b) & \\
 & \swarrow & & \swarrow & \\
\mathrm{H}_j(X_a^b) & & \xrightarrow{(f_a^b)_*} & & \mathrm{H}_j(Y_a^b) \\
 & \nwarrow & & \nwarrow & \\
 & \mathrm{H}_j(X_a) & \xrightarrow{(f_a)_*} & \mathrm{H}_j(Y_a). &
\end{array}$$

It follows that $\mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})$ and $\mathrm{H}_j(\mathrm{Y}_{\{a,b,c,d\}})$ have the same interval decomposition. Consequently, ${}_j\mu^*_{\mathrm{X}}(R) = {}_j\mu^*_{\mathrm{Y}}(R)$. □

To prove Theorem 5.10 we can use the same argument in combination with Corollary 3.8.

Theorem 5.10. *Let* X *and* Y *be isomorphic in the category of parametrized spaces. If* X *has a well-defined parametrized homology in* $\overline{\mathscr{H}}$*, then* Y *has a well-defined parametrized homology in* $\overline{\mathscr{H}}$ *and* ${}_\infty \mathrm{Par}_{\mathrm{H}_*}(\mathrm{X}) = {}_\infty \mathrm{Par}_{\mathrm{H}_*}(\mathrm{Y})$.

Next we show an equivalent of Lemma 5.7 for parametrized homology. In our setting, a parametrized space X where every levelset and slice is a compact set is embedded in a parametrized Euclidean space $\mathbb{R}^{\mathrm{n}+1}$. Whenever we write a parametrized space $\mathbb{R}^{\mathrm{n}+1}$, this means a pair $(\mathbb{R}^{n+1}, p_{\mathbb{R}^{n+1}})$, where

$$p_{\mathbb{R}^{n+1}}(x_1, x_2, \ldots, x_{n+1}) = x_{n+1}.$$

Lemma 5.11. *Let X be a parametrized space, where every levelset and slice is a compact set and let* $\mathrm{f}\colon \mathrm{X} \hookrightarrow \mathbb{R}^{\mathrm{n}+1}$ *be a parametrized embedding. We write* $\mathrm{f}\colon \mathrm{X} \hookrightarrow \mathbb{R}^{\mathrm{n}+1}$ *as a pair* (f, p_X)*. We define* $i_{n+1}\colon \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ *by* $i_{n+1}(u) = (0, u)$*. If* $\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f, p_X))$ *has a well-defined parametrized homology, then* $\mathbb{R}^{\mathrm{n}+2} \setminus \mathrm{Im}(i_{n+1} \circ (f, p_X))$ *also has a well-defined parametrized homology. As a result,*

$$\mathrm{Par}_{\mathrm{H}_*}(\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f, p_X))) = \mathrm{Par}_{\mathrm{H}_{*+1}}(\mathbb{R}^{\mathrm{n}+2} \setminus \mathrm{Im}(i_{n+1} \circ (f, p_X))).$$

Proof. By Corollary 3.5 it is sufficient to show that for all j and $* = \backslash\backslash, \vee, \wedge, //$,

$${}_j\mu^*_{\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f,p_X))}(R) = {}_{j+1}\mu^*_{\mathbb{R}^{\mathrm{n}+2} \setminus \mathrm{Im}(i_{n+1}\circ(f,p_X))}(R) \tag{5.2}$$

for all $R = [a, b] \times [c, d]$ with $-\infty < a < b < c < d < \infty$.

Consider the diagram below:

$$\begin{array}{ccc}
\mathrm{H}_j((\mathbb{R}^n - f(X))_d) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_d) \\
\swarrow & & \swarrow \\
\mathrm{H}_j((\mathbb{R}^n - f(X))_c^d) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_c^d) \\
\nwarrow & & \nwarrow \\
\mathrm{H}_j((\mathbb{R}^n - f(X))_c) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_c) \\
\swarrow & & \swarrow \\
\mathrm{H}_j((\mathbb{R}^n - f(X))_b^c) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_b^c) \\
\nwarrow & & \nwarrow \\
\mathrm{H}_j((\mathbb{R}^n - f(X))_b) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_b) \\
\swarrow & & \swarrow \\
\mathrm{H}_j((\mathbb{R}^n - f(X))_a^b) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_a^b) \\
\nwarrow & & \nwarrow \\
\mathrm{H}_j((\mathbb{R}^n - f(X))_a) & \longrightarrow & \mathrm{H}_{j+1}((\mathbb{R}^{n+1} - (0, f(X)))_a)).
\end{array}$$

By Lemma 5.7 and Proposition 5.11 the horizontal arrows are isomorphisms that arise from the Mayer-Vietoris sequence for homology groups. By Theorem 15.4 [24] all squares commute. Therefore, the two zigzag modules are isomorphic and have the same interval decomposition. Equation 5.2 follows. □

We can prove the following lemma using the same argument in combination with Corollary 3.8.

Lemma 5.12. *Let X be a parametrized space, whose every levelset and slice is a compact set and let* $\mathrm{f}\colon \mathrm{X} \hookrightarrow \mathbb{R}^{\mathrm{n}+1}$ *be a parametrized embedding. We write* $\mathrm{f}\colon \mathrm{X} \hookrightarrow \mathbb{R}^{\mathrm{n}+1}$ *as a pair* (f, p_X). *We also define* $i_{n+1}\colon \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ *by* $i_{n+1}(u) = (0, u)$. *If* $\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f, p_X))$ *has a well-defined parametrized homology in* $\overline{\mathscr{H}}$*, then* $\mathbb{R}^{\mathrm{n}+2} \setminus \mathrm{Im}(i_{n+1} \circ (f, p_X))$ *also has a well-defined parametrized homology in* $\overline{\mathscr{H}}$*. As a result,*

$$_\infty\,\mathrm{Par}_{\mathrm{H}_*}(\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f, p_X))) = {}_\infty\,\mathrm{Par}_{\mathrm{H}_{*+1}}(\mathbb{R}^{\mathrm{n}+2} \setminus \mathrm{Im}(i_{n+1} \circ (f, p_X))).$$

The following theorem indicates that the parametrized homology of the complement of a parametrized space with compact levelesets and slices does not depend on the parametrized embedding.

Theorem 5.13. *Let X be a parametrized space, whose every levelset and slice is a compact set. Let* $\mathrm{f}\colon \mathrm{X} \hookrightarrow \mathbb{R}^{\mathrm{n}+1}$ *and* $\mathrm{g}\colon \mathrm{X} \hookrightarrow \mathbb{R}^{\mathrm{m}+1}$ *be parametrized embeddings. We write* f *as a pair* (f, p_X) *and* g *as a pair* (g, p_X).

If X has a well-defined parametrized homology, then

$$\mathrm{Par}_{\mathrm{H}_{*-m}}(\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f, p_X))) = \mathrm{Par}_{\mathrm{H}_{*-n}}(\mathbb{R}^{\mathrm{m}+1} \setminus \mathrm{Im}((g, p_X))).$$

If X has a well-defined parametrized homology in $\overline{\mathscr{H}}$*, then*

$$_\infty\,\mathrm{Par}_{\mathrm{H}_{*-m}}(\mathbb{R}^{\mathrm{n}+1} \setminus \mathrm{Im}((f, p_X))) = {}_\infty\,\mathrm{Par}_{\mathrm{H}_{*-n}}(\mathbb{R}^{\mathrm{m}+1} \setminus \mathrm{Im}((g, p_X))).$$

Proof. Maps

$$i_n\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}, \quad (u, a) \mapsto (u, 0, a)$$

and

$$i_m\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}, \quad (v, a) \mapsto (0, v, a)$$

are parametrized embeddings and consequently $i_n \circ f$ and $i_m \circ g$ are parametrized embeddings of X into $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{m}} \times \mathbb{R}$.

Now we define:

$$G'(f(x), p_X(x)) = \; g(x) \quad \text{for } x \in X$$

and

$$F'(g(x), p_X(x)) = \; f(x) \quad \text{for } x \in X.$$

$\mathrm{Im}((f, p_X))$ is closed in $\mathbb{R}^{n+1}$ and $\mathrm{Im}((g, p_X))$ is closed in $\mathbb{R}^{m+1}$. By Tietze's extension theorem $G\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ and $F\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ exist such that $G|_{\mathrm{Im}(f)} = G'$ and $F|_{\mathrm{Im}(g)} = F'$.

Now we define

$$\Phi\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \quad \text{by} \quad (u, v, a) \mapsto (u, v + G(u, a), a).$$

This is a parametrized homeomorphism. Furthermore,

$$\Phi(f(x), 0, p_X(x)) = (f(x), g(x), p_X(x)),$$

so Φ maps $\mathrm{Im}(i_n \circ (f, p_X))$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ to $\mathrm{Im}((f, g, p_X))$.

Similarly, we define

$$\Psi\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \quad \text{by} \quad (u, v, a) \mapsto (u + F(v, a), v, a).$$

This is a parametrized homeomorphism that maps $\mathrm{Im}(i_m \circ (g, p_X))$ in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ to $\mathrm{Im}((f, g, p_X))$.

The composition $\Psi^{-1} \circ \Phi\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ is a parametrized homeomorphism that maps $\mathrm{Im}(i_n \circ (f, p_X))$ to $\mathrm{Im}(i_m \circ (g, p_X))$. This implies that $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \setminus \mathrm{Im}(i_n \circ (f, p_X))$ and $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \setminus \mathrm{Im}(i_m \circ (g, p_X))$ are isomorphic in the category of parametrized spaces. By Theorem 5.9

$$\mathrm{Par}_{\mathrm{H}_*}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \setminus \mathrm{Im}(i_n \circ (f, p_X))) = \mathrm{Par}_{\mathrm{H}_*}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \setminus \mathrm{Im}(i_m \circ (g, p_X)))$$

if these spaces have a well-defined parametrized homology. Repeatedly using Lemma 5.11 finishes the proof.

We prove the statement for parametrized homology in $\overline{\mathscr{H}}$ using Theorem 5.10 and Lemma 5.12 in a similar manner. □

5.3 Parametrized Alexander Duality

This section extends the Alexander Duality theorems to parametrized spaces.

The following theorem is the parametrized equivalent of Theorem 5.1.

Theorem 5.14. *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We assume that the levelsets X_a for $a \in \mathbb{R}$, and slices X_a^b for $a < b$ are compact and locally contractible. If $\mathrm{X} = (X, p|_X)$ has a well-defined parametrized cohomology, then the pair $\mathrm{Y} = (Y, p|_Y)$ has a well-defined reduced parametrized homology. Additionally, for all $j = 0, \ldots, n-1$:*

$$\begin{aligned}
\widetilde{\mathrm{D}}\mathrm{gm}_{n-j-1}^{\backslash\backslash}(\mathrm{Y}) &= \mathrm{Dgm}^{j//}(\mathrm{X}),\\
\widetilde{\mathrm{D}}\mathrm{gm}_{n-j-1}^{\vee}(\mathrm{Y}) &= \mathrm{Dgm}^{j\wedge}(\mathrm{X}),\\
\widetilde{\mathrm{D}}\mathrm{gm}_{n-j-1}^{\wedge}(\mathrm{Y}) &= \mathrm{Dgm}^{j\vee}(\mathrm{X}),\\
\widetilde{\mathrm{D}}\mathrm{gm}_{n-j-1}^{//}(\mathrm{Y}) &= \mathrm{Dgm}^{j\backslash\backslash}(\mathrm{X}).
\end{aligned}$$

This is a statement about the diagram in the half plane $\mathscr{H}$. If X is compact, there are no points at infinity and there is no need to work in the extended half plane $\overline{\mathscr{H}}$. In this case $\mathrm{Par}_{\mathrm{H}_*}$ and $\mathrm{Par}_{\mathrm{H}^*}$ capture all the information about homology and cohomology.

Remark 5.15. *From the proof we can deduce the following duality: if a j-dimensional cohomology cycle in* X *is killed (ceases to exist) at endpoint p, then there is a corresponding $(n-j-1)$-dimensional homology cycle in* Y*, which ceases to exist (is killed) beyond that same endpoint.*

Remark 5.16. *It follows from Theorem 4.9 that the conditions of the theorem are satisfied for $(X, p|_X)$, where:*

- *X is a compact submanifold of $\mathbb{R}^n \times \mathbb{R}$ (with or without boundary) and $p|_X$ is Morse;*
- *X is a finite simplicial complex and $p|_X$ is a piecewise-linear map;*

Example 5.17 presents a case where our theorem is applicable, but which is not covered by Edelsbrunner and Kerber's Land and Water Theorem [20]:

Example 5.17. *Let $S \subset \mathbb{R}^3$ be the Alexander horned sphere (see [27, Example 2B.2]). Let $X = S \times [-1, 1] \subset \mathbb{R}^3 \times \mathbb{R}$, let $Y = \mathbb{R}^3 \times \mathbb{R} \setminus X$, and let p be the projection onto the second factor. The conditions of Theorem 5.21 are satisfied, because S is locally contractible and compact.*

The proof of Theorem 5.14 requires two lemmas.

Lemma 5.18. *Let X, Y, and p be as in the theorem. Consider the following diagram of vector spaces and maps:*

$$
\begin{array}{ccccc}
 & & \widetilde{\mathrm{H}}_{n-j-1}(Y_a^b) & & \\
 & \overset{i_a}{\nearrow} & & \overset{i_b}{\nwarrow} & \\
\widetilde{\mathrm{H}}_{n-j-1}(Y_a) & & & & \widetilde{\mathrm{H}}_{n-j-1}(Y_b) \\
\uparrow D_a & & & & \uparrow D_b \\
\mathrm{H}^j(X_a) & & & & \mathrm{H}^j(X_b) \\
 & \underset{i^a}{\nwarrow} & & \underset{i^b}{\nearrow} & \\
 & & \mathrm{H}^j(X_a^b) & &
\end{array}
$$

Maps i^a, i^b, i_a, and i_b are induced by the inclusions $X_a \hookrightarrow X_a^b$, $X_b \hookrightarrow X_a^b$, $Y_a \hookrightarrow Y_a^b$, and $Y_b \hookrightarrow Y_a^b$. Isomorphisms D_a and D_b are Alexander Duality isomorphisms in $\mathbb{R}^n \times \{a\}$ and $\mathbb{R}^n \times \{b\}$. Then $Im(D_a i^a \oplus D_b i^b) = Ker(i_a - i_b)$.

Remark 5.19. *This lemma holds even if we do not assume field coefficients.*

Proof. We look at the long exact sequence for homology groups of the pair $(Y_a^b, Y_a \cup Y_b)$:

$$\to \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) \to \mathrm{H}_{n-j-1}(Y_a \cup Y_b) \to \mathrm{H}_{n-j-1}(Y_a^b) \to \mathrm{H}_{n-j-1}(Y_a^b, Y_a \cup Y_b) \to \tag{5.3}$$

We claim that an isomorphism $\mathrm{H}^j(X_a^b) \to \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b)$ exists making the following diagram commute up to a sign

$$\begin{array}{ccccc} \mathrm{H}^j(X_a^b) & \xrightarrow{i^a \oplus i^b} & \mathrm{H}^j(X_a) \oplus \mathrm{H}^j(X_b) & & \\ \wr\!\parallel\downarrow & & \downarrow\wr\!\parallel & & \\ \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a) \oplus \mathrm{H}_{n-j-1}(Y_b) & \xrightarrow{i_a - i_b} & \mathrm{H}_{n-j-1}(Y_a^b). \end{array}$$

The case $j \neq n-1$ follows immediately by virtue of the exactness of the bottom line. We analyze the case when $j = n-1$ separately.

Let H_c^* denote cohomology with compact supports. According to Hatcher [27, Chapter 3, Problem 35], the following diagram, where the horizontal lines are long exact sequences of the corresponding pairs and the vertical arrows are Poincaré Duality isomorphisms, commutes up to a sign.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{H}_c^{j+1}(Y_a^b) & \longrightarrow & \mathrm{H}_c^{j+1}(Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_c^{j+2}(Y_a^b, Y_a \cup Y_b) & \to & \cdots \\ & & \wr\!\parallel\downarrow & & \wr\!\parallel\downarrow & & \wr\!\parallel\downarrow & & \\ \cdots & \to & \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a^b) & \longrightarrow & \cdots \end{array}$$

Let $\{U_i\}_i$ be a nested sequence of neighborhoods of X_a^b in $\mathbb{R}^n \times [a, b]$, such that U_1 retracts onto X_a^b and $\cap_i U_i = X_a^b$. Such sequences exist since X_a^b is compact and locally contractible by Theorem A.7 [27]. Further let $\{B_i\}$ be an increasing sequence of closed balls centered at the origin and containing X_a^b such that $\cup_i B_i = \mathbb{R}^{n+1}$. We may assume that $\overline{U}_1 \subset \mathrm{Int} B_1$ so that $\overline{U}_i \cap \overline{B_i^C} = \emptyset$ for all i. Now $U_1 \cap \mathbb{R}^n \times \{a\}$ is open in $\mathbb{R}^n \times \{a\}$ in the subspace topology and contains X_a. Since X_a is compact and locally contractible, we can find a neighborhood U^a of X_a which retracts onto X_a (again using Theorem A.7 [27]). Pick a nested sequence of neighborhoods U_i^a of X_a such that $U_i^a \subset U^a \cap U_i$ for each i and $\cap_i U_i^a = X_a$. In a similar manner we obtain a system of neighborhoods for X_b.

Let A^C denote the complement of A (where the ambient set is clear from the context). By cofinality, we have

$$\begin{array}{rcl} \mathrm{H}_c^{j+1}(Y_a^b) & \cong & \operatorname{colim} \mathrm{H}^{j+1}(Y_a^b, Y_a^b \setminus B_i \cap U_i^C), \\ \mathrm{H}_c^{j+1}(Y_a \cup Y_b) & \cong & \operatorname{colim} \mathrm{H}^{j+1}(Y_a, Y_a \setminus B_i \cap (U_i^a)^C) \oplus \mathrm{H}^{j+1}(Y_b, Y_b \setminus B_i \cap (U_i^b)^C). \end{array}$$

Moreover, the restriction $\mathrm{H}_c^{j+1}(Y_a^b) \to \mathrm{H}_c^{j+1}(Y_a \cup Y_b)$ is the the colimit of the corresponding morphisms.

Using the notation $(B_i^C)_a^b = B_i^C \cap \mathbb{R}^n \times [a, b]$, $(B_i^C)_a = B_i^C \cap \mathbb{R}^n \times \{a\}$, and $(B_i^C)_b = B_i^C \cap \mathbb{R}^n \times \{b\}$, we rewrite the expressions

$$\begin{array}{rcl} \mathrm{H}^{j+1}(Y_a^b, Y_a^b \setminus B_i \cap U_i^C) & = & \mathrm{H}^{j+1}(Y_a^b, (B_i^C)_a^b \cup (U_i \setminus X_a^b)), \\ \mathrm{H}^{j+1}(Y_a, Y_a \setminus B_i \cap (U_i^a)^C) & = & \mathrm{H}^{j+1}(Y_a, (B_i^C)_a \cup (U_i^a \setminus X_a)), \\ \mathrm{H}^{j+1}(Y_b, Y_b \setminus B_i \cap (U_i^b)^C) & = & \mathrm{H}^{j+1}(Y_b, (B_i^C)_b \cup (U_i^b \setminus X_b)). \end{array}$$

We have

$$\begin{array}{rcl}
\mathrm{H}^{j+1}(Y_a^b, (B_i^C)_a^b \cup (U_i \setminus X_a^b)) & \cong & \mathrm{H}^{j+1}(\mathbb{R}^n \times [a,b], (B_i^C)_a^b \cup U_i) \\
& \cong & \widetilde{\mathrm{H}}^j((B_i^C)_a^b \cup U_i), \\
\mathrm{H}^{j+1}(Y_a, (B_i^C)_a \cup (U_i^a \setminus X_a)) & \cong & \mathrm{H}^{j+1}(\mathbb{R}^n \times \{a\}, (B_i^C)_a \cup U_i^a) \\
& \cong & \widetilde{\mathrm{H}}^j((B_i^C)_a \cup U_i^a), \\
\mathrm{H}^{j+1}(Y_b, (B_i^C)_b \cup (U_i^a \setminus X_b)) & \cong & \mathrm{H}^{j+1}(\mathbb{R}^n \times \{b\}, (B_i^C)_b \cup U_i^b) \\
& \cong & \widetilde{\mathrm{H}}^j((B_i^C)_b \cup U_i^b).
\end{array}$$

The left-hand isomorphisms follow from excision and the right-hand isomorphisms from the long exact sequence of a pair.

By the naturality of the above isomorphisms and by the commutativity of the Poincaré Duality ladder, the following diagram commutes up to a sign.

$$\begin{array}{ccc}
\operatorname{colim} \widetilde{\mathrm{H}}^j((B_i^C)_a^b \cup U_i) & \longrightarrow & \operatorname{colim}(\widetilde{\mathrm{H}}^j((B_i^C)_a \cup U_i^a) \oplus \widetilde{\mathrm{H}}^j((B_i^C)_b \cup U_i^b)) \\
\wr\!\!\parallel \downarrow & & \downarrow \wr\!\!\parallel \\
\mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a \cup Y_b).
\end{array} \tag{5.4}$$

Arguing as Hatcher, in Theorem 3.44 [27], we infer

$$\begin{array}{rcl}
\operatorname{colim} \widetilde{\mathrm{H}}^j((B_i^C)_a^b \cup U_i) & \cong & \operatorname{colim}(\widetilde{\mathrm{H}}^j(S^{n-1})) \oplus \operatorname{colim}(\mathrm{H}^j(U_i)) \\
& \cong & \widetilde{\mathrm{H}}^j(S^{n-1}) \oplus \mathrm{H}^j(X_a^b).
\end{array} \tag{5.5}$$

Similarly

$$\operatorname{colim}(\widetilde{\mathrm{H}}^j((B_i^C)_a \cup U_i^a)) \cong \widetilde{\mathrm{H}}^j(S^{n-1}) \oplus \mathrm{H}^j(X_a), \tag{5.6}$$

and

$$\operatorname{colim}(\widetilde{\mathrm{H}}^j((B_i^C)_b \cup U_i^b)) \cong \widetilde{\mathrm{H}}^j(S^{n-1}) \oplus \mathrm{H}^j(X_b). \tag{5.7}$$

If $j \neq n-1$, we have $\widetilde{\mathrm{H}}^j(S^{n-1}) = 0$. We insert (5.5), (5.6) and (5.7) into (5.4) and get the desired commutative square

$$\begin{array}{ccc}
\mathrm{H}^j(X_a^b) & \longrightarrow & \mathrm{H}^j(X_a) \oplus \mathrm{H}^j(X_b) \\
\wr\!\!\parallel \downarrow & & \downarrow \wr\!\!\parallel \\
\mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a) \oplus \mathrm{H}_{n-j-1}(Y_b).
\end{array}$$

This finishes the proof for $j \neq n-1$.

Now let $j = n-1$. In this case $\widetilde{\mathrm{H}}^j(S^{n-1}) \cong \mathbf{k}$. Observe that

$$\mathrm{H}_0(Y_a \cup Y_b) \cong \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_a) \oplus \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_b) \quad \text{and} \quad \mathrm{H}_0(Y_a^b) \cong \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_a^b).$$

Taking this into account, inserting (5.7), (5.6) and (5.5) into (5.4), and extending the bottom line by an extra term from (5.3), we get the following commutative diagram

$$\begin{array}{ccccc}
\mathbf{k} \oplus \mathrm{H}^{n-1}(X_a^b) & \longrightarrow & \mathbf{k} \oplus \mathrm{H}^{n-1}(X_a) \oplus \mathbf{k} \oplus \mathrm{H}^{n-1}(X_b) & & \\
\wr\!\!\parallel \downarrow & & \downarrow \wr\!\!\parallel & & \\
\mathrm{H}_1(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_a) \oplus \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_b) & \longrightarrow & \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_a^b).
\end{array}$$

Once again, the vertical maps are isomorphisms. Since the additional copies of **k** get mapped to corresponding copies of **k**, exactness of the diamond diagram from the statement of the lemma now follows for $j = n - 1$. □

Now we can state and prove a version of the Alexander Duality theorem for levelset zigzag persistence.

Theorem 5.20 (Alexander Duality for Levelset Zigzag Persistence)**.** *Let a parametrized space $X \subset \mathbb{R}^n \times \mathbb{R}$, $n \geq 2$, have compact and locally contractible levelsets and slices, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We fix a discretization*

$$-\infty < s_1 < s_2 < \ldots < s_m < \infty,$$

where $m \geq 2$. Assume that the levelsets X_{s_k} where $k = 1, \ldots, m$ and slices $X_{s_k}^{s_{k+1}}$ where $k = 1, \ldots, m-1$ are locally contractible. There is then a partial bijection between the set of intervals that appear in the interval modules decomposition of $\mathrm{H}^j(\mathrm{X}_{\{s_1,s_2,\ldots,s_m\}})$ and those that appear in the interval modules decomposition of $\widetilde{\mathrm{H}}_{n-j-1}(\mathrm{Y}_{\{s_1,s_2,\ldots,s_m\}})$ for $j = 0, 1, \ldots, n-1$. For a fixed j the intervals are matched according to the following rules:

- *For $k = 1, \ldots, m-1$ the intervals of type $[2k, 2k]$ are unmatched;*
- *For $i < k < m$ the intervals $[2i, 2k]$ are matched with intervals $[2i+1, 2k-1]$ and vice versa;*
- *For $i < k < m$ the intervals $[2i, 2k-1]$ are matched with intervals $[2i+1, 2k]$ and vice versa;*

  For $k < m$ the intervals $[2k, 2m-1]$ are matched with intervals $[2k+1, 2m-1]$ and vice versa;
- *For $1 < i < k < m$ the intervals $[2i-1, 2k-1]$ are matched with intervals $[2i-2, 2k]$ and vice versa;*

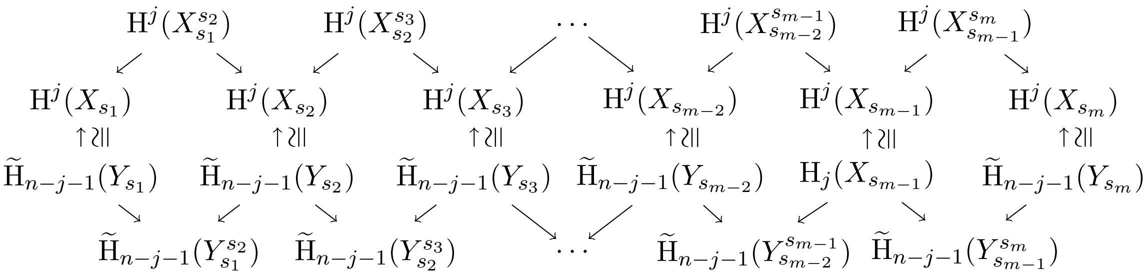
  For $k < m$ the intervals $[1, 2k-1]$ are matched with intervals $[1, 2k]$ and vice versa;

  For $1 < k$ the intervals $[2k-1, 2m-1]$ are matched with intervals $[2k-2, 2m-1]$ and vice versa;

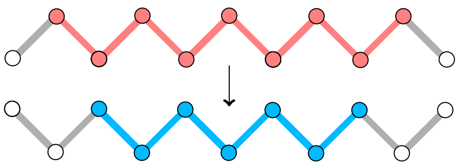
  Intervals $[1, 2m-1]$ are matched with intervals $[1, 2m-1]$ and vice versa;
- *For $1 < i < k < m$ the intervals $[2i-1, 2k]$ are matched with intervals $[2i-2, 2k-1]$ and vice versa;*

  For $k < m$ the intervals $[1, 2k]$ are matched with intervals $[1, 2k-1]$ and vice versa.

Proof. By Lemma 5.18 all the diamonds in the following diagram are exact:



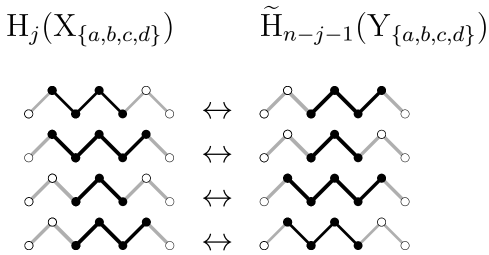
Now suppose that the interval $[2, 2m-2]$ appears in the interval decomposition of $\mathrm{H}^j(\mathrm{X}_{\{s_1,s_2,\ldots,s_m\}})$. By applying the Diamond Principle for each diamond, the following change occurs:



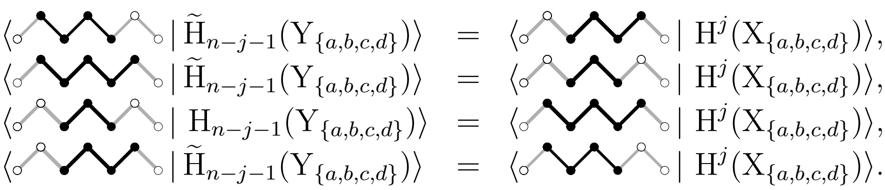
The same argument works in other cases. □

Proof of Theorem 5.14. $(X, p|_X)$ has a well-defined parametrized cohomology, so ${}^j\mu_{\mathrm{X}}^{\backslash\backslash}$, ${}^j\mu_{\mathrm{X}}^{\vee}$, ${}^j\mu_{\mathrm{X}}^{\wedge}$, and ${}^j\mu_{\mathrm{X}}^{//}$ are finite r-measures for $j = 0, \ldots, n-1$.

Let $R = [a,b] \times [c,d]$ with $-\infty < a < b < c < d < \infty$. For this particular discretization it follows by Theorem 5.20 that the four indecomposable summands change as follows:



From here we conclude that



Consequently,

$$\begin{aligned}
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\backslash\backslash}(R) &= {}^j\mu_{\mathrm{X}}^{//}(R),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\vee}(R) &= {}^j\mu_{\mathrm{X}}^{\wedge}(R),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\wedge}(R) &= {}^j\mu_{\mathrm{X}}^{\vee}(R),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{//}(R) &= {}^j\mu_{\mathrm{X}}^{\backslash\backslash}(R).
\end{aligned}$$

Since ${}^j\mu^*_{\mathrm{X}}$ are finite r-measures for $j = 1, \dots n-1$, $* = \backslash\backslash, \vee, \wedge, //$, it follows that ${}_{n-j-1}\widetilde{\mu}^*_{\mathrm{Y}}$ are finite r-measures for $j = 1, \dots, n-1$ and $* = \backslash\backslash, \vee, \wedge, //$. Therefore Y has a well-defined parametrized homology. Since the measures are the same on all the rectangles, the associated decorated diagrams are also the same by Corollary 3.5. This proves Theorem 5.14. □

Parametrized homology matches levelset zigzag persistence in most real-world situations. Since an algorithm exists to compute the latter, the following theorem is useful.

Theorem 5.21. *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We assume that the levelsets X_a for $a \in \mathbb{R}$, and slices X_a^b for $a < b$ are compact and locally contractible. If $(X, p|_X)$ has a well-defined parametrized homology, then the pair $(Y, p|_Y)$ has a well-defined reduced parametrized homology. Additionally, for $j = 0, \dots, n-1$:*

$$\begin{aligned}
\widetilde{\mathrm{D}}\mathrm{gm}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{//}_j(\mathrm{X}),\\
\widetilde{\mathrm{D}}\mathrm{gm}^{\vee}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{\wedge}_j(\mathrm{X}),\\
\widetilde{\mathrm{D}}\mathrm{gm}^{\wedge}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{\vee}_j(\mathrm{X}),\\
\widetilde{\mathrm{D}}\mathrm{gm}^{//}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{\backslash\backslash}_j(\mathrm{X}).
\end{aligned}$$

This statement follows from Theorem 5.14 and a parametrized version of the Universal Coefficient Theorem.

Lemma 5.22. *Let $\mathrm{Z} = X_1 \leftrightarrow X_2 \leftrightarrow \dots \leftrightarrow X_n$ be a sequence of topological spaces and continuous maps between them. Let $\mathrm{H}_j(\mathrm{Z})$ and $\mathrm{H}^j(\mathrm{Z})$ be the zigzag modules we get by applying the H_j and H^j functors, respectively. Then the barcodes for $\mathrm{H}_j(\mathrm{Z})$ and $\mathrm{H}^j(\mathrm{Z})$ are equal as multisets of intervals.*

Proof. $\mathrm{H}_j(\mathrm{Z})$ is decomposable by Gabriel's Theorem, therefore b_i, d_i exist for $1 \leq i \leq n$ such that

$$\mathrm{H}_j(\mathrm{Z}) \cong \oplus_{1 \leq i \leq n} \mathbb{I}(b_i, d_i).$$

If the type of $\mathrm{H}_j(\mathrm{Z})$ is τ, then applying the contravariant functor $\mathrm{Hom}_{\tau\text{-Mod}}(;\mathbf{k})$ gives an isomorphism

$$\mathrm{Hom}_{\tau\text{-Mod}}(\mathrm{H}_j(\mathrm{Z}); \mathbf{k}) \cong \oplus_{1 \leq i \leq n} \mathbb{I}(b_i, d_i)$$

in the category τ^{op}-Mod.

By the Universal Coefficient Theorem [27] we have $\mathrm{H}_{\tau\text{-Mod}}(\mathrm{H}_j(\mathrm{Z}); \mathbf{k}) \cong \mathrm{H}^j(\mathrm{Z})$. It follows that the barcodes for $\mathrm{H}_j(\mathrm{Z})$ and $\mathrm{H}^j(\mathrm{Z})$ are equal as multisets of intervals. □

Proposition 5.23. *Let X be a parametrized space. It has a well-defined parametrized homology $\mathrm{Par}_{\mathrm{H}_*}(\mathrm{X})$ if and only if X has a well-defined parametrized cohomology $\mathrm{Par}_{\mathrm{H}^*}(\mathrm{X})$. In that case, $\mathrm{Par}_{\mathrm{H}_*}(\mathrm{X}) = \mathrm{Par}_{\mathrm{H}^*}(\mathrm{X})$.*

Proof. By Lemma 5.22 $\mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})$ and $\mathrm{H}^j(\mathrm{X}_{\{a,b,c,d\}})$ have the same interval decomposition for $j = 1, \ldots, n-1$. This implies that ${}_j\mu^*_{\mathrm{X}}(R) = {}^j\mu^*_{\mathrm{X}}(R)$ for any $R = [a,b] \times [c,d]$ with $-\infty < a < b < c < d < \infty$ and $* = \backslash\backslash, \vee, \wedge, //$. This proves the first claim. If X has a well-defined parametrized homology then by the Equivalence Theorem $\mathrm{Dgm}^*_j(\mathrm{X}) = \mathrm{Dgm}^{j*}(\mathrm{X})$ for all j, proving the second claim. □

Example 5.24. *Revisiting Example 4.17, let c_X denote the 0-dimensional homology cycle in* X *that ceases to exist beyond a_1 and is killed at a_2 (see Figure 5.2).*

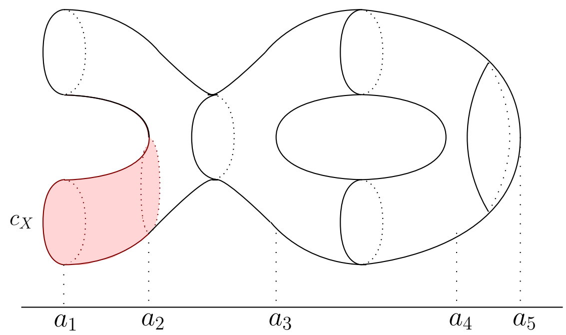


Figure 5.2: A 0-homology cycle c_X in X ceases to exist beyond a_1 and is killed at a_2.

The parametrized homology of X *and the reduced parametrized homology of* Y *can be seen in Figure 5.3.*

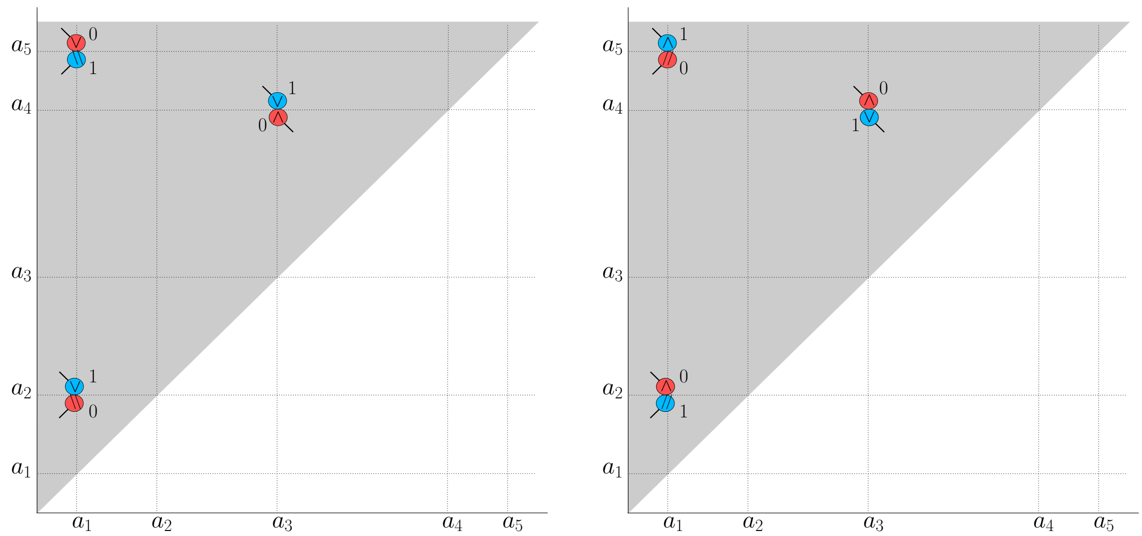


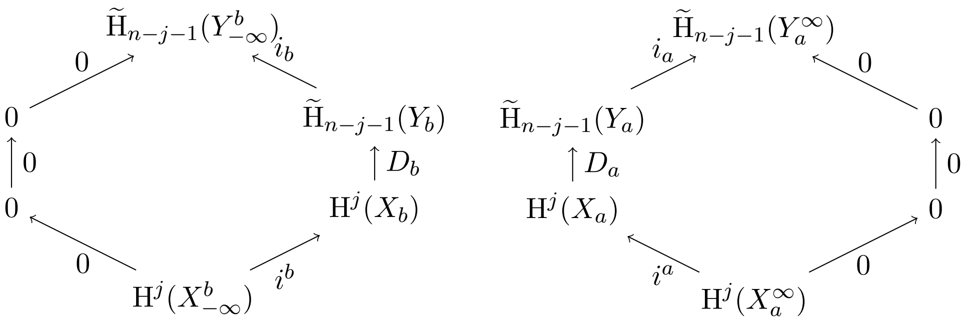
Figure 5.3: The parametrized homology of X is on the left and the one of Y on the right.

The red point in the diagram representing the cycle c_X indicates its dimension, whereas the symbol $\backslash\backslash$ designates the way it perishes at endpoints. By Theorem 5.21 *we know that there is a corresponding 1-dimensional homology cycle c_Y in* Y*. Its dimension is indicated in blue. In addition to the change of dimension, we observe the following:*

- *Since cycle c_Y persists over $[a_1, a_2)$ like c_X, the decorations of the points representing these two cycles are the same;*

- *In contrast to c_X, the cycle c_Y is killed at a_1 and ceases to exist beyond a_2. This is expressed by the symbol //.*

Next we explore if there is a parametrized version of Alexander Duality for diagrams in the extended half-plane $\overline{\mathscr{H}}$. To do this we need to account for the points at infinity. The following two diamonds come into play.



These two diamonds are not necessarily exact. Nor can we say much about the relationship between $\mathrm{H}^j(X^\infty_a)$ and $\widetilde{\mathrm{H}}_{n-j-1}(Y^\infty_a)$ or $\mathrm{H}^j(X^b_{-\infty})$ and $\widetilde{\mathrm{H}}_{n-j-1}(Y^b_{-\infty})$. See Figure 5.4.

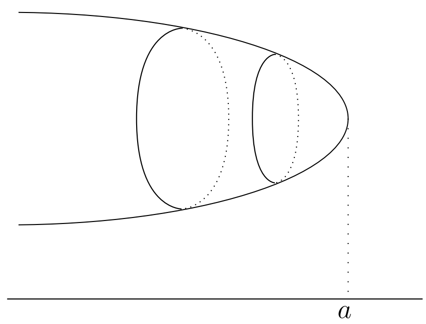


Figure 5.4: If $b < a$, we have $\widetilde{\mathrm{H}}_0(Y^b_{-\infty}) \cong \mathrm{H}^1(X^b_{-\infty}) \cong \mathbf{k}$. However, $\widetilde{\mathrm{H}}_0(Y^a_{-\infty}) \cong \mathbf{k}$ and $\mathrm{H}^1(X^a_{-\infty}) \cong 0$.

In order for a theorem to exist, we need additional constraints on the parametrized space X, more specifically, on its behavior at infinity. As in Chapter 2, we assume that X and Y are cylindrical at infinity.

Theorem 5.25. *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We assume that the levelsets X_a for $a \in \mathbb{R}$, and slices X_a^b for $-\infty < a < b < \infty$ are compact and locally contractible. Additionally, we assume that X and Y are cylindrical at infinity. Let $\mathrm{X} = (X, p|_X)$ have a well-defined parametrized cohomology and $\mathrm{Y} = (Y, p|_Y)$ a well-defined reduced parametrized homology. Then, for all $j = 0, \ldots, n-1$:*

$$\begin{array}{rcl}
\widetilde{\mathrm{Dgm}}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}) & = & \mathrm{Dgm}^{j//}(\mathrm{X}), \\
\widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}) & = & \mathrm{Dgm}^{j\wedge}(\mathrm{X}), \\
\widetilde{\mathrm{Dgm}}^{\wedge}_{n-j-1}(\mathrm{Y}) & = & \mathrm{Dgm}^{j\vee}(\mathrm{X}), \\
\widetilde{\mathrm{Dgm}}^{//}_{n-j-1}(\mathrm{Y}) & = & \mathrm{Dgm}^{j\backslash\backslash}(\mathrm{X}).
\end{array}$$

Points at infinity are matched according to the following rules:

$$\begin{array}{lcl}
(-\infty, \infty) \in \mathrm{Dgm}^{j\vee}(\mathrm{X}) & \Leftrightarrow & (-\infty, \infty) \in \widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}), \\
(-\infty, q^*) \in \mathrm{Dgm}^{j\vee}(\mathrm{X}) & \Leftrightarrow & (-\infty, q^*) \in \widetilde{\mathrm{Dgm}}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}), \\
(-\infty, q^*) \in \mathrm{Dgm}^{j\backslash\backslash}(\mathrm{X}) & \Leftrightarrow & (-\infty, q^*) \in \widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}), \\
(p^*, \infty) \in \mathrm{Dgm}^{j//}(\mathrm{X}) & \Leftrightarrow & (p^*, \infty) \in \widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}), \\
(p^*, \infty) \in \mathrm{Dgm}^{j\vee}(\mathrm{X}) & \Leftrightarrow & (p^*, \infty) \in \widetilde{\mathrm{Dgm}}^{//}_{n-j-1}(\mathrm{Y}).
\end{array}$$

Proof. We proved the first statement in Theorem 5.14. We now examine what happens with quivers associated with infinite rectangles.

By assumption X and Y are cyllindrical at infinity. By definition an $a > 0$ exists such that the following pairs are homeomorphic in the category of parametrized spaces: $X_a^\infty \cong X_a \times [a, \infty)$, $Y_a^\infty \cong Y_a \times [a, \infty)$, $X_{-\infty}^{-a} \cong X_{-a} \times (-\infty, -a]$, and $Y_{-\infty}^{-a} \cong Y_{-a} \times (-\infty, -a]$. If $b > a$, then

$$\mathrm{H}^j(X_b^\infty) \cong \mathrm{H}^j(X_b) \cong \widetilde{\mathrm{H}}_{n-j-1}(Y_b) \cong \widetilde{\mathrm{H}}_{n-j-1}(Y_b^\infty).$$

If $b < -a$, then

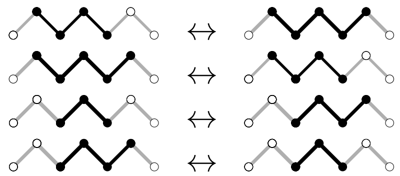
$$\mathrm{H}^j(X_{-\infty}^b) \cong \mathrm{H}^j(X_b) \cong \widetilde{\mathrm{H}}_{n-j-1}(Y_b) \cong \widetilde{\mathrm{H}}_{n-j-1}(Y_{-\infty}^b).$$

First we check how the four types of generators change on horizontal strips $H = (-\infty, b] \times [c, d]$ where $b < -a$. The associated quivers are

$$\begin{array}{ccccccccc}
 & & & & & \mathrm{H}^j(X_b^c) & & \mathrm{H}^j(X_c^d) & \\
 & & & & \swarrow & & \searrow \quad \swarrow & & \searrow \\
0 & \longleftarrow & \mathrm{H}^j(X_{-\infty}^b) & \longrightarrow & \mathrm{H}^j(X_b) & & \mathrm{H}^j(X_c) & & \mathrm{H}^j(X_d) \\
\uparrow \wr\| & & \uparrow \wr\| & & \uparrow \wr\| & & \uparrow \wr\| & & \uparrow \wr\| \\
0 & \longrightarrow & \widetilde{\mathrm{H}}_{n-j-1}(Y_{-\infty}^b) & \longleftarrow & \widetilde{\mathrm{H}}_{n-j-1}(Y_b) & & \widetilde{\mathrm{H}}_{n-j-1}(Y_c) & & \widetilde{\mathrm{H}}_{n-j-1}(Y_d), \\
 & & & & \searrow & & \swarrow \quad \searrow & & \swarrow \\
 & & & & & \widetilde{\mathrm{H}}_{n-j-1}(Y_b^c) & & \widetilde{\mathrm{H}}_{n-j-1}(Y_c^d) &
\end{array}$$

where the diamonds on the right are exact. The indecomposable summands change as follows:

$$\mathrm{H}_j(\mathrm{X}_{\{-\infty,b,c,d\}}) \qquad \widetilde{\mathrm{H}}_{n-j-1}(\mathrm{Y}_{\{-\infty,b,c,d\}})$$



From here we conclude that

$$\begin{aligned}
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\vee}(H) &= {}^{j}\mu_{\mathrm{X}}^{\backslash\backslash}(H),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\backslash\backslash}(H) &= {}^{j}\mu_{\mathrm{X}}^{\vee}(H),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{//}(H) &= {}^{j}\mu_{\mathrm{X}}^{\wedge}(H),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\wedge}(H) &= {}^{j}\mu_{\mathrm{X}}^{//}(H).
\end{aligned}$$

Similarly we get

$$\begin{aligned}
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\vee}(V) &= {}^{j}\mu_{\mathrm{X}}^{//}(V), & {}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\vee}(Q) &= {}^{j}\mu_{\mathrm{X}}^{\vee}(Q),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\backslash\backslash}(V) &= {}^{j}\mu_{\mathrm{X}}^{\wedge}(V), & {}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\backslash\backslash}(Q) &= {}^{j}\mu_{\mathrm{X}}^{\backslash\backslash}(Q),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{//}(V) &= {}^{j}\mu_{\mathrm{X}}^{\vee}(V), & {}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{//}(Q) &= {}^{j}\mu_{\mathrm{X}}^{//}(Q),\\
{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\wedge}(V) &= {}^{j}\mu_{\mathrm{X}}^{\backslash\backslash}(V), & {}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\wedge}(Q) &= {}^{j}\mu_{\mathrm{X}}^{\wedge}(Q),
\end{aligned}$$

for vertical strips $V = [p,b] \times [c,\infty)$ $(c > a)$ and $Q = (-\infty,b] \times [c,\infty)$ $(c > a$ and $b < -a)$.

We show the that $(-\infty, q^+) \in \mathrm{Dgm}^{j\vee}(\mathrm{X}) \Leftrightarrow (-\infty, q^+) \in \widetilde{\mathrm{Dgm}}_{n-j-1}^{\backslash\backslash}(\mathrm{Y})$. Let $(-\infty, q^+) \in \overline{\mathscr{H}}$. By definition

$$m_{{}^{j}\mu_{\mathrm{X}}^{\vee}}(-\infty, q^+) = \min\{{}^{j}\mu_{\mathrm{X}}^{\vee}((-\infty,b] \times [q,d])\}.$$

We can assume that $b < -a$. On these horizontal strips ${}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\backslash\backslash} = {}^{j}\mu_{\mathrm{X}}^{\vee}$. Therefore,

$$m_{{}^{j}\mu_{\mathrm{X}}^{\vee}}(-\infty, q^+) = m_{{}_{n-j-1}\widetilde{\mu}_{\mathrm{Y}}^{\backslash\backslash}}(-\infty, q^+).$$

So $(-\infty, q^+)$ appears in $\widetilde{\mathrm{Dgm}}_{n-j-1}^{\backslash\backslash}(\mathrm{Y})$ with the same multiplicity as $(-\infty, q^+)$ in $\mathrm{Dgm}^{j\vee}(\mathrm{X})$ and vice versa.

We prove other equivalences using a similar argument. □

The parametrized homology from Example 4.18 contains points at infinity. We analyze the parametrized homology of the complement using Theorem 5.30.

Example 5.26. *Consider the surface X from Example 4.18 (see Figure 5.5). Both* X *and* Y *have a well-defined parametrized homology and are cylindrical at infinity. The parametrized homology of* X *and* Y *is represented in Figure 5.6.*

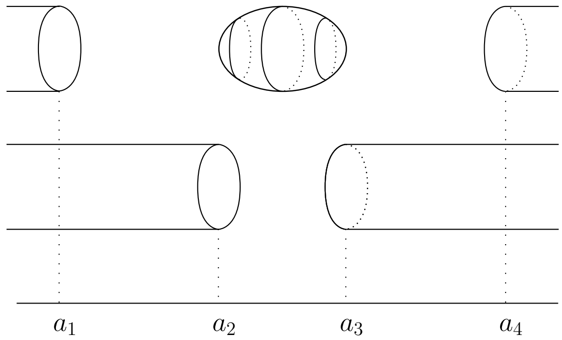


Figure 5.5: The complement Y is well-behaved, so Theorem Theorem 5.30 applies.

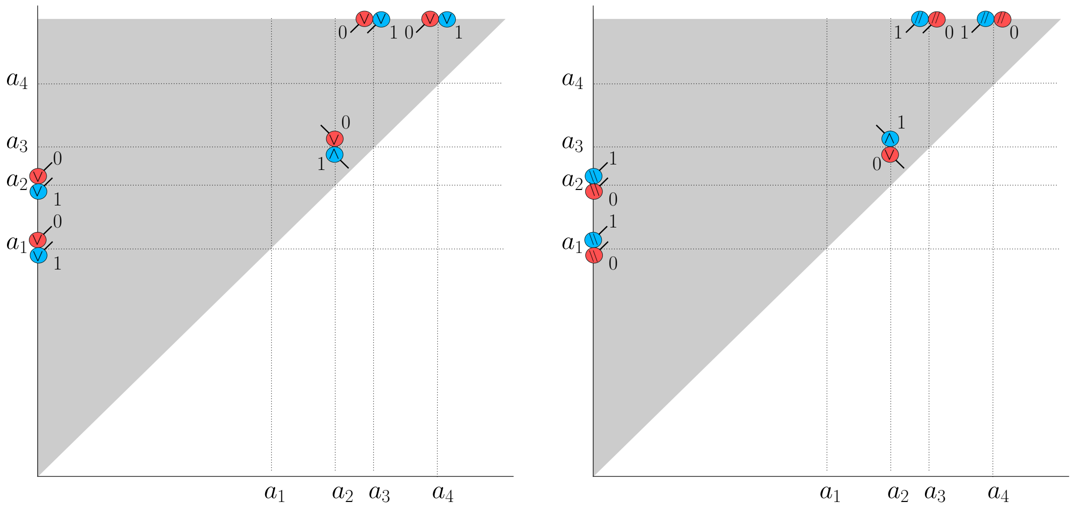


Figure 5.6: The parametrized homology of X on the left and that of Y on the right.

The last theorem in this chapter generalizes Alexander Duality for Čech cohomology to the parametrized setting.

Theorem 5.27. *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$ be parametrized space with compact levelsets and slices, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. If $(X, p|_X)$ has a well-defined parametrized Čech cohomology, then the pair $(Y, p|_Y)$ has a well-defined reduced parametrized homology. Additionally, for $j = 0, \ldots, n-1$:*

$$\begin{aligned}
\widetilde{\mathrm{Dgm}}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}) &= \check{\mathrm{Dgm}}^{j\,//}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}) &= \check{\mathrm{Dgm}}^{j\,\wedge}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{\wedge}_{n-j-1}(\mathrm{Y}) &= \check{\mathrm{Dgm}}^{j\,\vee}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{//}_{n-j-1}(\mathrm{Y}) &= \check{\mathrm{Dgm}}^{j\,\backslash\backslash}(\mathrm{X}).
\end{aligned}$$

Remark 5.28. *It follows from Theorem 4.16 that the conditions of the theorem are satisfied for $(X, p|_X)$, where X is a locally compact triangulable space and $p|_X$ is a proper continuous map.*

Lemma 5.29. *Let X, Y, and p be as in the theorem. Consider the following diagram of vector spaces and maps:*

$$\begin{array}{ccccc} & & \widetilde{\mathrm{H}}_{n-j-1}(Y_a^b) & & \\ & {\scriptstyle i_a}\nearrow & & \nwarrow{\scriptstyle i_b} & \\ \widetilde{\mathrm{H}}_{n-j-1}(Y_a) & & & & \widetilde{\mathrm{H}}_{n-j-1}(Y_b) \\ \uparrow D_a & & & & \uparrow D_b \\ \check{\mathrm{H}}^j(X_a) & & & & \check{\mathrm{H}}^j(X_b) \\ & \nwarrow{\scriptstyle i^a} & & {\scriptstyle i^b}\nearrow & \\ & & \check{\mathrm{H}}^j(X_a^b) & & \end{array}$$

Maps i^a, i^b, i_a, and i_b are induced by the inclusions $X_a \hookrightarrow X_a^b$, $X_b \hookrightarrow X_a^b$, $Y_a \hookrightarrow Y_a^b$, and $Y_b \hookrightarrow Y_a^b$. Isomorphisms D_a and D_b are Alexander Duality isomorphisms in $\mathbb{R}^n \times \{a\}$ and $\mathbb{R}^n \times \{b\}$. Then $Im(D_a i^a \oplus D_b i^b) = Ker(i_a - i_b)$.

Proof. As seen in the proof of Theorem 5.21, it is sufficient to show that an isomorphism $\check{\mathrm{H}}^j(X_a^b) \to \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b)$ exists making the following diagram commute up to a sign

$$\begin{array}{ccccc} \mathrm{H}^j(X_a^b) & \xrightarrow{i^a \oplus i^b} & \mathrm{H}^j(X_a) \oplus \mathrm{H}^j(X_b) & & \\ \wr\!\| \downarrow & & \downarrow \wr\!\| & & \\ \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a) \oplus \mathrm{H}_{n-j-1}(Y_b) & \xrightarrow{i_a - i_b} & \mathrm{H}_{n-j-1}(Y_a^b). \end{array} \tag{5.8}$$

The case $j \neq n-1$ follows immediately. We analyze the case when $j = n-1$ separately.

According to Hatcher [27, Chapter 3, Problem 35], the following diagram commutes up to a sign.

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{H}_c^{j+1}(Y_a^b) & \longrightarrow & \mathrm{H}_c^{j+1}(Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_c^{j+2}(Y_a^b, Y_a \cup Y_b) & \to \cdots \\ & \wr\!\| \downarrow & & \wr\!\| \downarrow & & \wr\!\| \downarrow & \\ \cdots \to & \mathrm{H}_{n-j}(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a \cup Y_b) & \longrightarrow & \mathrm{H}_{n-j-1}(Y_a^b) & \longrightarrow \cdots \end{array}$$

The horizontal lines are long exact sequences of the corresponding pairs and the vertical arrows are Poincaré duality isomorphisms.

By cofinality we can pick a countable sequence $\{U_i\}_i$ such that $\operatorname{colim} \mathrm{H}^j(U_i) = \mathrm{H}^j(X)$. Let $\{B_i\}_i$ be an increasing sequence of closed balls centered at the origin, containing X_a^b such that $\cup_i B_i = \mathbb{R}^{n+1}$.

Sets $\{(K_i)_a^b\}_i$, where $(K_i)_a^b = U_i^C \cap (B_i)_a^b$ exhaust Y_a^b. We have an isomorphism of directed systems $(U_i)_a^b \cup (B_i^C)_a^b \mapsto (K_i)_a^b$. Similarly, $\{(K_i)_a\}_i$, where $(K_i)_a = U_i^C \cap (B_i)_a$ exhaust Y_a and we have an isomorphism of directed systems $(U_i)_a \cup (B_i^C)_a \mapsto (K_i)_a$.

First we look at the long exact sequence of pairs $((\mathbb{R}^{n+1})^b_a, (U_i)^b_a \cup (B_i^C)^b_a)$, and $((\mathbb{R}^{n+1})_a, (U_i)_a \cup (B_i^C)_a)$ for reduced cohomology. For simplicity we write

$$\mathrm{H}^{j+1}((\mathbb{R}^{n+1})^b_a, (\mathbb{R}^{n+1})^b_a \setminus K^b_a) = \mathrm{H}^{j+1}((\mathbb{R}^{n+1})^b_a \,|\, K^b_a)$$

and

$$\mathrm{H}^{j+1}((\mathbb{R}^{n+1})_a, (\mathbb{R}^{n+1})_a \setminus (K_a)) = \mathrm{H}^{j+1}((\mathbb{R}^{n+1})_a \,|\, K_a).$$

We have a family of inclusions $((\mathbb{R}^{n+1})_a, (U_i)_a \cup (B_i^C)_a) \hookrightarrow ((\mathbb{R}^{n+1})^b_a, (U_i)^b_a \cup (B_i^C)^b_a)$ and by naturality the following commuting diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^j((\mathbb{R}^{n+1})_a) & \to & \mathrm{H}^j((U_i)_a \cup (B_i^C)_a) & \longrightarrow & \mathrm{H}^{j+1}((\mathbb{R}^{n+1})_a \,|\, K_a) & \to & \mathrm{H}^{j+1}((\mathbb{R}^{n+1})_a) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{H}^j((\mathbb{R}^{n+1})^b_a) & \longrightarrow & \mathrm{H}^j(U_i \cup (B_i^C)^b_a) & \longrightarrow & \mathrm{H}^{j+1}((\mathbb{R}^{n+1})^b_a \,|\, K^b_a) & \to & \mathrm{H}^{j+1}((\mathbb{R}^{n+1})^b_a).
\end{array}$$

Since $(\mathbb{R}^{n+1})_a$ and $(\mathbb{R}^{n+1})^b_a$ are contractible, their reduced cohomology groups are trivial. Therefore, we have the following commutative diagram for all $j = 0, \dots, n-1$:

$$\begin{array}{ccccc}
\mathrm{H}^j((U_i)_a \cup (B_i^C)_a) & \xrightarrow{\cong} & \mathrm{H}^{j+1}((\mathbb{R}^{n+1})_a \,|\, K_a) & \xrightarrow{\cong \text{ exc}} & \mathrm{H}^{j+1}((Y)_a \,|\, K_a) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{H}^j(U_i \cup (B_i^C)^b_a) & \xrightarrow{\cong} & \mathrm{H}^{j+1}((\mathbb{R}^{n+1})^b_a \,|\, K^b_a) & \xrightarrow{\cong \text{ exc}} & \mathrm{H}^{j+1}(Y^b_a \,|\, K^b_a).
\end{array}$$

Now let $j \neq n-1$. Taking the colimit of the diagram above, we get

$$\begin{array}{ccc}
\check{\mathrm{H}}^j(X_a) & \xrightarrow{\cong} & \mathrm{H}_c^{j+1}(Y_a) \\
i^a \uparrow & & \uparrow \\
\check{\mathrm{H}}^j(X^b_a) & \xrightarrow{\cong} & \mathrm{H}_c^{j+1}(Y^b_a).
\end{array}$$

We can repeat the argument for the levelset over b. Taking direct sums of Čech cohomology groups at a and b yields the following commutative diagram:

$$\begin{array}{ccc}
\check{\mathrm{H}}^j(X_a) \oplus \check{\mathrm{H}}^j(X_b) & \xrightarrow{\cong} & \mathrm{H}_c^{j+1}(Y_a) \oplus \mathrm{H}_c^{j+1}(Y_b) \\
i^a \oplus i^b \uparrow & & \uparrow \\
\check{\mathrm{H}}^j(X^b_a) & \xrightarrow{\cong} & \mathrm{H}_c^{j+1}(Y^b_a).
\end{array}$$

Diagram 5.8 is the result of composing vertical isomorphisms with the appropriate Poincaré isomorphism. This finishes the proof for the case of $j \neq n-1$.

Now let $j = n-1$. Since $\mathrm{H}^{n-1}((U_i)^b_a \cup (B_i^C)^b_a) \cong \mathrm{H}^{n-1}((U_i)^b_a) \oplus \mathrm{H}^{n-1}((B_i^C)^b_a)$ and $\mathrm{H}^{n-1}((U_i)_a \cup (B_i^C)_a) \cong \mathrm{H}^{n-1}((U_i)_a) \oplus \mathrm{H}^{n-1}((B_i^C)_a)$, taking the colimit of the diagram above, we get

$$\begin{array}{ccc}
\check{\mathrm{H}}^{n-1}(X_a) \oplus \mathbf{k} & \xrightarrow{\cong} & \mathrm{H}_c^{n}(Y_a) \\
\uparrow & & \uparrow \\
\check{\mathrm{H}}^{n-1}(X^b_a) \oplus \mathbf{k} & \xrightarrow{\cong} & \mathrm{H}_c^{n}(Y^b_a).
\end{array}$$

Once again taking direct sums (as in the previous case), produces the following commutative diagram:

$$\begin{array}{ccc} \check{\mathrm{H}}^{n-1}(X_a) \oplus \mathbf{k} \oplus \check{\mathrm{H}}^{n-1}(X_b) \oplus \mathbf{k} & \xrightarrow{\cong} & \mathrm{H}^n_c(Y_a) \\ \uparrow & & \uparrow \\ \check{\mathrm{H}}^{n-1}(X_a^b) \oplus \mathbf{k} & \xrightarrow{\cong} & \mathrm{H}^n_c(Y_a^b). \end{array}$$

Composing vertical arrows with appropriate Poincaré duality isomorphisms and extending the bottom line we get

$$\begin{array}{ccccc} \mathbf{k} \oplus \check{\mathrm{H}}^{n-1}(X_a^b) & \longrightarrow & \mathbf{k} \oplus \check{\mathrm{H}}^{n-1}(X_a) \oplus \mathbf{k} \oplus \check{\mathrm{H}}^{n-1}(X_b) & & \\ \wr\| \downarrow & & \downarrow \wr\| & & \\ \mathrm{H}_1(Y_a^b, Y_a \cup Y_b) & \longrightarrow & \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_a) \oplus \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_b) & \longrightarrow & \mathbf{k} \oplus \widetilde{\mathrm{H}}_0(Y_a^b). \end{array}$$

Additional copies of **k** get mapped to corresponding copies of **k**, so exactness also follows for $j = n - 1$.

□

Proof of Theorem 7.16. We use the Diamond Principle and the Equivalence Theorem as in the proof of Theorem 5.14. □

An equivalent of Theorem 5.30 also holds for Čech cohomology.

Theorem 5.30. *Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We assume that the levelsets X_a for $a \in \mathbb{R}$, and slices X_a^b for $-\infty < a < b < \infty$ are compact and locally contractible. Additionally, we assume that X and Y are cylindrical at infinity. Let $\mathrm{X} = (X, p|_X)$ have a well-defined parametrized Čech cohomology and $\mathrm{Y} = (Y, p|_Y)$ a well-defined reduced parametrized homology. Then, for all $j = 0, \ldots, n-1$:*

$$\begin{aligned} \check{\mathrm{D}}\mathrm{gm}^{j\,//}(\mathrm{X}) &= \widetilde{\mathrm{D}}\mathrm{gm}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}), \\ \check{\mathrm{D}}\mathrm{gm}^{j\,\wedge}(\mathrm{X}) &= \widetilde{\mathrm{D}}\mathrm{gm}^{\vee}_{n-j-1}(\mathrm{Y}), \\ \check{\mathrm{D}}\mathrm{gm}^{j\,\vee}(\mathrm{X}) &= \widetilde{\mathrm{D}}\mathrm{gm}^{\wedge}_{n-j-1}(\mathrm{Y}), \\ \check{\mathrm{D}}\mathrm{gm}^{j\,\backslash\backslash}(\mathrm{X}) &= \widetilde{\mathrm{D}}\mathrm{gm}^{//}_{n-j-1}(\mathrm{Y}). \end{aligned}$$

Points at infinity are matched according to the following rules:

$$\begin{aligned} (-\infty, \infty) \in \check{\mathrm{D}}\mathrm{gm}^{j\vee}(\mathrm{X}) &\Leftrightarrow (-\infty, \infty) \in \widetilde{\mathrm{D}}\mathrm{gm}^{\vee}_{n-j-1}(\mathrm{Y}), \\ (-\infty, q^*) \in \check{\mathrm{D}}\mathrm{gm}^{j\vee}(\mathrm{X}) &\Leftrightarrow (-\infty, q^*) \in \widetilde{\mathrm{D}}\mathrm{gm}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}), \\ (-\infty, q^*) \in \check{\mathrm{D}}\mathrm{gm}^{j\backslash\backslash}(\mathrm{X}) &\Leftrightarrow (-\infty, q^*) \in \widetilde{\mathrm{D}}\mathrm{gm}^{\vee}_{n-j-1}(\mathrm{Y}), \\ (p^*, \infty) \in \check{\mathrm{D}}\mathrm{gm}^{j//}(\mathrm{X}) &\Leftrightarrow (p^*, \infty) \in \widetilde{\mathrm{D}}\mathrm{gm}^{\vee}_{n-j-1}(\mathrm{Y}), \\ (p^*, \infty) \in \check{\mathrm{D}}\mathrm{gm}^{j\vee}(\mathrm{X}) &\Leftrightarrow (p^*, \infty) \in \widetilde{\mathrm{D}}\mathrm{gm}^{//}_{n-j-1}(\mathrm{Y}). \end{aligned}$$

Chapter 6

The Evasion Problem

The evasion problem is one of the motivations for extending Alexander Duality to parametrized spaces. In Section 6.1 we state a version of the evasion problem building on de Silva and Ghrist [26]. In Section 6.2 we describe a partial answer to the problem [26]. We conclude by discussing a levelset zigzag persistence criterion [1] that requires parametrized Alexander Duality in Section 6.3.

6.1 Setting the stage

A sensor is a device that identifies certain inputs from the physical environment. Most basically, sensors respond to the presence of a stimulus. For example, motion sensors in home security systems detect the presence of movement. The output is a signal that can be simple as a binary flag, as with metal detectors, and as complicated as a video recording, requiring sophisticated analysis. For simplicity, our discussion deals exclusively with sensors of the former type.

Suppose $D \subset \mathbb{R}^n$, $n \geq 2$ is a domain homeomorphic to the unit ball. We have a finite set of sensors $X_1, \ldots, X_m$ that move continuously in this domain over the time interval $I = [0, 1]$. At any given time a sensor is represented by a node in the domain that covers the surrounding neighborhood. We assume that these neighborhoods are unit balls about the nodes, even though most statements hold more generally. We also assume that the boundary of the domain ∂D is covered.

We adopt the following notation. For $j = 1, \ldots, m$, $X_j(t)$ denotes the position of the sensor X_j and $K_j(t) = \{x \in D \,|\, ||x - X_j(t)|| \leq 1\}$ the region it covers at time t. The covered region at time t is $K_t = \cup_{j=1}^{m} K_j(t) \subset D \times \{t\}$ and the region covered by sensors in spacetime is $K = \cup_{t \in I} K_t \subset D \times I$. Similarly, $U_t = D \times \{t\} - K_t$ denotes the uncovered region at time t and $U = D \times I - K$ the uncovered region in spacetime. Both K and U are parametrized spaces with respect to the projection onto I (see Figure 6.1).

Definition 6.1. *The pair* $X = (D \times I, \{(X_j(t), K_j(t))_{t \in [0,1]}\}_{j=1}^{m})$ *is called a* sensor network*.*

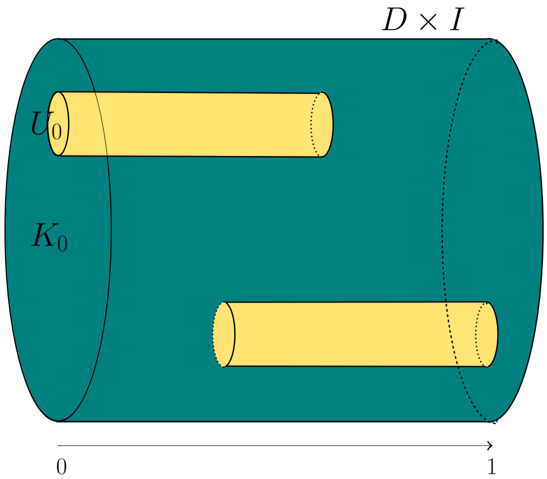


Figure 6.1: A sensor network with domain $D \subset \mathbb{R}^2$ on the vertical and time I on the horizontal axis. The uncovered region U in spacetime is drawn in yellow. The leftmost slice shows the domain at time 0 with covered region K_0 and uncovered region U_0.

The security and defense industries are faced with the problem of 'wandering' loss of coverage. For example, an intruder may be able to escape detection by moving continuously within the domain of a sensor network. If this is possible, then an evasion path exists in the network.

Definition 6.2. *Let X be a sensor network and let $p_T \colon D \times I \to I$ be a projection map. An* evasion path *in X is a continuous map $e \colon I \to U \subset D \times I$ such that $p_T \circ e = \mathrm{Id}_I$.*

Remark 6.3. *Omitting the condition $p_T \circ e = \mathrm{Id}_I$ would allow time travel.*

Example 6.4. *In Figure 6.1 no evasion path exists.*

Now we are ready to state a slightly different version of the evasion problem [26].

The Evasion Problem

Given the isomorphism class of the covered region K of a sensor network in the category of parametrized spaces, can we determine the existence of an evasion path?

6.2 Homology criterion

De Silva and Ghrist give a partial answer to the evasion problem in Theorem 11.2 [26]. They rely on the idea that a 'sheet' separating the uncovered areas from each other can help in sensor detection. We reformulate and prove their theorem in a different setting.

Theorem 6.5. *Let X be a sensor network. If $c \in \mathrm{H}_n(K, K \cap \partial D \times I)$ exists such that $0 \neq \partial c \in \mathrm{H}_{n-1}(\partial D \times I)$, then no evasion path exists in X.*

In this theorem, we think of c as a 'sheet' separating the uncovered areas. Figure 6.2 presents two examples of sensor networks and their uncovered regions. In the first case, the turquoise 'sheet' separates the uncovered regions. The theorem applies and no evasion path exists. However, this criterion is not sharp. In the second case, the theorem does not apply, since such an α does not exist. Additionally, no evasion path exists, since staying within the uncovered region would require traveling back in time.

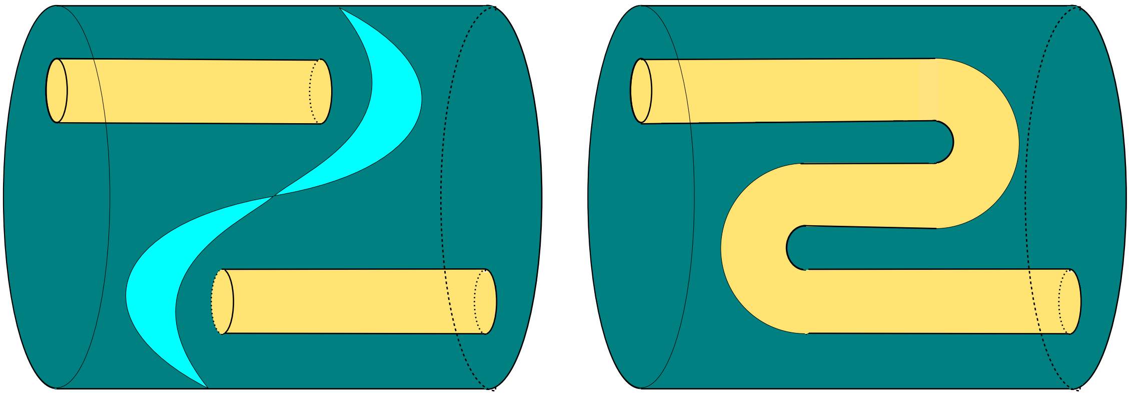


Figure 6.2: The theorem can be successfully applied in the situation in the left. The right side presents an example of a sensor network in which the criterion does not apply.

Proof. By our assumption a $c \in \mathrm{H}_n(K, \partial D \times I)$ exists such that $0 \neq \partial c \in \mathrm{H}_{n-1}(\partial D \times I)$. By construction $\mathrm{H}_n(K, \partial D \times I) \to \mathrm{H}_{n-1}(\partial D \times I)$ is non-trivial.

Let us suppose that an evasion path $e \colon I \to U$ exists. Since $e(t) \notin K(t)$ for all t, the map $\mathrm{H}_n(K, \partial D \times I) \to \mathrm{H}_n(D \times I, \partial D \times I)$ factors through $\mathrm{H}_n(D \times I - e(I), \partial D \times I)$. Now let $A = D \times I - e(I)$ and let B be a neighborhood of $e(I)$, such that $A \cap B$ is an annular tube homotopic to S^{n-1}. Let $A' = G$ and $B' = \emptyset$. Using the relative version of Mayer-Vietoris for the pair $(A \cup B, A' \cup B')$, we get

$$\mathrm{H}_n(S^{n-1}) \xrightarrow{\phi_*} \mathrm{H}_n(A, \partial D \times I) \oplus \mathrm{H}_n(B) \xrightarrow{\psi_*} \mathrm{H}_n(D \times I, \partial D \times I) \xrightarrow{\partial_*} \mathrm{H}_{n-1}(S^{n-1})$$

Since $\mathrm{H}_n(D \times I, \partial D \times I) \cong \mathrm{H}_n(D, \partial D) \cong \mathbf{k}$ and that consequently ∂_* is an isomorphism, we obtain

$$0 \longrightarrow \mathrm{H}_n(A, \partial D \times I) \oplus 0 \longrightarrow \mathrm{H}_n(D \times I, \partial D \times I) \xrightarrow{\cong} \mathbf{k}$$

By exactness it follows that $\mathrm{H}_n(D \times I - e(I), \partial D \times I) \cong \mathrm{H}_n(A, \partial D \times I) = 0$. This implies that $\mathrm{H}_n(K, \partial D \times I) = 0$, which leads to a contradiction. So no evasion path exists in X. □

6.3 Bringing parametrized homology into the discussion

At first it seemed that parametrized homology could be used to sharpen the criterion in the previous section. The hypothesis was that an evasion path exists if and only if there exists a cycle in $\mathrm{H}_{n-1}(K)$ that persists from time 0 to time 1. This does not turn out to be the case as Henry Adams' examples in Figure 6.3 show.

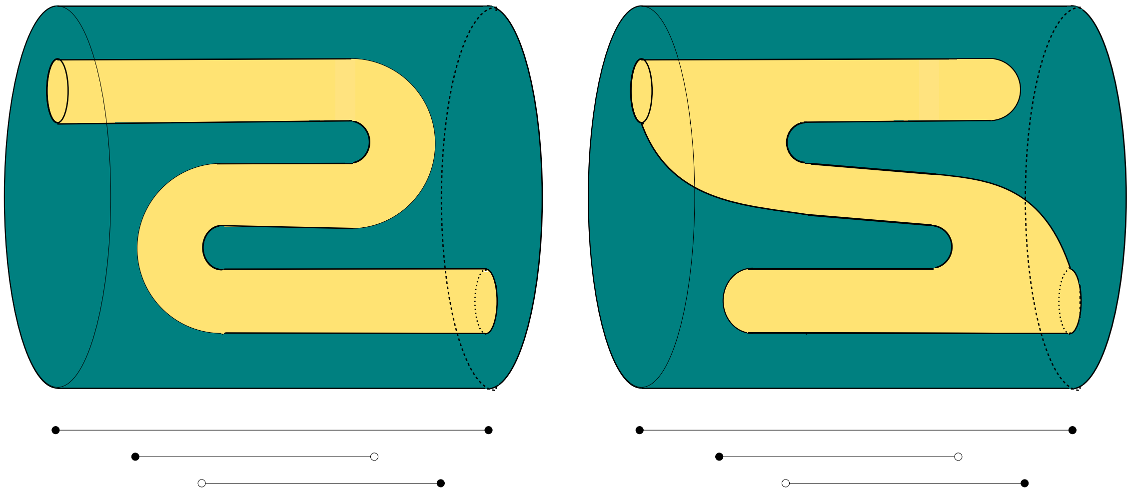


Figure 6.3: In the sensor networks in the picture, the covered regions are isomorphic in the category of parametrized spaces. Below the pictures we see the barcodes in dimension $n-1$ of the covered region. In both a cycle exists in $\mathrm{H}_{n-1}(K)$ that persists over I. However, in the network in the left picture there is no evasion path, whereas in the one on the right there is.

It turns out that an evasion path implies the existence of a persistent cycle. We reformulate the claim that was posed by Henry Adams, Gunnar Carlsson and Vin de Silva so that it fits into our setting. The proof of this statement requires Theorem 5.21.

Theorem 6.6. *Let X be a sensor network with a covered region of Morse-type with critical points $a_1 = 0 < a_2 < \ldots < a_d < a_{d+1} = 1$. We select a set of indices s_i which satisfy*

$$0 < s_1 < a_2 < \ldots < s_d < 1.$$

If an evasion path exists in X, then there is a full-length interval $[1, 2d-1]$ in the barcode for $\mathrm{H}_{n-1}(\mathrm{K}_{\{s_1,s_2,\ldots,s_d\}})$.

Proof. First we show that $\widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus \mathrm{K}_{\{s_1,\ldots,s_d\}})$ and $\mathrm{H}_0(\mathrm{U}_{\{s_1,\ldots,s_d\}})$ are isomorphic zigzag modules. For all $t \in I$ we have $(\mathbb{R}^{n+1} \setminus K)_t = U_t \sqcup (\mathbb{R}^{n+1} \setminus D)_t$. This is indeed a disjoint union since we assume that ∂D is covered for $t \in I$. The same holds for slices. Applying the reduced homology functor in dimension 0 we get

$$\widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus K)_t) = \widetilde{\mathrm{H}}_0(U_t \sqcup (\mathbb{R}^{n+1} \setminus D)_t) \cong \mathrm{H}_0(U_t) \oplus \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus D)_t) = \mathrm{H}_0(U_t)$$

since $\mathbb{R}^n \times \{t\} \setminus D_t$ is connected for every t. By naturality of reduced homology functor it follows that

$$\begin{array}{ccccccccc} & \mathrm{H}_0(U^{s_2}_{s_1}) & & & \cdots & & & \mathrm{H}_0(U^{s_d}_{s_{d-1}}) & \\ \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\ \mathrm{H}_0(U_{s_1}) & & & \mathrm{H}_0(U_{s_2}) & & \mathrm{H}_0(U_{s_{d-1}}) & & & \mathrm{H}_0(U_{s_d}) \end{array}$$

and

$$\begin{array}{ccccccccc} & \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus \mathrm{K})^{s_2}_{s_1}) & & & \cdots & & & \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus \mathrm{K})^{s_d}_{s_{d-1}}) & \\ \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\ \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus \mathrm{K})_{s_1}) & & & \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus \mathrm{K})_{s_2}) & & \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus \mathrm{K})_{s_{d-1}}) & & & \widetilde{\mathrm{H}}_0((\mathbb{R}^{n+1} \setminus \mathrm{K})_{s_d}) \end{array}$$

are isomorphic as zigzag modules and therefore have the same interval decomposition.

Zigzag modules $\mathrm{H}_{n-1}(\mathrm{K}_{\{s_1,s_2,\ldots,s_d\}})$ and $\widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus \mathrm{K}_{\{s_1,\ldots,s_d\}})$ have the same interval decomposition by Theorem 5.20 and Lemma 5.22.

From these two observations it follows that $\mathrm{H}_0(\mathrm{U}_{\{s_1,\ldots,s_d\}})$ and $\mathrm{H}_{n-1}(\mathrm{K}_{\{s_1,s_2,\ldots,s_d\}})$ have the same barcodes. So it is sufficient to prove that there is a full length interval in $\mathrm{H}_0(\mathrm{U}_{\{s_1,\ldots,s_d\}})$.

Let $e\colon I \to U$ be an evasion path in X. We have the following commuting diagram

$$I \xrightarrow{\;e\;} U \xrightarrow{\;p\;} I, \qquad p \circ e = \mathrm{Id}_I .$$

Taking zigzag diagrams and applying H_0 gives the following diagram.

$$\begin{array}{ccccc}
\mathbf{k} & \longrightarrow & \mathrm{H}_0(U_{s_d}) & \longrightarrow & \mathbf{k} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{k} & \longrightarrow & \mathrm{H}_0(U^{s_d}_{s_{d-1}}) & \longrightarrow & \mathbf{k} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{k} & \longrightarrow & \mathrm{H}_0(U_{s_{d-1}}) & \longrightarrow & \mathbf{k} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{k} & \longrightarrow & \cdots & \longrightarrow & \mathbf{k} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{k} & \longrightarrow & \mathrm{H}_0(U_{s_2}) & \longrightarrow & \mathbf{k} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{k} & \longrightarrow & \mathrm{H}_0(U^{s_2}_{s_1}) & \longrightarrow & \mathbf{k} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{k} & \longrightarrow & \mathrm{H}_0(U_{s_1}) & \longrightarrow & \mathbf{k}.
\end{array}$$

At every level the composition of two vertical maps is the identity map since $p \circ e = \mathrm{Id}_I$. This implies that there is a full-length interval $[1, 2d-1]$ in the barcode for $\mathrm{H}_0(\mathrm{U}_{\{s_1,\ldots,s_d\}})$. □

Chapter 7

Slovenski povzetek

Posledica bliskovitega razvoja informacijske tehnologije v zadnjih desetletjih je zbiranje in shranjevanje ogromne količine podatkov, ki jih želimo analizirati. Podatkov je tipično preveč za popolno obdelavo, poleg tega pa se vprašanja, ki nas zanima, morda sploh ne tičejo vsi podatki (ne vemo pa vnaprej, kateri). V praksi velikokrat potrebujemo zanesljivost pri odločitvi, da sta dva nabora podatkov 'različna'. Na primer, pri naboru podatkov, ki ustrezata dvema posnetkoma obrazov, ki jih posname varnostna kamera (z različnih zornih kotov, pri različni oddaljenosti in osvetljenosti), želimo vedeti le, ali gre za isti obraz ali ne. Tedaj iščemo kvalitativne lastnosti, ki dobro razlikujejo med različnimi možnostmi in, enako pomembno, jih je mogoče hitro in zanesljivo implementirati algoritmično.

Med metode, ki na podlagi vzorca (ali vzorcev) iz podatkov sklepajo na kvalitativne lastnosti, vse bolj prodirajo metode algebraične topologije. (Glej [31] in [19] za širše področje računske topologije.) Topologija je pri interpretaciji podatkov naravna, saj so morebitne koordinate, ki jih vpeljemo v procesu računališke predstavitve podatkov (na primer nabor števil, prirejen molekuli DNK), morda vsiljene in niso intrinzične za dani problem. Nasprotno pa moramo znati opredeliti vsaj 'bližino', če želimo pojave razlikovati. Algebraična topologija kot prirejanje algebraičnih invariant (na primer homoloških grup) topološkemu prostoru je tudi naravna, saj je neobčutljiva za majhne deformacije in v tem smislu stabilno predstavlja kvaliteto. Edini resni problem se pojavi pri občutljivosti na šum: podatki v praksi odstopajo od dejanskih vrednosti, tovrstna odstopanja pa prav lahko povzročijo spremembo algebraičnih invariant. Edelsbrunner, Letscher in Zomorodian [22] so ta problem učinkovito rešili z vpeljavo vztrajne homologije. Poglobitev in trdno teoretično podlago predstavlja članek Zomorodiana in Carlssona [13].

Metode vztrajne homologije so zelo uporabne. Med pomembnejše uporabe sodi izdelava algoritma, ki z vztrajno homologijo odloči o (ne)pokritosti danega senzorskega omrežja [26]. Dalje je vztrajna homologija priljubljena kot osnova za vizualizacijske tehnike [18, 23]. Aplikacije v problemih analize oblike so porodile vpeljavo črtnih kod [11, 12, 10] in kasneje vztrajnega diagrama [16]. Črtne kode in diagram sta ne le ugodna načina za predstavitev oziroma vizualizacijo vztrajne homologije, ampak je vpeljava vztrajnega diagrama pripomogla k dokazu stabilnosti vztrajne homologije [16] glede na primeren pojem perturbacije.

Vztrajna homologija je področje, ki se zelo živahno razvija. Med nedavnimi pomembnejšimi posplošitvami je cikcak vztrajnost [7].

7.1 Cikcak vztrajnost

Delamo nad fiksnim obsegom $\mathbf{k}$. Vsi vektorski prostori so končno dimenzionalni.

Definicija 7.1. *Cikcak modul V je zaporedje vektorskih prostorov in linearnih preslikav dolžine n:*

$$V_1 \overset{p_1}{\longleftrightarrow} V_2 \overset{p_2}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n.$$

Vsaka $\overset{p_i}{\longleftrightarrow}$ *predstavlja bodisi* $\overset{f_i}{\longrightarrow}$ *ali* $\overset{g_i}{\longleftarrow}$.

Zaporedje simbolov f ali g je *tip* V. Na primer, diagram tipa $\tau = ffgg$ izgleda takole:

$$V_1 \to V_2 \to V_3 \leftarrow V_4 \leftarrow V_5.$$

Dolžina tipa τ je dolžina kateregakoli diagrama tipa τ. Diagram $ffgg$ ima dolžino 5. Ponavadi bomo imeli v mislih cikcak module fiksnega tipa τ dolžine n. Tem diagramom pravimo τ-moduli, razred τ-modulov označimo s τ-Mod. Cikcak moduli tvorijo kategorijo na naraven način.

Primer 7.2. Vztrajni moduli *so cikcak moduli, pri katerih vse puščice kažejo naprej; z drugimi besedami, kjer $\tau = ff\dots f$. Na vztrajne module lahko gledamo kot na stopničene module nad kolobarjem polinomov $\mathbf{k}[t]$. Ta opazka nam precej olajša analizo vztrajnih modulov.*

Cikcak module poskušamo razumeti tako, da jih razcepimo na enostavnejše dele.

Definicija 7.3. *Direktna vsota τ-modulov V in W je τ-modul s prostori $V_i \oplus W_i$ in preslikavami $f_i \oplus h_i$ ali $g_i \oplus k_i$, kjer so h puščice naprej, k pa puščice nazaj v diagramu W.*

τ-modul V je *razcepen*, če ga lahko zapišemo kot direktno vsoto podmodulov in *nerazcepen* sicer. Vsak τ-modul V ima *Remakovo dekompozicijo*, z drugimi besedami, zapišemo ga lahko kot $V = W_1 \oplus \dots \oplus W_N$, pri čemer so sumandi W_j nerazcepni. Te dekompozicije same po sebi niso enolične, ampak Krull-Schmidtov princip iz komutativne algebre nam pove, da so sumandi v Remakovi dekompoziciji enolični do preureditve natančno.

Izrek 7.4 (Krull-Remak-Schmidt)**.** *Recimo, da ima τ-modul V Remakovi dekompoziciji*

$$V = a_1V_1 \oplus \dots \oplus a_nV_n \quad \text{in} \quad V = b_1W_1 \oplus \dots \oplus b_mW_m.$$

Potem je $n = m$ in obstaja permutacija π elementov $\{1, \dots n\}$, tako da $V_i \cong W_{\pi(i)}$ za vse $1 \leq i \leq n$ in $a_i = b_{\pi(i)}$ za vse i.

S tem principom postane multimnožica $\{(V_i, a_i)\}$ izomorfnostna invarianta V. Da lahko to uporabimo, moramo najprej identificirati množico nerazcepnih τ-modulov. Poglejmo sedaj naraven primer nerazcepnih modulov - vsakemu podintervalu $[b, d]$ zaporedja števil $\{1, \ldots n\}$, lahko priredimo τ-modul.

Definicija 7.5. *Naj bo τ tip dolžine n in $1 \leq b \leq d \leq n$. Intervalni τ-modul, ki se rodi ob času b in umre ob času d označimo z $\mathbb{I}_\tau(b, d)$, je podan z vektorskimi prostori*

$$(\mathbb{I}(b, d))_i = \begin{cases} \mathbf{k} & \textit{če } b \leq i \leq d, \\ 0 & \textit{sicer;} \end{cases}$$

in identičnimi preslikavami med sosednjimi kopijami $\mathbf{k}$ *ter ničelnimi preslikavami sicer.*

Primer 7.6. *Če je $\tau = ffgg$, potem je $\mathbb{I}(2, 3)$ cikcak modul*

$$0 \xrightarrow{0} \mathbf{k} \xrightarrow{1} \mathbf{k} \xleftarrow{0} 0 \xleftarrow{0} 0.$$

Temelj teorije cikcak vztrajnosti je Gabrielov izrek [25].

Izrek 7.7 (Gabriel)**.** *Nerazcepni τ-moduli so natanko intervalni moduli $\mathbb{I}(b, d)$, kjer $1 \leq b \leq d \leq n =$ dolžina(τ). Ekvivalentno, vsak τ-modul lahko zapišemo kot direktno vsoto intervalov.*

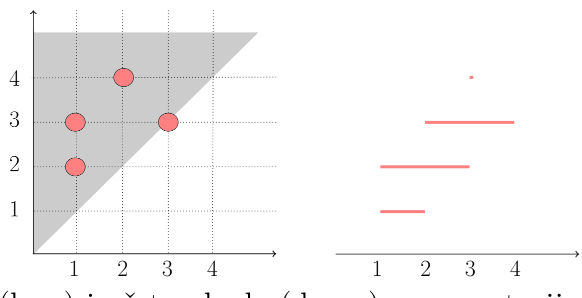
Posledično je vsak τ-modul do izomorfizma natanko določen z neurejenim seznamom intervalov $[b, d]$, ki ustrezajo nerazcepnim sumandom. To je v skladu s posebnim primerom vztrajne homologije, kjer je ta rezultat sorazmerno enostavno dokazati - gre se samo za klasifikacijo končno generiranih stopničastih modulov nad kolobarjem polinomov $\mathbf{k}[t]$.

Filozofija, ki se skriva za tem je, da je dekompozicijska teorija reprezentacij grafov neodvisna od orientacije povezav - če sprejmemo ta princip, potem posplošitev od običajne vztrajne homologije do cikcak vztrajne homologije ni presenetljiva.

Definicija 7.8. *Naj bo V cikcak modul poljubnega tipa.* Cikcak vztrajnost V *je multimnožica*

$$\mathrm{Pers}(V) = \{[b_j, d_j] \subset \{1, \ldots, n\} \mid j = 1, \ldots, n\}$$

celoštevilskih intervalov, ki jih dobimo iz dekompozicije $V \cong \mathbb{I}(b_1, d_1) \oplus \ldots \oplus \mathbb{I}(b_n, d_n)$.



Vztrajni diagram (levo) in črtna koda (desno) reprezentacije cikcak vztrajnosti $\{[1, 2], [1, 3], [3, 3], [2, 4]\}$ cikcak modula dolžine 4.

Grafično lahko $\mathrm{Pers}(V)$ predstavimo kot množico daljic (črtna koda) ali kot množico točk v ravnini $\mathbb{R}^2$, ki leži na diagonali ali nad njo v pozitivnem kvadrantu (vztrajni diagram).

Cikcak vztrajnost je najsplošnejša različica enorazsežne vztrajnosti v smislu, da dopušča podoben klasifikacijski izrek kot vztrajna homologija [13]. V okviru filtracije, ki je prirejena funkciji $f\colon X \to \mathbb{R}$, v cikcak razširitvi ne gledamo le inkluzij, ampak gledamo 'cikcak' inkluzije $f^{-1}(r) \subset f^{-1}[r,s] \supset f^{-1}(s)$.

Eno izmed ključnih orodij pri dokazu parametriziranih različic Aleksandrove dualnosti je *karo princip.*

7.1.1 Karo princip

Oglejmo si naslednji diagram

$$
\begin{array}{ccccccccccccc}
 & & & & & & W_k & & & & & & \\
 & & & & & {\scriptstyle f_{k-1}}\nearrow & & \nwarrow{\scriptstyle g_k} & & & & & \\
V_1 & \xleftrightarrow{p_1} & \cdots & \xleftrightarrow{p_{k-3}} & V_{k-2} \xleftrightarrow{p_{k-2}} & V_{k-1} & & V_{k+1} & \xleftrightarrow{p_{k+1}} & V_{k+2} & \xleftrightarrow{p_{k+2}} \cdots \xleftrightarrow{p_{n-1}} & V_n. \\
 & & & & & {\scriptstyle g_{k-1}}\nwarrow & & \nearrow{\scriptstyle f_k} & & & & & \\
 & & & & & & U_k & & & & & &
\end{array}
$$

Naj V^+ in V^- označujeta cikcak modula, vsebovana v diagramu:

$$V^+ = \quad V_1 \xleftrightarrow{p_1} \cdots \xleftrightarrow{p_{k-2}} V_{k-1} \xrightarrow{f_{k-1}} W_k \xleftarrow{g_k} V_{k+1} \xleftrightarrow{p_{k+1}} \cdots \xleftrightarrow{p_{n-1}} V_n,$$

$$V^- = \quad V_1 \xleftrightarrow{p_1} \cdots \xleftrightarrow{p_{k-2}} V_{k-1} \xleftarrow{g_{k-1}} U_k \xrightarrow{f_k} V_{k+1} \xleftrightarrow{p_{k+1}} \cdots \xleftrightarrow{p_{n-1}} V_n.$$

Radi bi primerjali $\mathrm{Pers}(V^+)$ in $\mathrm{Pers}(V^-)$, posebej pri indeksih $\{k-1, k, k+1\}$. To zahteva nek pogoj na štiri preslikave v karu.

Pravimo, da je karo zgoraj *eksakten*, če $\mathrm{Im}(D_1) = \mathrm{Ker}(D_2)$ v naslednjem zaporedju

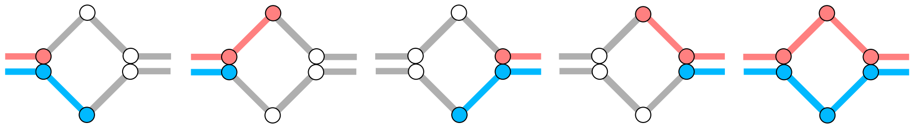
$$U_k \xrightarrow{D_1} V_{k-1} \oplus V_{k+1} \xrightarrow{D_2} W_k,$$

kjer $D_1(u) = g_{k-1}(u) \oplus f_k(u)$ in $D_2(v \oplus v') = f_{k-1}(v) - g_k(v')$.

Izrek 7.9 (Karo princip)**.** *Naj bosta V^+ in V^- kot zgoraj in predpostavimo, da je karo v sredini eksakten. Potem lahko intervale v $\mathrm{Pers}(V^+)$ in $\mathrm{Pers}(V^-)$ parimo po naslednjih pravilih:*

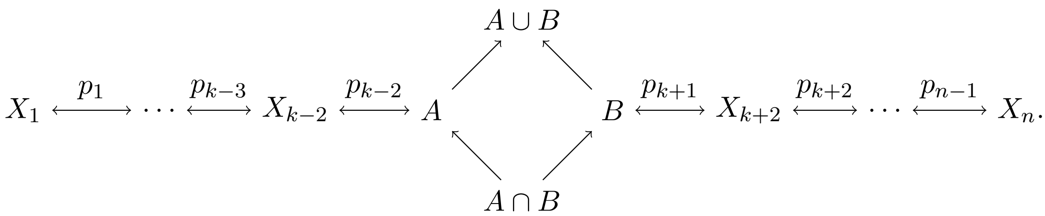
- *Med intervali tipa $[k,k]$ ni povezave;*
- *Intervali $[b,k]$ ustrezajo intervalom $[b,k-1]$ in obratno za $b \leq k-1$;*
- *Intervali $[k,d]$ ustrezajo intervalom $[k+1,d]$ in obratno za $d \geq k+1$;*

- *Intervali* $[b, d]$ *ustrezajo intervalom* $[b, d]$ *v vseh ostalih primerih.*



Carlsson in de Silva sta dokazala karo princip v [7]. Mi uberemo drugačen pristop in dokažemo izrek z zrcalnim funktorjem Bernsteina, Gelfanda in Ponomareva (BGP zrcalni princip) [4].

Karo princip lahko uporabimo na sledečem diagramu topoloških prostorov in zveznih preslikav



Z X^+ in X^- označimo zgornji in spodnji cikcak diagram. Če uporabimo funktor singularne homologije H_j s koeficienti $\mathbf{k}$ na X^+, dobimo cikcak modul $\mathrm{H}_j(X^+)$. Družino $\mathrm{Pers}(\mathrm{H}_j(X^+))$ po vseh j označimo s $\mathrm{Pers}(\mathrm{H}_*(X^+))$. Podobne oznake uporabimo pri X^-.

Izrek 7.10 (Krepki karo princip)**.** *Za* X^+ *in* X^- *kot sta definirana zgoraj obstaja bijekcija med* $\mathrm{Pers}(\mathrm{H}_*(X^+))$ *in* $\mathrm{Pers}(\mathrm{H}_*(X^-))$*. Intervale parimo po naslednjih pravilih:*

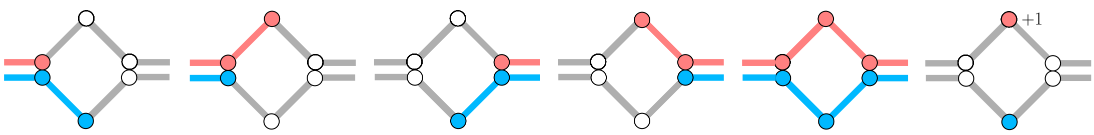
- $[k, k] \in \mathrm{Pers}(\mathrm{H}_{j+1}(X^+))$ *se ujema z* $[k, k] \in \mathrm{Pers}(\mathrm{H}_j(X^-))$.

Pri ostalih primerih se homološka dimenzija ohrani:

- *Intervali* $[b, k]$ *ustrezajo intervalom* $[b, k-1]$ *in obratno za* $b \leq k-1$.

- *Intervali* $[k, d]$ *ustrezajo intervalom* $[k+1, d]$ *in obratno za* $d \geq k+1$.

- *Intervali* $[b, d]$ *ustrezajo intervalom* $[b, d]$ *v vseh ostalih primerih.*



7.2 Parametrizirani prostori in cikcak vztrajnost nivojnic

Parametrizirani prostor je par (X, p_X), kjer je $p|_X \colon \mathrm{X} \to \mathbb{R}$ zvezna funkcija na topološkem prostoru X. Ta funkcija določa nivojnice $X_a = f^{-1}(a)$ in nivojske rezine $X_a^b = f^{-1}([a,b])$ za različne intervale $[a,b] \subset \mathbb{R}$.

Za dano diskretizacijo

$$a = s_0 \leq \ldots \leq s_n = b$$

intervala $[a,b] \subseteq \mathbb{R}$ zgradimo cikcak diagram, ki modelira nivojsko rezino X_a^b:

$$\begin{array}{ccccccccccccc} & & X_{s_0}^{s_1} & & & & X_{s_1}^{s_2} & & \cdots & & X_{s_{n-2}}^{s_{n-1}} & & X_{s_{n-1}}^{s_n} & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow & \\ X_{s_0} & & & & X_{s_1} & & & X_{s_2} & & X_{s_{n-2}} & & X_{s_{n-1}} & & & X_{s_n}. \end{array}$$

Na tem diagramu uporabimo funktor singularne homologije H_j:

$$\begin{array}{ccccccccccccc} & & \mathrm{H}_j(X_{s_0}^{s_1}) & & & & \mathrm{H}_j(X_{s_1}^{s_2}) & & \cdots & & \mathrm{H}_j(X_{s_{n-2}}^{s_{n-1}}) & & \mathrm{H}_j(X_{s_{n-1}}^{s_n}) & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow & \\ \mathrm{H}_j(X_{s_0}) & & & & \mathrm{H}_j(X_{s_1}) & & & \mathrm{H}_j(X_{s_2}) & & \mathrm{H}_j(X_{s_{n-2}}) & & \mathrm{H}_j(X_{s_{n-1}}) & & & \mathrm{H}_j(X_{s_n}). \end{array}$$

Ta diagram označimo z $\mathrm{H}_j(\mathrm{X}_{\{s_0,s_1,\ldots,s_n\}})$.

Cilj tega podpoglavja je razumeti, kako se spreminja homologija nivojnic s parametrom. Seveda tega ne bo mogoče določiti za splošne parametrizirane prostore.

Definicija 7.11. *Parametriziran prostor* $\mathrm{X} = (X, p_X)$ *je* Morsovega tipa*, če obstaja končna množica homoloških kritičnih vrednosti* $a_1 < a_2 < \ldots < a_n$*, za katere je nad vsakim intervalom*

$$I \in \{(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, \infty)\}$$

nivojska rezina nad I *homeomorfna produktu oblike* $Y \times I$ *s projekcijo* p_X *na faktor* I*. Dodatno predpostavimo, da ima vsaka nivojska rezina* X_I *končno generirano homologijo. Končno predpostavimo, da lahko vsak homeomorphizem* $Y \times I \to X_I$ *razširimo do zvezne funkcije* $Y \times \overline{I} \to X_{\overline{I}}$*, kjer* $\overline{I}$ *označjue zaprtje* I *v* $\mathbb{R}$*.*

Naj bo X Morsovega tipa. Izberemo množico indeksov s_i, ki zadoščajo

$$-\infty < s_0 < a_1 < s_1 < a_2 < \ldots < s_{n-1} < a_n < s_n < \infty.$$

Ker imajo vse nivojske rezine X_I končno generirano homologijo, je $\mathrm{H}_j(\mathrm{X}_{\{s_0,s_1,\ldots,s_n\}})$ cikcak modul.

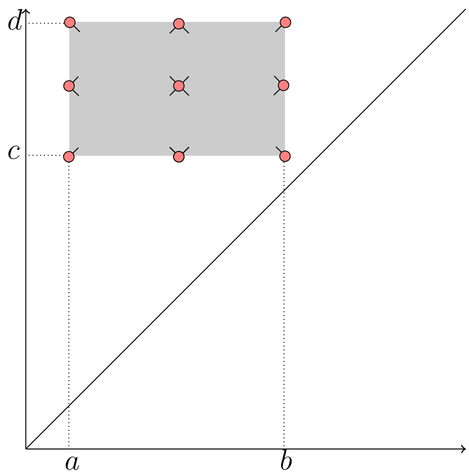
Cikcak vztrajnost nivojnic parametriziranega prostora X v dimenziji j je cikcak vztrajnost $\mathrm{H}_j(\mathrm{X}_{\{s_0,s_1,\ldots,s_n\}})$. Družina teh po vseh j je cikcak vztrajnost nivojnic X.

7.3 Mere in vztrajnost

Standardno predstavimo cikcak vztrajnost kot črtno kodo ali kot vztrajni diagram. Slednji ne razlikuje med različnimi tipi intervalov ($[p,q]$, $[p,q)$, $(p,q]$ in (p,q)), zato Chazal et al. [14] uvedejo okrašene točke. Vsako okrašeno točko lahko predstavimo v polravnini nad diagonalo skupaj s črtico, ki nakazuje okras:

$[p,q)$	pišemo kot	(p^-,q^-)	in narišemo kot
$[p,q]$	pišemo kot	(p^-,q^+)	in narišemo kot
(p,q)	pišemo kot	(p^+,q^-)	in narišemo kot
$(p,q]$	pišemo kot	(p^+,q^+)	in narišemo kot .

Naj bo $R = [a,b] \times [c,d]$, kjer $a < b < c < d$, pravokotnik v ravnini in naj bo (p^*,q^*) okrašena točka. Potem je $(p^*,q^*) \in R$, če $[b,c] \subset (p^*,q^*) \subset (a,d)$. To se zgodi natanko takrat, ko sta (p,q) in črtica vsebovani v R.



Chazal et al. [14] so predstavili nov način za predstavitev vztrajnosti, ki je še posebej primeren, ko imamo opravka s prostori parametrizirani z realnimi števili. Osnovna ideja je, da če vemo, koliko točk diagrama je vsebovanih v vsakem pravokotniku v zgornji polravnini, potem lahko določimo diagram.

Naj bo $\mathscr{H} = \{(p,q) \in \mathbb{R}^2 \mid p < q\}$ odprta polravnina, ki leži nad diagonalo. Množica pravokotnikov v $\mathscr{H}$

$$\mathrm{Rect}(\mathscr{H}) = \{[a,b] \times [c,d] \subset \mathscr{H} \mid a < b < c < d\}.$$

Pravokotna mera ali r-mera na $\mathscr{H}$ je funkcija

$$\mu\colon \mathrm{Rect}(\mathscr{H}) \to \{0,1,2,3,\dots\} \cup \{\infty\},$$

ki je aditivna glede na vodoravni in navpični razcep. Natančneje, ko velja $\mu(R) = \mu(R_1) + \mu(R_2)$ za

$$\boxed{R} = \boxed{R_1}\boxed{R_2} \text{ ali } \boxed{R} = \begin{array}{|c|}\hline R_1 \\ \hline R_2 \\ \hline \end{array}\,.$$

Sledi Izrek o ekvivalenci, ki utemelji bijektivno korespondenco med okrašenimi diagrami in r-merami. Od sedaj naprej lahko tako okrašen vztrajni diagram podamo z mero na pravokotnikih.

Izrek 7.12 (Izrek o ekvivalenci)**.** *Imamo bijektivno korespondenco med:*

- *Končnimi r-merami μ na $\mathscr{H}$. Tukaj 'končna' pomeni, da $\mu(R) < \infty$ za vsak $R \in Rect\mathscr{H}$.*
- *Lokalno končnimi multimnožicami okrašenih točk v $\mathscr{H}$. Tukaj 'lokalno končna' pomeni, da $card(A|_R) < \infty$ za vsak $R \in Rect\mathscr{H}$.*

Za mero μ, ki ustreza multimnožici A, velja

$$\mu(R) = card(A|_R)$$

za vsak $R \in Rect(\mathscr{H})$.

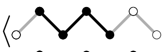
7.4 Parametrizirana homologija

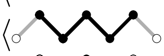
Carlsson, de Silva in Morozov v neobjavljenem delu kombinirajo pristop z merami in vztrajno homologijo nivojnic [8], da definirajo parametrizirano homologijo.

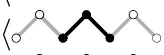
Naj bo H_j funktor singularne homologije s koeficienti v obsegu $\mathbf{k}$. Za dana števila $a < b < c < d$ nas zanimajo homološke značilnosti, ki vztrajajo na rezini X_b^c, pa ne presegajo rezine X_a^d. Natančneje, v cikcak diagramu

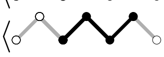
$$\begin{array}{ccccccc} & \mathrm{H}_j(X_a^b) & & \mathrm{H}_j(X_b^c) & & \mathrm{H}_j(X_c^d) & \\ \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \\ \mathrm{H}_j(X_a) & & \mathrm{H}_j(X_b) & & \mathrm{H}_j(X_c) & & \mathrm{H}_j(X_d) \end{array}$$

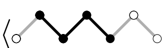
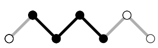
nas zanimajo tisti nerazcepni sumandi, ki vztrajajo na $\mathrm{H}_j(X_b) \to \mathrm{H}_j(X_b^c) \leftarrow \mathrm{H}_j(X_c)$, pa ne segajo niti do $\mathrm{H}_j(X_a)$ na levi niti do $\mathrm{H}_j(X_d)$ na desni strani. Ker obstajajo štirje tipi takih nerazcepnih sumandov, lahko na podlagi tega definiramo štiri označene vztrajne mere za prostor X:

${}_j\mu_{\mathrm{X}}^{\backslash\backslash}(R) = \langle$  $\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle,$

${}_j\mu_{\mathrm{X}}^{\vee}(R) = \langle$  $\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle,$

${}_j\mu_{\mathrm{X}}^{\wedge}(R) = \langle$  $\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle,$

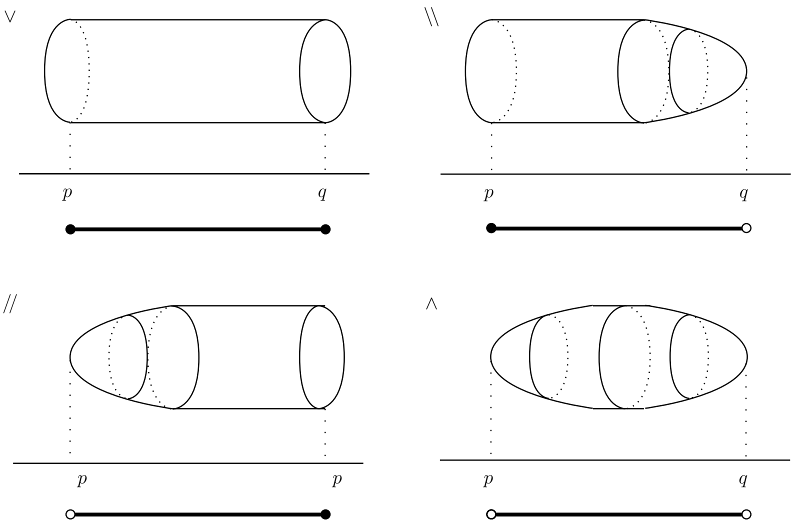
${}_j\mu_{\mathrm{X}}^{//}(R) = \langle$  $\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle.$

Tukaj je $\langle$  $\mid \mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})\rangle$ število pojavitev sumanda  v razcepu na intervale cikcak modula $\mathrm{H}_j(\mathrm{X}_{\{a,b,c,d\}})$.

Definicija 7.13. *Parametriziran prostor* X *ima dobro definirano parametrizirano homologijo, če so vse štiri zgoraj definirane vrednosti r-mere. Po izreku o ekvivalenci vsaka določa okrašen vztrajni diagram. Naj bo* $Dgm_j^*(\mathrm{X})$ *diagram prirejen* ${}_j\mu^*$. Parametrizirana homologija X, $\mathrm{Par}_{\mathrm{H}_*}(\mathrm{X})$, *je družina* $Dgm_j^{\backslash\backslash}(\mathrm{X})$, $Dgm_j^{\vee}(\mathrm{X})$, $Dgm_j^{\wedge}(\mathrm{X})$ *in* $Dgm_j^{//}(\mathrm{X})$ *po vseh* j.

Ti štirje diagrami povedo, kako homološki cikli izginejo v robnih točkah intervala (ali so j-dimenzionalni cikli *ubiti* v homologiji, ko nalepimo $(j+1)$-dimenzionalne verige ali preprosto *nehajo obstajati*).

- $\mathrm{Dgm}_j^{\backslash\backslash}(\mathrm{X})$ vsebuje okrašene točke (p^*, q^*), ki ustrezajo homološkim j-ciklom, nehajo obstajati pri p in so ubiti v q;
- $\mathrm{Dgm}_j^{\vee}(\mathrm{X})$ vsebuje okrašene točke (p^*, q^*), ki ustrezajo homološkim j-ciklom, nehajo obstajati pri p in q;
- $\mathrm{Dgm}_j^{\wedge}(\mathrm{X})$ vsebuje okrašene točke (p^*, q^*), ki ustrezajo homološkim j-ciklom, ki so ubiti v p in q;
- $\mathrm{Dgm}_j^{//}(\mathrm{X})$ vsebuje okrašene točke (p^*, q^*), ki ustrezajo homološkim j-ciklom, so ubiti v p in nehajo obstajati pri q.

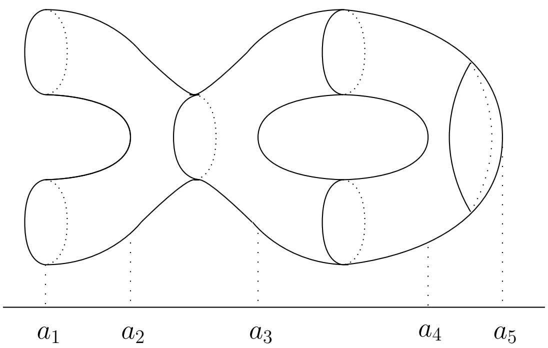


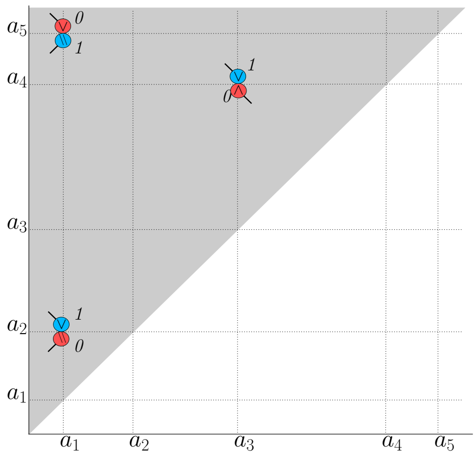
1-dimenzionalni cikel levo zgoraj neha obstajati pri obeh robnih točkah, medtem ko ta zgoraj desno neha obstajati pri p in je ubit z diskom v q.

Parametriziran prostor $\mathrm{X} = (X, p_X)$ ima dobro definirano parametrizirano homologijo, če je:

- X kompaktna mnogoterost in p_X Morsova;
- X končen simplicialni kompleks in p_X kosoma linearna preslikava.

Primer 7.14. *Naj bo X ploskev prikazana na sliki. Ker je projekcija na zadnjo komponento Morsova funkcija, ima X dobro definirano parametrizirano homologijo.*





Podobno definiramo parametrizirane različice drugih homoloških teorij.

7.5 Parametrizirana Aleksandrova dualnost

Če je X podprostor kartezičnega produkta $\mathbb{R}^n \times \mathbb{R}$ in je $p_X\colon X \to \mathbb{R}$ projekcija na zadnjo koordinato, lahko definiramo še parametrizirani prostor $Y = \mathbb{R}^n \times \mathbb{R} \setminus X$ (skupaj s projekcijo na zadnjo koordinato). Če so nivojnice X_t kompaktne in lokalno kontraktibilne, obstaja za vsak t in vsak obseg $\mathbf{k}$ izomorfizem $\widetilde{\mathrm{H}}_j(Y_t;\mathbf{k}) \cong \mathrm{H}^{n-j-1}(X_t;\mathbf{k})$ (Aleksandrova dualnost), kjer je H^* singularna kohomologija, $\widetilde{\mathrm{H}}$ pa reducirana singularna homologija.

Zanima nas, pri kakšnih pogojih in za katere pare kohomoloških oziroma homoloških teorij je mogoče konstruirati dualnostni izomorfizem med homološkimi cikcak vztrajnostmi ter kohomološkimi cikcak vztrajnostmi za vsak nabor števil $a < b < c < d$ ter posledično izomorfizem med homološkimi ter kohomološkimi črtnimi kodami.

Izrek 7.15. *Naj bo $X \subset \mathbb{R}^n \times \mathbb{R}$, $n \geq 2$, naj $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$ in naj bo p projekcija na zadnjo komponento. Predpostavimo še, da so nivojnice X_a za $a \in \mathbb{R}$ in nivojske rezine X_a^b za $a < b$ kompaktne in lokalno kontraktibilne. Če ima $\mathrm{X} = (X, p|_X)$ dobro definirano parametrizirano homologijo, potem ima $\mathrm{Y} = (Y, p|_Y)$ dobro definirano reducirano parametrizirano homologijo. Za vse $j = 0, \ldots, n-1$ velja:*

$$
\begin{aligned}
\widetilde{\mathrm{Dgm}}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{//}_{j}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{\wedge}_{j}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{\wedge}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{\vee}_{j}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{//}_{n-j-1}(\mathrm{Y}) &= \mathrm{Dgm}^{\backslash\backslash}_{j}(\mathrm{X}).
\end{aligned}
$$

Ta izrek velja za pare $(\mathrm{X}, p|_X)$, kjer je:

- X kompaktna mnogoterost in $p|_X$ Morsova;
- X končen simplicialni kompleks in $p|_X$ kosoma linearna preslikava.

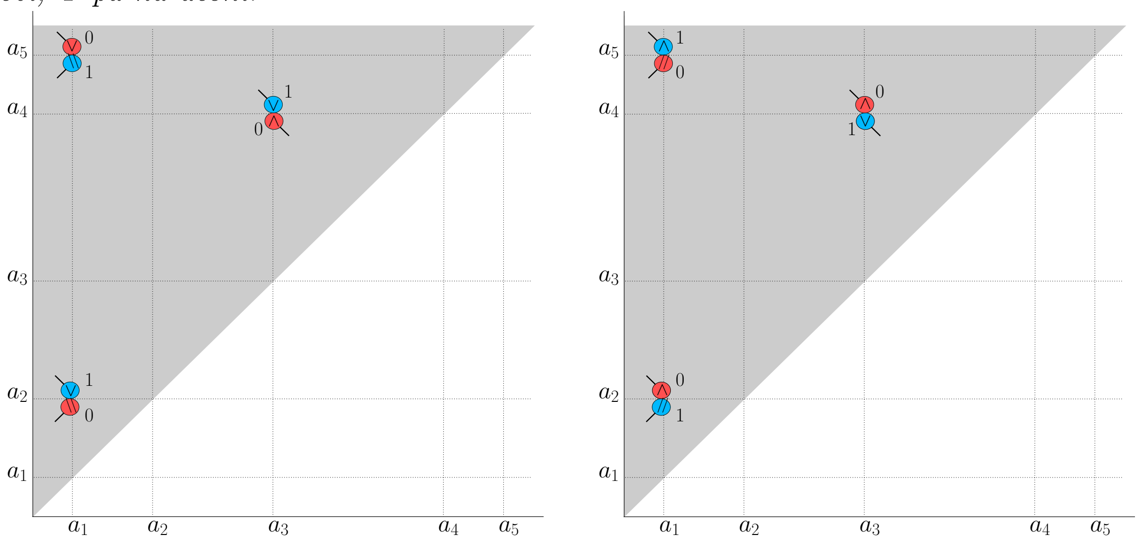
Splošnejši izrek velja, če namesto singularne vzamemo Čechovo kohomologijo.

Izrek 7.16. *Naj bo $X \subset \mathbb{R}^n \times \mathbb{R}$, $n \geq 2$, naj $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$ in naj bo p projekcija na zadnjo komponento. Predpostavimo še, da so nivojnice X_a za $a \in \mathbb{R}$ in nivojske rezine X_a^b za $a < b$ kompaktne. Če ima $\mathrm{X} = (X, p|_X)$ dobro definirano parametrizirano Čechovo kohomologijo, potem ima $\mathrm{Y} = (Y, p|_Y)$ dobro definirano reducirano parametrizirano homologijo. Za vse $j = 0, \dots, n-1$ velja:*

$$\begin{aligned}
\widetilde{\mathrm{Dgm}}^{\backslash\backslash}_{n-j-1}(\mathrm{Y}) &= \breve{\mathrm{Dgm}}^{j\,//}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{\vee}_{n-j-1}(\mathrm{Y}) &= \breve{\mathrm{Dgm}}^{j\,\wedge}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{\wedge}_{n-j-1}(\mathrm{Y}) &= \breve{\mathrm{Dgm}}^{j\,\vee}(\mathrm{X}),\\
\widetilde{\mathrm{Dgm}}^{//}_{n-j-1}(\mathrm{Y}) &= \breve{\mathrm{Dgm}}^{j\,\backslash\backslash}(\mathrm{X}).
\end{aligned}$$

Ta izrek velja za pare $(X, p|_X)$, kjer je X lokalno kompakten triangulabilen in $p|_X$ prava zvezna preslikava.

Primer 7.17. *Spomnimo se Primera 7.14. Parametrizirana homologija* X *je prikazana na levi,* Y *pa na desni.*



7.6 Problem vsiljivca

Poleg teoretičnega pomena so moje raziskave prispevale k razumevanju t.i. 'problema vsiljivca': v danem območju D, ki ga pokrivajo premični senzorji, se giblje vsiljivec, ki bi rad od dane točke v nepokritem območju ob času 0 prišel do točke v nepokritem območju ob času 1 in sicer tako, da bi se izognil senzorjem. V tem primeru imamo za vsak čas $t \in [0,1]$ območje pokritosti K_t in območje nepokritosti $U_t = D \setminus K_t$. Ta območja določajo parametriziran prostor $D \times [0,1]$ skupaj s podprostoroma K in U.

Sledeči kriterij uporablja različico parametrizirane Aleksandrove dualnosti.

Izrek 7.18. *Naj bo X senzorsko omrežje, pri katerem je pokrit prostor parametriziran prostor Morsovega tipa s kritičnimi točkami $a_1 = 0 < a_2 < \ldots < a_d < a_{d+1} = 1$. Izberemo indekse s_i, ki zadoščajo*

$$0 < s_1 < a_2 < \ldots < s_d < 1.$$

Če se vsiljivec lahko izmuzne nadzoru, potem interval $[1, 2d-1]$ nastopa v črtni kodi $\mathrm{H}_{n-1}(\mathrm{K}_{\{s_1,s_2,\ldots,s_d\}})$.

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