Equivalent Mixing of Experimental Models in Dynamic Substructuring

Dissertation

Submitted for The Degree of Doctor of Philosophy to The University of Ljubljana, Faculty of Mechanical Engineering

Tomaž Bregar

Ljubljana, November 2020
UNIVERSITY OF LJUBLJANA
Faculty of Mechanical Engineering

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Tomaž Bregar

Mentor: Professor Miha Boltežar
Co-mentor: Associate Professor Gregor Čepon

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ODLOČBO

Komisija za doktorski študij, Univerze v Ljubljani je na svoji 20. seji dne 15.10.2019 kandidatu

Tomažu Bregarju

1. sprejela temo doktorske disertacije z naslovom:
   Ekvivalentno mešanje eksperimentalnih modelov v dinamiki podstruktur
2. imenovala mentorja: prof. dr. Miha Boltežar
3. in smentorja: izr. prof. dr. Gregor Čepon
4. ter odobrila pisanje doktorske disertacije v angleškem jeziku

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1. was approved the topic of the dissertation entitled
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2. appointed as mentor: Prof. Dr. Miha Boltežar
3. appointed as co-mentor: Assoc. Prof. Dr. Gregor Čepon
4. approved the writing of the dissertation in English

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Current number: DR III/201
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Prof. dr. Mitja Kalin,
dekan/dean
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Evaluation of structural dynamics is an essential step in the design of modern complex mechanical systems. The ever increasing demand on optimization of vibro-acoustic performance has led to numerous advancements in noise, vibration and harshness engineering in the past decade. One of the most promising frameworks is dynamic substructuring, within which complex dynamic systems can be analyzed by characterizing each subsystem separately. This work covers an experimental approach to frequency based substructuring and transfer path analysis. Through experimental modelling the real dynamic properties of each subsystem can be obtained directly, without the need for a complex numerical model. Several approaches to experimentally model an interface within frequency based substructuring are introduced. An expansion to virtual point transformation is presented, where directly measured rotation response is included in the transformation. An novel transformation method named the singular vector transformation is presented on a coupling and decoupling application. Furthermore, an approach to identifying a set of full-field frequency response functions based on system equivalent model mixing is presented. Finally, an application of component-based transfer path analysis is presented on the source characterization of an electric motor. The main limitations and advantages of each methodology are outlined. Based on the provided observations, both frequency based substructuring and transfer path analysis can be an integral part of the development process.
Ekvivalentno mešanje eksperimentalnih modelov v dinamiki podstruktur

Tomaž Bregar

Ključne besede: strukturna dinamika
eksperimentalno modeliranje
podstrukturiranje v frekvenčni domeni
analiza prenosnih poti
transformacija virtualne točke
transformacija singularnih vektorjev

Vrednotenje strukturne dinamike je bistven korak pri načrtovanju sodobnih mehan- 
skih sistemov. Vedno večje povpraševanje k optimizaciji vibro-akustičnih karakteris-
tik je v zadnjem desetletju privedlo do številnih napredkov na področju obvladovanja 
hrupa in vibracij. Eno najbolj obetavnih področij je dinamika podstrukturn, ki omogoča 
analizo kompleksnih dinamskih sistemov z dekompozicijo na manjše podsisteme. Dok-
torska naloga obravnava frekvenčno podstrukturiranje in analize prenosnih poti na 
eksperimentalno pridobljenih dinamskih modelih. Z eksperimentalnim modeliranjem 
postanevnamega podsistema lahko neposredno pridobimo resnične dinamske lastnosti, brez 
potrebe po zapleteni numeričnih modelih. Raziskanih je več pristopov eksperimental-
ega modeliranja povezave znotraj frekvenčnega podstrukturiranja. Predstavljena 
je razširitev transformacije virtualne točke z direktno izmerjenim rotacijskim odzivom. 
Predstavljena je nova metoda transformacije singularnih vektorjev na aplikaciji sklapl-
janja in razsklapljanja. Predstavljena je tudi predikcija polnega polja frekvenčnega 
prenosnih funkcij iz šumnih meritev hitre kamere na podlagi metode kombiniranja ekvi-
valentnih modelov. Nazadnje pa je obravnavana karakterizacija izvora elektromotorja 
a podlagi analize prenosnih poti. Opisane so glavne omejitve in prednosti posamezne 
metodologije. Na podlagi predstavljenih rezultatov sta lahko tako podstrukturiranje 
kot analiza prenosnih poti, ob pravilni uporabi, sestavni del razvojnega procesa.
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<td>f</td>
<td>N</td>
<td>force vector</td>
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<td>g</td>
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<td>interface force vector</td>
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<tr>
<td>m</td>
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<td>(\star_b)</td>
<td>boundary/interface DoFs</td>
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<tr>
<td>(\star_g)</td>
<td>combined boundary and internal DoFs</td>
<td></td>
</tr>
<tr>
<td>(\star_i)</td>
<td>internal DoFs</td>
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<td>(\star_c)</td>
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<td>(\star_f)</td>
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<td>(\star_{ov})</td>
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<td>(\star_{ps})</td>
<td>pseudo-forces</td>
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<td>(\star_{int})</td>
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<td>(\star_{par})</td>
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<td>removed model</td>
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<td>(\star_{1})</td>
<td>internal/source excitation DoF</td>
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<td>(\star_{2})</td>
<td>interface DoF</td>
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<td>(\star_{3})</td>
<td>internal/receiver DoF</td>
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<td>(\star_{i})</td>
<td>indicator DoF</td>
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<td></td>
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<td>(\star^{AB})</td>
<td>pertaining to assembly AB</td>
<td></td>
</tr>
<tr>
<td>(\star^B)</td>
<td>pertaining to substructure B</td>
<td></td>
</tr>
<tr>
<td>(\star^{TS})</td>
<td>pertaining to transmission simulator TS</td>
<td></td>
</tr>
<tr>
<td>(\star^{op})</td>
<td>operational response</td>
<td></td>
</tr>
<tr>
<td>(\star^{meas})</td>
<td>measured response</td>
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</table>
## List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>CMIF</td>
<td>complex mode indicator function</td>
</tr>
<tr>
<td>CMS</td>
<td>component mode synthesis</td>
</tr>
<tr>
<td>DS</td>
<td>dynamic substructuring</td>
</tr>
<tr>
<td>DoF</td>
<td>degree of freedom</td>
</tr>
<tr>
<td>EMA</td>
<td>experimental modal analysis</td>
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<tr>
<td>EMPC</td>
<td>equivalent multi point connection</td>
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<td>FBS</td>
<td>frequency based substructuring</td>
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<td>FEM</td>
<td>finite element analysis</td>
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<td>FRF</td>
<td>frequency response function</td>
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<td>ICA</td>
<td>independent component analysis</td>
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<td>ICC</td>
<td>interface completeness criterion</td>
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<td>IDM</td>
<td>interface deformation/displacement mode</td>
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<tr>
<td>IRF</td>
<td>impulse response function</td>
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<td>LM</td>
<td>Lagrange multipliers</td>
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<td>LSCF</td>
<td>least-square complex frequency</td>
</tr>
<tr>
<td>LSFD</td>
<td>least-square frequency domain</td>
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<tr>
<td>NVH</td>
<td>noise, vibration and harshness</td>
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<tr>
<td>OPAX</td>
<td>operational path analysis with exogenous inputs</td>
</tr>
<tr>
<td>OTPA</td>
<td>operational transfer path analysis</td>
</tr>
<tr>
<td>PCA</td>
<td>principal component analysis</td>
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<tr>
<td>PRF</td>
<td>principal response function</td>
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<td>SEMM</td>
<td>system equivalent model mixing</td>
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<tr>
<td>SEREP</td>
<td>system equivalent reduction expansion process</td>
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<tr>
<td>SGBOF</td>
<td>simplified gradient-based optical flow</td>
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<tr>
<td>SVD</td>
<td>singular value decomposition</td>
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<td>SVT</td>
<td>singular vector transformation</td>
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<td>TPA</td>
<td>transfer path analysis</td>
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<td>TS</td>
<td>transmission simulator</td>
</tr>
<tr>
<td>VIKING</td>
<td>variability improvement of key inaccurate node groups</td>
</tr>
<tr>
<td>VP</td>
<td>virtual point</td>
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<td>VPT</td>
<td>virtual point transformation</td>
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1 Introduction

Vibro-acoustic comfort in households is becoming increasingly more important. Silent and peaceful environment is often connected with a higher quality of life. Since home appliances are often located near the living area, they should be as silent as possible when operating. In addition, the vibro-acoustic performance of appliance (along with appliance’s visual design) contributes to the customer’s perception of the product’s premium quality and, with that, to its overall value on the market [1]. Complex products such as home appliances are nowadays designed modularly, that is, in such a way that each component can be either developed in-house or outsourced to an external supplier. Therefore, it is beneficial to analyse the dynamics of each subsystem separately. Dynamic Substructuring (DS) enables us to assemble dynamic properties of subsystems, and, accordingly, to predict the dynamic response of a complete system [2], as schematically depicted in Fig. 1.1.

Figure 1.1: A schematic depiction of the dynamic substructuring approach to obtaining an assembled response from the characterization of separate subsystems.
**1.1 Motivation**

Experimental and numerical modelling are essential tools in the modern engineering. After determining the acceptable target-settings, which are based on market research and benchmarking, the whole product is subdivided into subsystems. Each subsystem is then delegated to either an in-house R&D department or to an external supplier. As is depicted in the V-model in Fig. 1.2, numerical modelling is primarily used in the early stage of product development [3,4]. After the components are designed and manufactured, experimental testing is used for the integration and final validation.

![V-model life-cycle of modern product development.](image)

Figure 1.2: The V-model life-cycle of modern product development.

Nowadays, with time-to-market becoming increasingly shorter, the performance of a product must be determined in advance, that is, already in the early stage of development. In order to achieve the target performance, the Noise, Vibration and Harshness (NVH) engineering should be incorporated right from the start of development. However, many subcomponents are made by an external supplier and proper numerical modelling can become tedious and some times even impossible due to the sheer complexity or lack of detailed information. Fortunately, with experimental modelling, the correct model of each substructure can be obtained straight-away. After obtaining the precise model of each substructure, Dynamics Substructuring [2] methods can be used for a proper NVH prediction.

**1.2 Research context**

Several ways of representing the dynamic of substructures have been developed. Five different domains can be defined, within which a linear dynamic system can be modelled (Fig. 1.3). The first is the physical domain, where the dynamics are expressed with the displacement at a specific nodal location. The discrete representation is typically obtained using a Finite Element Modelling (FEM) [3] and is often referred to as a numerical model. However, the solution can also be continuous by solving partial differential equations describing the continuous dynamics of the substructure. The
second one is the modal domain, where the substructure dynamics are expressed by a combination of vectors of a subspace obtained by mode superposition. The methods within the modal domain such as the Craig-Bampton [6] or Rubin [7] method are often referred to the field of Component Mode Synthesis (CMS) [8]. The third is the fre-

Figure 1.3: Five domains of representation for structural dynamic simulation [9].

quency domain where the Frequency Response Functions (FRFs) are used to describe the dynamics of the substructure. The Frequency-Based Substructuring (FBS) method was introduced by Jetmundsen et. al. [10] and was reformulated as the Lagrange Multiplier FBS method (LM FBS) by de Klerk et. al. [11]. The fourth is the time domain where the response is simulated based on the Impulse Response Functions (IRFs) [12]. The last is the state-space domain where the advanced system identification methods from control theory and signal processing are used for structural dynamic identification [13]. Despite the distinct differences between each representation, there are two conditions which must be satisfied on the interface between substructures and which are mathematically equivalent throughout different domains. The first condition is coordinate compatibility and the second is force equilibrium.

Even with careful planning and precise Noise, Vibration, and Harshness (NVH) engineering, a mistake in the design can happen, and, with that, the predetermined target-setting can deviate from the predetermined value. When this happens, a set of proper diagnostic tools is essential for a quick and precise determination of the source of the problem. Transfer Path Analysis (TPA) relates to a wide family of diagnostic tools for analysing NVH problems based on a substructuring approach [14]. Usually, TPA is used for identifying the NVH problems in existing designs. The most common
classical TPA methods are: mount-stiffness [15] and matrix-inverse [16]. Operational TPA (OTPA) [17] and Operational Path Analysis with exogenous inputs (OPAX) [18]. What these methods all have in common is that the excitation is not a characteristic of the source alone. For that reason, classical or transmissibility-based TPA cannot predict the effect of subsystem modification (the whole TPA needs to be repeated). However, an interesting class of TPA methods is also the so-called component based TPA [9] which tries to characterise the excitation as a property of the active component only. The dynamic interaction with the passive side is added later on, which enables design optimization for dynamic substructuring [14] as well as for virtual acoustic prototyping [19].

1.3 Research problem

Achieving the specified target-setting regarding vibro-acoustic comfort can be particularly challenging. One can imagine countless connections between the subsystems. With each connection, the complexity increases, and with it, the uncertainty of the final prediction. In fact, active mechatronic components tend to be even too complex to model solely with a numerical approach [9]. However, a hybrid approach where a numerical model is coupled with experimental results can resolve this problem. A promising framework for developing hybrid dynamic models is the System Equivalent Model Mixing (SEMM) introduced by Klaassen et. al. [20]. It enables the mixing of different models of the same component into one hybrid model based on the LM FBS method [11]. For example, with SEMM, the quality of numerical models can be improved with relatively limited experimental data, resulting in a physically correct hybrid model [20]. The SEMM could also be regarded as an expansion method, where the experimental model is projected on an unmeasured DoF-set. There are also several additional expansions methods which use the numerical model as their foundation, such as Guyan-, IRS- or Dynamic-expansion [21,23] in physical domain, or the Variability Improvement of Key Inaccurate Node Groups VIKING [24] or System Equivalent Reduction Expansion Process SEREP [25] expansions in the modal domain.

However, because of a lack of time or even the sheer complexity of a problem, the numerical modelling is sometimes not feasible. Consider a simple mechatronic system, where a complex multiphysics simulation would be required to acquire consistent results. On the other hand, through experimental modelling the real dynamic properties of the whole system can be obtained directly, without the need for a complex numerical model. A promising pure experimental approach, the Virtual Point Transformation (VPT), was introduced by Seijs et. al. [26]. The VPT is an upgrade of the equivalent multiple point connection (EMPC) [27] method. A geometrical transformation projects the measured translation DoFs on the so-called Interface Deformation Modes (IDMs), which are assumed to describe the dynamics of the interface. The interface can be modelled as rigid or as extended with proper flexible interface mode as shown in [28]. The whole transformation can also be interpreted as a minimization procedure [29].

The main drawback of the VPT is that only interfaces resembling point connections, which behave adequately rigid in the vicinity of the interface, can be modelled. For
modelling a continuous interface a promising concept named the transmission simulator was introduced by Allen et. al. [30]. The method uses the temporary attachment of a flexible structure called the transmission simulator, which enables improved measurement and excitation of the rotational and flexible modes of the interface. The transmission simulator is mounted to the measured structure in exactly the same way as the system will ultimately be assembled. Because of this, the resulting experimental model inherently includes the linearised stiffness and damping of the interface [31,32].

Despite the distinct divergences between the different experimental substructuring methods, all of them share a common problem. Even a small error in measurement can cause the assembled model to be completely erroneous and, consequently, meaningless [33,34]. The uncertainties are even more pronounced for lightly damped structures [35,36]. However, even with all the problems and drawbacks the tremendous potential of experimental substructuring outweighs the current limitations associated with it.

### 1.4 Scientific hypotheses

The primary focus of the doctoral thesis is the characterisation, development and analysis of frequency based substructuring methods. Both numerical and experimental models will be used in the characterisation of substructures. The proposed methods will be validated on a real structures with experimental case studies.

The main research hypotheses of the doctoral thesis are outlined below.

1. **Hypothesis**: The use of a directly measured rotational response in virtual point transformation can reduce uncertainties associated with deviations in sensor position and sensitivity.

2. **Hypothesis**: Expanded virtual point transformation with a directly measured rotational response can increase the consistency of coupled FRFs.

3. **Hypothesis**: The dynamic substructuring methods can be used to improve the full-field FRFs estimation from noisy high-speed camera data.
1.5 Thesis structure and contributions

The basic theory required for structural dynamics in frequency domain and frequency based substructuring is provided in Chapter 2. In Chapter 3, a transfer path analysis is introduced from a substructuring perspective. Next Chapter 4 introduces current state-of-the-art methods in frequency based substructuring and reviews various approaches in component-based transfer path analysis. Chapter 5 provides a practical applications of dynamic substructuring methods on both simple numerical examples, as well as, on complex real-life experimental examples.

The following developments may extend the current state-of-the-art in the field of frequency based substructuring:

Extended virtual point transformation: The effect of measurement bias on the virtual point transformation is presented and analysed in Section 4.1.3. A proposed expansion to virtual point transformation that reduces the effect of measurement bias with the use of directly measured rotational response is introduced, which was reported in [37]. The extended VPT is first applied on a simple numerical and experimental models, followed by an application on a complex automotive test structure in Section 5.1. It can be observed that the expansion directly reduces the sensitivity of the transformation, which yield more consistent dynamic substructuring results [38].

Full-field FRFs estimation: The use of high-speed camera in combination with experimental dynamic substructuring methods is presented in Section 5.2. A consistent set of full-field FRFs can be obtained through equivalent mixing of two experimental models [39]. The first model is the reconstructed full-field FRFs that contribute the full-field DoF set and the second model is the accelerometer measurements that provide accurate dynamic characteristics. The final hybrid model has shown to have an increase accuracy, throughout the whole frequency range (Section 5.2).

Coupling application with singular vector transformation: The use of a singular vector transformation was recently introduced on a decoupling example in [40]. The SVT formulation is here extended to include weighting matrices in the transformation (Section 4.2). The final coupling application on a numerical example is shown in Section 5.3.3.

Application of in-situ source characterization: The in-situ transfer path analysis is used for the determination of equivalent forces on an electric motor commonly used in a washing machine in combination with substructuring methods (Section 5.4).
2 Introduction to Frequency Based Substructuring

The idea behind substructuring appeared already in 1890, when Schwarz confirmed the existence and uniqueness of the solution for a Poisson problem on a domain that was a combination of a square and a circle [41]. The method itself had only a small practical usage at the time and was limited mainly to the academic sphere. The practical applicability of the methodology was demonstrated by Hurty in 1960, who introduced the method of dynamic substructuring [42, 43]. Until the end of the 20th century, substructuring was primarily used in numerical cases, and it was not until 1988 when Jetmundsen et. al. [10] introduced a formulation based on frequency response functions called Frequency Based Substructuring (FBS), that enabled assembly of experimental models.

The FBS enabled the assembly of experimentally modelled substructures. Although the methodology could be used on real problems, the application of it was still demanding and time-consuming. By introducing Lagrange multipliers in the frequency based substructuring LM-FBS (Lagrange Multiplier Frequency Based Substructuring), de Klerk et. al. [11] simplified the assembly of substructures. The LM-FBS method is nowadays one of the most commonly used method for coupling experimentally obtained data in the frequency domain [9, 44, 45].

Dynamic substructuring is not only limited to the assembly and disassembly of complex dynamic systems. Since the interface dynamics have to be well defined for the assembling substructures; also the overall transmission of vibration can also be defined. The contribution of different transfer paths can be evaluated, which can be highly beneficial when an NVH optimization of a complex product has to be performed [14].

In this chapter, the basic theory for frequency based substructuring is provided. First, the formulation of structural dynamics in frequency domain is introduced [2.1]. Next, the required interface conditions for assembling of substructures are outlined [2.2]. In the third and forth section (2.3 and 2.4) the coupling and decoupling concepts are introduced. Next, the weakening of the interface problem in LM-FBS is shown [2.5].
2.1 Structural dynamics in frequency domain

Frequency domain is one of the five domains in which the structural dynamics of substructures can be modelled. For consistent modelling of dynamic systems within the frequency domain the following assumptions have to be valid:

1. **Linearity** - response amplitude is linearly proportional to the excitation amplitude.
2. **Time invariance** - mass, stiffness and damping characteristics are time-independent.
3. **Passivity** - the energy flow in the system is always positive or equal to zero.
4. **Reciprocity** - the response of the structure remains the same if the excitation and response location are exchanged.
5. **Stability** - the response of the system is bounded, if the excitation of the system is bounded.

In frequency domain a time function of response $u(t)$ can be written in terms of harmonic functions $u(\omega)$. If we consider a linear dynamic system, the following equation of motion can be written in the frequency domain as follows:

$$\mathbf{M}\ddot{\mathbf{u}}(\omega) + \mathbf{C}\dot{\mathbf{u}}(\omega) + \mathbf{K}\mathbf{u}(\omega) = \mathbf{f}(\omega),$$

(2.1)

where $\mathbf{M}$ denotes the systems mass matrix, $\mathbf{C}$ the damping matrix and $\mathbf{K}$ the stiffness matrix. All three system matrices are commonly obtained from a Finite Element (FE) modelling. By utilizing the orthogonality property of harmonic functions the response $\mathbf{u}(\omega)$ can be computed individually from the equation of motion:

$$(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})\mathbf{u}(\omega) = \mathbf{f}(\omega).$$

(2.2)

The left-hand side can be written as a single frequency-dependent matrix as follows:

$$\mathbf{Z}(\omega)\mathbf{u}(\omega) = \mathbf{f}(\omega),$$

(2.3)

where $\mathbf{Z}(\omega)$ is the dynamic stiffness matrix (often referred to as mechanical impedance or apparent mass). By inverting the mechanical impedance an admittance notation can be obtained:

$$\mathbf{u}(\omega) = \mathbf{Y}(\omega)\mathbf{f}(\omega) \quad \text{where} \quad \mathbf{Y}(\omega) = (\mathbf{Z}(\omega))^{-1} = (-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K})^{-1}.$$

(2.4)

The $\mathbf{Y}(\omega)$ denotes the Frequency Response Function (FRF) matrix and is often referred to as admittance or dynamic flexibility.

Based on the quantity used to described the motion $u(\omega)$, various denotations can be used. Table 2.1 lists the most commonly used terminology for both admittance $\mathbf{Y}$ and impedance $\mathbf{Z}$.

---

1. Assumptions are valid for all dynamic systems in this thesis.
2. The letter $\mathbf{H}$ is also used to denote an FRF matrix; however, the denotation $\mathbf{Y}$ is commonly used within frequency based substructuring.
Table 2.1: Typical notation of impedance and admittance FRFs.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Impedance</th>
<th>Z</th>
<th>Admittance</th>
<th>Y</th>
</tr>
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<tbody>
<tr>
<td>Displacement</td>
<td>Dynamic stiffness</td>
<td>$f/u$</td>
<td>Receptance</td>
<td>$u/f$</td>
</tr>
<tr>
<td>Velocity</td>
<td>Mechanical impedance</td>
<td>$f/\dot{u}$</td>
<td>Mobility</td>
<td>$\dot{u}/f$</td>
</tr>
<tr>
<td>Acceleration</td>
<td>Apparent mass</td>
<td>$f/\ddot{u}$</td>
<td>Accelerance</td>
<td>$\ddot{u}/f$</td>
</tr>
</tbody>
</table>

2.2 Interface conditions

Consider a simple assembly of two substructures A and B, connected through a shared interface as depicted in Fig. 2.1. The interface DoF are denoted with $u_2$, and the internal DoFs for the A and B substructures are denoted with $u_1$ and $u_3$. First a definition of the whole set of DoFs $u(\omega)$, applied forces $f(\omega)$ and interface forces $g(\omega)$:

$$
\mathbf{u} \triangleq \begin{bmatrix} u_1^A \\ u_2^A \\ u_2^B \\ u_3 \end{bmatrix} ; \quad \mathbf{f} \triangleq \begin{bmatrix} f_1^A \\ f_2^A \\ f_2^B \\ f_3^A \end{bmatrix} ; \quad \mathbf{g} \triangleq \begin{bmatrix} 0 \\ g_2^A \\ g_2^B \\ 0 \end{bmatrix} ; \quad \mathbf{u}, \mathbf{f}, \mathbf{g} \in \mathbb{R}^n .
$$

(2.5)

In order to couple the two substructures, the coordinate compatibility and the force equilibrium have to be satisfied on the interface DoFs.

Figure 2.1: An assembly of two substructures A and B: a) assembled configuration; b) disassembled configuration.

2.2.1 Coordinate compatibility

The coordinate compatibility states that each collocated interface DoF must have the same value and direction on both substructures:

$$
u_2^A = u_2^B .
$$

(2.6)

By introducing a signed Boolean matrix on a full set of DoFs, a systematic notation can be introduced as follows [11]:

$$
\mathbf{B} \mathbf{u} = 0 \quad \rightarrow \quad \mathbf{u}^A - \mathbf{u}^B = 0 \quad \text{where} \quad \mathbf{B} \triangleq \begin{bmatrix} 0 & -\mathbf{I} & \mathbf{I} & 0 \end{bmatrix} .
$$

(2.7)

An explicit dependency on frequency is omitted in order to simplify the notation.
By discarding a non-unique DoFs at the interface after the assembly, a set of generalised coordinates $\mathbf{q}$ can be defined. The following mapping can be used between the two sets of DoFs:

$$
\mathbf{u} = \mathbf{Lq} \quad \rightarrow \quad \begin{cases}
\mathbf{u}_1^A = \mathbf{q}_1 \\
\mathbf{u}_2^A = \mathbf{q}_2 \\
\mathbf{u}_2^B = \mathbf{q}_2 \\
\mathbf{u}_3^B = \mathbf{q}_3 
\end{cases}
$$

where $\mathbf{L} \triangleq \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}$.

The $\mathbf{L}$ denotes the Boolean localisation matrix, which maps the physical DoFs of all substructures $\mathbf{u}$ to generalised DoFs $\mathbf{q}$. A useful relation can be derived between the signed Boolean matrix and the Boolean localisation matrix:

$$
\mathbf{Bu} = \mathbf{BLq} = 0; \quad \forall \mathbf{q} \quad \rightarrow \quad \begin{cases}
\mathbf{L} = \text{null} (\mathbf{B}) \\
\mathbf{B}^T = \text{null} (\mathbf{L}^T)
\end{cases}.
$$

By definition, the matrices $\mathbf{B}$ and $\mathbf{L}$ are each other’s nullspace. Therefore, one can be derived based on Eq. (2.9) by knowing the other.

### 2.2.2 Force equilibrium

The second condition at the interface is the force equilibrium on the collocated interface DoFs. The interface forces have to be equal, but opposite in the direction:

$$
\mathbf{g}_2^A = -\mathbf{g}_2^B.
$$

By equating the sum of forces on the interface with zero and pre-multiplying the interface forces with the transpose of the Boolean localisation matrix $\mathbf{L}$, the following relation can be acquired:

$$
\mathbf{L}^T \mathbf{g} = 0 \quad \rightarrow \quad \begin{cases}
\mathbf{g}_1^A = 0 \\
\mathbf{g}_2^A + \mathbf{g}_2^B = 0 \\
\mathbf{g}_3^B = 0
\end{cases},
$$

as the interface forces are only present at the interface the first and the last line are redundant. However, they are required to maintain the same systematic notation.

In order to write the force equilibrium with the signed Boolean matrix, a set of Lagrange multipliers $\lambda$ is introduced:

$$
\mathbf{g} = -\mathbf{B}^T \lambda \quad \rightarrow \quad \begin{cases}
\mathbf{g}_1^A = 0 \\
\mathbf{g}_2^A = \lambda \\
\mathbf{g}_2^B = -\lambda \\
\mathbf{g}_3^B = 0
\end{cases}.
$$

Similarly as with the coordinate compatibility the nullspace property is also valid for the force equilibrium:

$$
\mathbf{L}^T \mathbf{g} = -\mathbf{L}^T \mathbf{B}^T \lambda = 0; \quad \forall \mathbf{q}.
$$
2.3 Substructure coupling

By applying the interface conditions, two dynamic systems can be assembled. Consider two simple substructures, sharing one interface, as schematically depicted in Fig. 2.2.

![Figure 2.2: Coupling of substructures A and B (i.e. $A \oplus B = AB$).](image)

The subsystem admittance for $A$ can be defined as follows:

$$Y^A \triangleq \begin{bmatrix} Y_{11}^A & Y_{12}^A \\ Y_{21}^A & Y_{22}^A \end{bmatrix}; \quad u^A, f^A \in \mathbb{R}^{n^A}, \tag{2.14}$$

the same can be written for the subsystem admittance of $B$:

$$Y^B \triangleq \begin{bmatrix} Y_{11}^B & Y_{12}^B \\ Y_{21}^B & Y_{22}^B \end{bmatrix}; \quad u^B, f^B \in \mathbb{R}^{n^B}. \tag{2.15}$$

The subsystem admittances can be arranged in a block-diagonal form:

$$Y^{A\mid B} = \begin{bmatrix} Y_{11}^A & Y_{12}^A & 0 & 0 \\ Y_{21}^A & Y_{22}^A & 0 & 0 \\ 0 & 0 & Y_{22}^B & Y_{23}^B \\ 0 & 0 & Y_{32}^B & Y_{33}^B \end{bmatrix}. \tag{2.16}$$

The equations of motion for the uncoupled system can therefore be written as follows:

$$u = Y^{A\mid B}(f + g). \tag{2.17}$$

Since the interface forces $g$ are unknown, the substructures remain uncoupled. By applying the interface conditions the substructures can finally be assembled.

The equations of motion for the uncoupled system could also be defined also with the subsystem impedance:

$$Z^{A\mid B} u = f + g. \tag{2.18}$$

Based on the chosen definition, the interface problem can either be solved by primal assembly, which involves unique interface displacements, or by a dual assembly, which involves introducing interface forces satisfying the interface equilibrium.

Both primal and dual assembly are mathematically equivalent. However, the interpretation of both forms from a mechanical standpoint is different and can provide additional information. In frequency based substructuring the dual assembly is most commonly used. In dual assembly, both DoF and interface forces are defined separately per substructure. In this section only the derivation of dual assembly is provided. For derivation of primal assembly, the interested reader is referred to [9,46].
2.3.1 Lagrange-multiplier FBS

The most popular approach in frequency based substructuring is the Lagrange-multiplier FBS (LM-FBS) [11]. The method allows for a seamless assembly of multiple substructures. It is, in fact, a dual assembly, in which an admittance notation is used to define the subsystem dynamics. The interface problem is solved with dual assembly by introducing Lagrange multipliers as the interface force amplitudes (Eq. (2.12)) [11]:

\[
\begin{align*}
\mathbf{u} &= \mathbf{Y}^{\mathcal{A}\mathcal{B}} (\mathbf{f} - \mathbf{B}^T \lambda) \\
\mathbf{B} \mathbf{u} &= 0
\end{align*}
\] (2.19)

The first line can be substituted into the compatibility condition and solved for \( \lambda \):

\[
\begin{align*}
\mathbf{B} \mathbf{l} (\mathbf{f} - \mathbf{B}^T \lambda) &= 0 \\
\mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{f} &= \mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{B}^T \lambda \\
\lambda &= (\mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{B}^T)^{-1} \mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{f}.
\end{align*}
\] (2.20)

The final coupled response can be obtained by substituting the derived Lagrange multipliers back into the Eq. (2.19) [11]:

\[
\begin{align*}
\mathbf{u} &= \mathbf{Y}^{\mathcal{A}\mathcal{B}} (\mathbf{f} - \mathbf{B}^T \lambda) \\
\mathbf{u} &= \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{f} - \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{B}^T (\mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{B}^T)^{-1} \mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{f}.
\end{align*}
\] (2.21)

The equation can be rewritten by collecting the admittance terms; in this way, and a dually assembled FRF matrix of the coupled response is obtained:

\[
\begin{align*}
\hat{\mathbf{u}} &= \hat{\mathbf{Y}}^{\mathcal{A}\mathcal{B}} \mathbf{f} \\
\hat{\mathbf{Y}}^{\mathcal{A}\mathcal{B}} &\triangleq \left[ \mathbf{I} - \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{B}^T (\mathbf{B} \mathbf{Y}^{\mathcal{A}\mathcal{B}} \mathbf{B}^T)^{-1} \mathbf{B} \right] \mathbf{Y}^{\mathcal{A}\mathcal{B}}.
\end{align*}
\] (2.22)

The DoFs are defined separately per substructure in the dual formulation. Therefore, the FRFs at the interface are identical for both substructures and are in fact redundant. Unique FRFs can be manually extracted, or - by utilizing the property of Boolean localisation matrix - a primal FRF matrix can be obtained with a set of unique coordinates \( \mathbf{q} \) and \( \mathbf{p} \):

\[
\begin{align*}
\mathbf{q} &= \mathbf{L}^+ \mathbf{u} \\
\mathbf{p} &= \mathbf{L}^+ \mathbf{f} \\
\rightarrow \hat{\mathbf{Y}}^{\mathcal{A}\mathcal{B}} &= \mathbf{L}^+ \hat{\mathbf{Y}}^{\mathcal{A}\mathcal{B}} (\mathbf{L}^T)^+.
\end{align*}
\] (2.23)

2.3.2 Application of LM-FBS

Consider the application of LM-FBS to assembling an arbitrary number of substructure \( N_s \). In practice the whole procedure can be separated into four distinct steps:

1. **Admittance matrix \( \mathbf{Y}^{(s)} \) for each substructure.**
   Depending on the type of the interface in the assembly, care should be taken that a sufficient number of interface DoFs \( \mathbf{u} \) is obtained. Both the observability (responses \( \mathbf{u}_2 \)) and the controllability (forces \( \mathbf{f}_2 \)) of the interface have to be satisfied for a successful assembly.
2. **Global set of DoFs:**
A global set of DoFs has to be defined uniquely for all substructure. A global set of coordinates \( \mathbf{u} \) and forces \( \mathbf{f} \) in FRF matrices \( \mathbf{Y} \) has to be defined for each separate substructure:

\[
\mathbf{u} = \begin{bmatrix} \mathbf{u}^{(1)} & \ldots & \mathbf{u}^{(N_s)} \end{bmatrix}^T, \quad (2.24)
\]

\[
\mathbf{f} = \begin{bmatrix} \mathbf{f}^{(1)} & \ldots & \mathbf{f}^{(N_s)} \end{bmatrix}^T, \quad (2.25)
\]

\[
\mathbf{Y} = \text{diag}(\mathbf{Y}^{(1)}, \ldots, \mathbf{Y}^{(N_s)}). \quad (2.26)
\]

3. **Subsystem connectivity:**
The connectivity between substructures has to be defined through a signed Boolean matrix \( \mathbf{B} \).

4. **Apply LM-FBS:**
Finally, the assembled response is obtained by applying the Eq. (2.22).

The current derivation of LM-FBS was shown for full-admittance matrices, in which all response locations \( \mathbf{u} \) are also aligned with all excitation locations \( \mathbf{f} \). However, with experimental testing, certain response DoFs can only be used as a reference, without the need for excitation. Therefore, the size of the FRFs matrix is equal to \( n_c \times n_e \), where \( n_c \) is the number of response locations and \( n_e \) the number of excitation locations. In order to assemble such a system, signed Boolean matrices can be defined separately for compatibility and equilibrium conditions:

\[
\begin{cases}
\mathbf{B}_c \mathbf{u} = 0 \\
\mathbf{g} = -\mathbf{B}_c^T \mathbf{\lambda}
\end{cases} \quad (2.27)
\]

The LM-FBS can then be rewritten as follows (Eq. (2.22)) [11]:

\[
\mathbf{u} = \mathbf{\tilde{Y}}^{AB} \mathbf{f}; \quad \mathbf{\tilde{Y}} \triangleq \left[ \mathbf{I} - \mathbf{YB}_c^T (\mathbf{B}_c \mathbf{YB}_c^T)^{-1} \mathbf{B}_c \right] \mathbf{Y}. \quad (2.28)
\]

The quality of the final assembled response is highly dependent on the conditioning of the interface flexibility matrix \( \mathbf{Y}_{\text{int}} = \mathbf{B}_c \mathbf{YB}_c^T \). For low conditioning (i.e. a high condition number) of the interface flexibility matrix even the slightest error will result in erroneous assembly.

### 2.4 Substructure decoupling

An opposite procedure of assembling substructures is the process of disassembling substructures, i.e. the decoupling of substructures. In decoupling, the dynamic behaviour of a substructure can be identified through the dynamic characteristics of the entire system and the remaining substructures. Consider the decoupling example depicted in Fig. 2.3.

---

4 Even with different sizes of Boolean matrices for compatibility and equilibrium conditions, the DoFs at the interface have to be collocated.
Figure 2.3: Decoupling of substructure B from the assembly AB (i.e. \( AB \ominus B = A \)).

The influence of substructure B has to be removed from the assembled system AB. Therefore, interface forces which ensure the coordinate compatibility and which the B from AB have to identified.

The equations of motions can be written for the assembled system AB:

\[
\begin{align*}
\mathbf{u}^{\text{AB}} &= \mathbf{Y}^{\text{AB}} \mathbf{f}^{\text{AB}} + \mathbf{Y}^{\text{AB}} \mathbf{g}^{\text{AB}}, \\
\end{align*}
\]

(2.29)

and also for substructure B, where a negative sign is added to interface forces:

\[
\begin{align*}
\mathbf{u}^{\text{B}} &= -\mathbf{Y}^{\text{B}} \mathbf{g}^{\text{B}}, \\
\end{align*}
\]

(2.30)

The equations of motion of the assembled system AB and substructure B can be written with one equation, following the LM-FBS notation:

\[
\begin{bmatrix}
\mathbf{u}^{\text{AB}} \\
\mathbf{u}^{\text{A}} \\
\mathbf{u}^{\text{B}} \\
\mathbf{u}^{\text{1}} \\
\mathbf{u}^{\text{2}} \\
\end{bmatrix}
= \begin{bmatrix}
\mathbf{Y}^{\text{A}}_{11} & \mathbf{Y}^{\text{A}}_{12} & \mathbf{Y}^{\text{A}}_{13} & 0 & 0 \\
\mathbf{Y}^{\text{A}}_{21} & \mathbf{Y}^{\text{A}}_{22} & \mathbf{Y}^{\text{A}}_{23} & 0 & 0 \\
\mathbf{Y}^{\text{A}}_{31} & \mathbf{Y}^{\text{A}}_{32} & \mathbf{Y}^{\text{A}}_{33} & 0 & 0 \\
0 & 0 & 0 & -\mathbf{Y}^{\text{B}}_{22} & -\mathbf{Y}^{\text{B}}_{23} \\
0 & 0 & 0 & -\mathbf{Y}^{\text{B}}_{32} & -\mathbf{Y}^{\text{B}}_{33} \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{0} \\
\mathbf{f}^{\text{AB}} \\
\mathbf{g}^{\text{AB}} \\
\mathbf{0} \\
\mathbf{g}^{\text{B}} \\
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{0} \\
\mathbf{0} \\
\mathbf{0} \\
\mathbf{g}^{\text{B}} \\
\mathbf{0} \\
\end{bmatrix}
\]

(2.31)

It can be observed that the equation is the same here as for the dual assembly in the coupling example shown in the previous section. Therefore, the LM-FBS (Eq. (2.22)) can be used to decouple substructures by simply adding a negative sign to the admittances of substructures that are to be decoupled.

Extending the interface to internal DoFs

In the coupling case, the application of coordinate compatibility and force equilibrium is limited to the interface DoFs. In the decoupling case, the interface conditions can be extended to the internal DoFs, beyond the physical interface. Consider a decoupling application depicted in Figure 2.3, where the size of the interface flexibility matrix is equal to \( n_c \times n_c \) (the number of compatibility and equilibrium conditions). A number of interface DoFs are denoted with \( n_2 \) and a number of internal DoFs are denoted with \( n_1 \). The most common decoupling approaches described in [9, 47–50] are listed below:

Standard interface \([n_c = n_c = n_2]\):

The interface conditions are enforced only at the interface DoFs. With this, the interface flexibility matrix has the same dimensions as it would have for a coupling example.
Extended interface \([n_c = n_e = n_2 + n_1]\):

The interface conditions are extended to include additional internal DoFs. By adding the compatibility and equilibrium at \(n_1\) internal DoFs, the observability and controllability of the interface dynamics is increased. The interface flexibility matrix remains square, but with rank \(n_2\). Therefore, in the presence of noise, the matrix is badly conditioned and a truncated inversion has to be performed.

Non-collocated overdetermined \([n_c = n_2 + n_1, n_e = n_2]\):

Only the compatibility condition is extended to the internal DoFs. The linear problem \(Y_{\text{int}}\lambda = u_{\text{int}}\) is overdetermined. The solution is found in a least-square sense by minimizing the norm of compatibility error \(||B_u u = 0|||\).

Non-collocated underdetermined \([n_c = n_2, n_e = n_1 + n_2]\):

Only the equilibrium condition is extended to the internal DoFs. The linear problem \(Y_{\text{int}}\lambda = u_{\text{int}}\) is underdetermined. The solution is found by obtaining a minimum quadratic set of \(\lambda\), which satisfy the compatibility \(B_u u = 0\).

Non-collocated internal \([n_c = n_2, n_e = n_1]\) or \([n_c = n_1, n_e = n_2]\):

Only compatibility or equilibrium condition is imposed at the interface DoFs. The observability and controllability of the interface are guaranteed through internal DoFs. By choosing a higher number of internal than interface DoFs (\(n_1 > n_2\)) the linear problem \(Y_{\text{int}}\lambda = u_{\text{int}}\) can be underdetermined or overdetermined. Respectively, a solution of the linear problems is achieved by a constrained optimization or a least-square optimization.

2.5 Weakening the interface problem

The consistency of the final assembled or disassembled response from the LM-FBS is directly related to the conditioning of the interface flexibility matrix \(Y_{\text{int}}\), as already stated in the previous section. It is common that experimentally identified admittances contain measurement errors. Measurement errors, such as random noise from transducers, small deviations in transducer sensitivity, bias in excitation location, or bias in measurement location, are always present when performing experimental measurements.

With standard LM-FBS formulation, the compatibility and equilibrium conditions are strongly imposed on all matching interface DoFs. Therefore, the final response will always be affected by even the smallest measurement error. Additionally, relatively small errors can be dramatically amplified due to the required inverse of the interface flexibility matrix.

The compatibility and equilibrium conditions can be weakened by enforcing the interface conditions in a reduced space\[5\]. Consider a compatibility condition for a standard

\[5\] For further information on the weakening the interface problem, the interested reader is referred to Section 4.3.4 in [46].
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coupling example:

\[ u^A_2 - u^B_2 = 0. \]  \hfill (2.32)

Next, assume a good representation space \( R_{\Gamma^{AB}} \) for the response in the assembled system at the interface \( \Gamma^{AB} \). The reduction matrix \( R_{\Gamma^{AB}} \) consists of vectors which can approximate the response in the final assembled configuration. One of the most common reductions used in FBS is the assumption of a locally rigid behaviour around the interface.\(^6\) The response of each separate substructure can be expressed with the pre-assumed reduction space as follows \[46\]:

\[ u^A_2 \approx R_{\Gamma^{AB}} \beta^A \quad \text{and} \quad u^B_2 \approx R_{\Gamma^{AB}} \beta^B. \]  \hfill (2.33)

Where the \( \beta^A \) and \( \beta^B \) are the reduced coordinates in the subspace of \( R_{\Gamma^{AB}} \). The definition of the compatibility condition can now be written as follows:

\[ R_{\Gamma^{AB}} \beta^A - R_{\Gamma^{AB}} \beta^B = 0, \]  \hfill (2.34)

or, if the reduction basis \( R_{\Gamma^{AB}} \) is full rank, directly as:

\[ \beta^A - \beta^B = 0. \]  \hfill (2.35)

With this, the compatibility condition is enforced only on the unfiltered response from the subspace of \( R_{\Gamma^{AB}} \). The rest of the response at the interface is left uncoupled. For example if the reduction space includes only the locally rigid motion, the flexible motion of the interface is not included in the assembled configuration.

Similar to the weakening of the compatibility condition also the equilibrium condition can be weakened through an appropriate reduction basis. By including the weakened compatibility and equilibrium conditions the LM-FBS can be rewritten as follows \[46\]:

\[ u = Y_{\text{weak}} f; \quad Y_{\text{weak}} \triangleq \left[ I - Y B^T R_{\Gamma^{AB}} (R_{\Gamma^{AB}}^T Y B Y^T B R_{\Gamma^{AB}})^{-1} R_{\Gamma^{AB}} B \right] Y. \]  \hfill (2.36)

Here an explicit dependency on frequency was omitted from the derivation; however, if required the weakening of both equilibrium and compatibility condition can be frequency dependent.

\[ 2.6 \quad \text{Regularization of the interface flexibility matrix} \]

The consistency of coupling and decoupling applications is directly related to the conditioning of the interface flexibility matrix \( Y_{\text{int}} \). The condition number of the matrix can be reduced by applying a truncated singular value transformation. The interface

---

\(^6\) The reduction based on an assumption of local rigidity is called a virtual point transformation and it is thoroughly discussed in section 4.1.
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The flexibility matrix can be decomposed in terms of singular values and singular vectors with the SVD (Singular Value Decomposition) as follows [51,52]:

\[ B_u Y B_f = U \Sigma V^H, \]

(2.37)

where \( U \) are the left singular vectors, \( V \) the right singular vectors and \( \Sigma \) the respective singular values. The left- and right singular vector can be used as a reduction basis and the LM-FBS equation can be written as follows [40]:

\[
\begin{align*}
\mathbf{u} &= [\mathbf{I} - Y \mathbf{B}_f^T \mathbf{V}(\mathbf{U}^H \mathbf{U} \Sigma \mathbf{V}^H \mathbf{V})^{-1} \mathbf{U}^H \mathbf{B}_u] \mathbf{Y}_{\mathbf{f}} = \\
&= [\mathbf{I} - Y \mathbf{B}_f^T \mathbf{V}(\Sigma)^{-1} \mathbf{U}^H \mathbf{B}_u] \mathbf{Y}_{\mathbf{f}}.
\end{align*}
\]

(2.38)

If a full set of singular vectors is used as a reduction basis, no reduction is applied and the inverse of the flexibility matrix remains the same.

A truncated basis of singular vectors can be used for a reduction basis by using only the \( r \) dominant singular components. By utilizing a reduced set of singular vectors \( \mathbf{U}_r \) and \( \mathbf{V}_r \), the LM-FBS formulation can be written as:

\[
\begin{align*}
\mathbf{u} &= [\mathbf{I} - Y \mathbf{B}_f^T \mathbf{V}_r (\mathbf{U}_r^H \mathbf{U}_r \Sigma_r \mathbf{V}_r^H \mathbf{V}_r)^{-1} \mathbf{U}_r^H \mathbf{B}_u] \mathbf{Y}_{\mathbf{f}} = \\
&= [\mathbf{I} - Y \mathbf{B}_f^T \mathbf{V}_r (\Sigma_r)^{-1} \mathbf{U}_r^H \mathbf{B}_u] \mathbf{Y}_{\mathbf{f}}.
\end{align*}
\]

(2.39)

The application of the truncated singular value decomposition \( \mathbf{U}_r^H \mathbf{U}_r \Sigma_r \mathbf{V}_r^H \mathbf{V}_r = \Sigma_r \) on the interface flexibility matrix can be regarded as a reduction of the interface dynamics in the subspace that is spanned by both left- and right singular vectors.
Introduction to Frequency Based Substructuring
3 Introduction of Transfer Path Analysis

Transfer Path Analysis (TPA) is an invaluable engineering tool for troubleshooting and characterizing noise and vibration sources in complex dynamic systems [9,14]. In modern products there can be a number of active components (such as an electric motor, a compressor, a ventilator, etc.) that are connected to the passive structures. The TPA enables a representation of active sources by forces and responses located directly at the interface with the passive side. Therefore, the source excitation can be separated from the transfer characteristics of the passive side and the most dominant path of vibro-acoustic transmission can be identified. When the dominant paths are identified, the NVH performance can be optimized by making changes either to the source or to the passive side.

In this chapter, the basic theory for transfer path analysis is provided. First a transfer path problem is defined 3.1. Next, a classification of various TPA methods is shown 3.2. Finally, the component-based TPA methods are introduced and advantages, as well as disadvantages of each method is outlined 3.3.

3.1 Transfer path problem

To define a transfer path problem, consider the simplified dynamic system AB depicted in Fig. 3.1. The system can be separated in two domains: an active subsystem (the source) and a passive subsystem (the receiver). An operational excitation is applied at node 1, the subsystems are connected at interface node 2, and response DoFs at locations of interest at node 3. The three distinct sets of nodes are used throughout the derivation of TPA methodologies [9,14]:

- $u_1$ - source: internal DoFs pertaining to the domain of the active component.
- $u_2$ - interface: DoFs located on the interface between the active and passive side.
- $u_3$ - receiver: response DoFs located on the passive component.

If a decomposition between the active and passive part can be achieved, the simplified notation can be used to model a wide range of assemblies. The operational excitation on the active component at node 1 is, in practice, unmeasurable. The excitation is transmitted through the interface to the passive side. The transfer path can be
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Figure 3.1: The transfer path problem: a) based on the admittance of assembled configuration AB [9]; b) based on the admittances of subsystems A and B.

defined either from the assembled configuration AB (section 3.1.1) or from the separate subsystems A and B (section 3.1.2).

3.1.1 Transfer path from assembled admittance

The transfer path in the assembled configuration (Fig. 3.1a) can be defined from the individual contributions. The response at the receiver $u_3$ under operational excitation $f_1$ can be obtained by combining each separate contribution as follows [14]:

$$ u_i(\omega) = \sum_j Y_{ij}^{AB}(\omega) f_j(\omega), \quad (3.1) $$

from matrix multiplication the addition can be written equivalently as:

$$ u_3(\omega) = Y_{31}^{AB}(\omega) f_1(\omega). \quad (3.2) $$

Only structure-borne transfer paths are considered in the presented formulation. However, Eq. (3.2) can be extended with airborne paths, if required. For more information on airborne transfer path, the interested reader is referred to [14].

3.1.2 Transfer path from subsystem admittance

The same transfer can also be derived from admittances of the separated subsystems $Y^A$ and $Y^B$ (Fig. 3.1b). The equations of motion for both separate subsystems are first assembled in a block-diagonal form:

$$ [u_1^A, u_2^B, u_3^B] = \begin{bmatrix} Y_{11}^A & Y_{12}^A & 0 & 0 \\ Y_{21}^A & Y_{22}^A & 0 & 0 \\ 0 & 0 & Y_{22}^B & Y_{23}^B \\ 0 & 0 & Y_{32}^B & Y_{33}^B \end{bmatrix} \begin{bmatrix} f_1 \\ 0 \\ g_A^2 \\ g_B^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_A^2 \\ g_B^2 \end{bmatrix}. \quad (3.3) $$

The operational excitation $f_1$ is expanded with interface forces $g_2$ on both subsystems. The system of equations resembles the dual assembly or LM-FBS notation as discussed in section 2.3.1.
The interface forces can be replaced with a set of Lagrange multipliers $\lambda$, and both the equilibrium and compatibility condition have to be satisfied on the interface:

\begin{align}
\text{Equilibrium cond.: } & g_A^2 = -g_B^2 = \lambda; \\
\text{Compatibility cond.: } & u_A^2 = u_B^2,
\end{align}

(3.4)

the interface forces can be obtained by imposing the compatibility condition on Eq. (3.3) and solving for $\lambda$:

\begin{align}
Y_{32}^A f_1 + Y_{32}^B g_A^2 = Y_{22}^B g_B^2 \\
-(Y_{22}^A + Y_{22}^B) \lambda &= Y_{21}^A f_1 \\
-\lambda &= (Y_{22}^A + Y_{22}^B)^{-1} Y_{21}^A f_1 = g_B^2.
\end{align}

(3.5)

The response on the receiver side $u_3^B$ can be found by substituting Eq. (3.5) back into the last line of Eq. (3.3):

\begin{equation}
\begin{aligned}
 u_3^B &= Y_{32}^B g_2^B = \left[ Y_{32}^B (Y_{22}^A + Y_{22}^B)^{-1} Y_{21}^A \right] f_1. \\
 &= Y_{31}^{AB} f_1.
\end{aligned}
\end{equation}

(3.6)

By comparing Eq. (3.6) with Eq. (3.2), it can be observed that the admittance of the assembly $Y_{31}^{AB}$ can be obtained by coupling the subsytems admittances, as was shown with the dual assembly in LM-FBS.

### 3.2 Classification of TPA methods

The final response with the TPA is built from a certain description of a response measured at the interface $u_2$ and from an appropriate set of transfer functions relating to the response on the passive side $Y_{31}^{AB}$. Regardless of the TPA method used, the workflow can be divided into four distinct steps:

1. **Operational test:**
   First operational measurements have to be performed on the active components at various operational conditions of interest.

2. **FRF measurement:**
   Transfer characteristics of the passive subsystem have to be identified; most commonly, experimental FRF measurements are performed.

3. **Interface loads:**
   The operational interface loads at various operational conditions are determined based on the selected TPA method.

4. **Path contributions:**
   Finally, the path contributions can be identified, and the most dominant transfer paths can be evaluated.

Based on the used description of the transmission of vibration and the identification of interface forces three different classification of the TPA methods can be distinguished. Only a brief introduction into each family of TPA methods is provided below. For a detailed description, the interested reader is referred to [9,14].
3.2.1 Classical TPA

A family of classical TPA methods is intended to identify the critical transfer paths in existing products. Nowadays, it is already a standard practice to use classical TPA methods for troubleshooting NVH problems. First, operational measurements are performed on the assembled configuration AB to obtain interface forces between the source and the receiver. The receiver response is obtained by applying the passive-side interface forces directly on the interface of subsystem B:

\[ u_3 = Y_{32}^B g_2^B. \]  

Identified interface forces \( g_2^B \) are a characteristic of the assembled dynamics. Therefore, for even a small subsystem modification, a new operational test is required. Based on the identification of interface forces, the classical TPA methods can be divided as follows [14]:

**Direct force:** the interface forces are measured with force transducers mounted directly between the source and the receiver [53].

**Mount stiffness:** the interface forces are identified by placing resilient mounts (with known mechanical properties) between the source and the receiver [15].

**Matrix inverse:** the interface forces are obtained through a matrix-inverse of subsystem FRFs located in the proximity of the interface [16].

3.2.2 Component-based TPA

With component-based TPA methods the source excitation is characterized by a set of equivalent forces, which are an inherent property of the active component. This property is highly beneficial because iterative changes can be made on the passive side without the need to repeat source characterization. Furthermore, the active component can be evaluated on a separate test bench, where various operational measurements can be performed. The equivalent forces are defined in such a way that applying them to the assembled system with the source deactivated yields the same response as when the source is active:

\[ u_3 = Y_{32}^{AB} f_2^{eq}. \]  

The potential benefits of component-based TPA lead to the development of various techniques, which were mostly derived independently from one another. Based on the approach used to identify equivalent forces the methods can be presented as follows [14]:

**Blocked force:** equivalent forces are identified by rigidly fixing the source at the interface [54,55].

**Free velocity:** equivalent forces are identified by leaving the source interface completely free [56,57].

**Hybrid interface:** equivalent forces are identified by attaching the source to the test-bench, where the interface is anywhere between being rigidly fixed and left free [58].
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**In-situ**: equivalent forces are identified directly from an arbitrary assembly \[59,60\].

**Pseudo-forces**: concept similar to the in-situ TPA, where the equivalent forces are identified from an assembled configuration; however, unlike the equivalent forces from the in-situ TPA, the pseudo-forces are not limited to the interface domain \[61–63\].

### 3.2.3 Transmissibility-based TPA

The last set of TPA methods are transmissibility-based and are conceptually different from classical and component TPA. The first two families try to model the transmission in a physically correct way by identifying as many forces and moments as are required to define the source excitation in full. The transmissibility-based TPA does not characterize interface forces, but it only determines the transmissibility between sensors directly from the operational measurements. This can be highly beneficial, since only operational measurements are required to evaluate contribution from multiple sources. Two different methods can be distinguished when evaluating the transmissibility \[14\]:

**Operational transfer path analysis (OTPA)**: the transmissibility matrix is estimated statistically by correlating the response around the interface with the response on the passive side \[17,64,65\].

**Operational mount identification (OPAX)**: can be regarded as a hybrid TPA method. The classical mount-stiffness and matrix-inverse TPA principles are extended with the possibility to estimate mount stiffness parameters from operational tests \[18\].

### 3.3 Overview of component-based TPA

The primary advantage of component-based TPA methods is that the identified forces are a characteristic of the source alone and not of the assembled dynamics. With this characteristic the identified forces can easily be transferred to an assembly with a different receiving side. Various subsystem modifications can be tested and evaluated with the same identified forces.

This is not the case for classical TPA methods, where the identified interface forces are a characteristic of the assembled dynamics. Therefore, if any part of the assembly is changed, all of the operational measurements have to be repeated. This can become an elaborate process, as many design changes can be expected during the development of complex systems.

The main objective of component-based TPA is to identify a set of equivalent forces \(f_{eq}^{2}\) (also known as blocked forces), which are a property of the active component alone. The response at the receiving side can be estimated by applying the identified forces to the assembled system (with the active part turned off). Therefore, the source characterization can be performed on a separate test setup, which may be more suited to perform a range of operational measurements.
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Definition of equivalent forces

Equivalent forces $f^q_2$ are defined in a way that when they are applied to the interface of the assembled system $AB$, they yield the correct response in $u_3$ (the same response as when the active part is operating). The response $u_3$ can be formulated with the assembly admittances $Y^{AB}$ or with the subsystem admittances $Y^A$ and $Y^B$ as follows:

$$u_3 = Y^{AB} f^q_2 = \left[ Y^B_{32} (Y^A_{22} + Y^B_{22})^{-1} Y^A_{22} \right] f^q_2. \quad (3.9)$$

The response $u_3$ can also be defined when the active part is running and the assembly is excited with operational forces $f_1$ (Eq. (3.6)). By comparing this with Eq. (3.9) the equivalent forces can be defined as [14]:

$$\left[ Y^B_{32} (Y^A_{22} + Y^B_{22})^{-1} Y^A_{22} \right] f^q_2 = \left[ Y^B_{32} (Y^A_{22} + Y^B_{22})^{-1} Y^A_{21} \right] f_1$$

$$f^q_2 = (Y^A_{22})^{-1} Y^A_{21} f_1. \quad (3.10)$$

It can be observed that the defined equivalent forces are in fact only a property of the active component $A$. This property of equivalent forces is exploited in each component-based TPA method and it also offers a great practical applicability.

The main limitation associated with equivalent forces is that they only properly represent the operational response at the receiving part and at the interface. The obtained response at the source is different by definition and it cannot be used. The main reason behind this is the fact that the response on the receiver side $B$ is only excited by forces on the interface. However, the response at the source is a result of both directly operational excitation $f_1$ and its reflection through the coupled receiver side $B$.

From a practical standpoint there are various ways of how the equivalent forces can be determined from operational measurements. The following sections briefly summarize the most commonly used methods [14].

### 3.3.1 Blocked force

The most commonly known variant of component-based TPA is the blocked-force TPA.\(^1\) Consider the source subsystem $A$, which is rigidly fixed at the interface, as

---

\(^1\)That is also the reason why equivalent forces are sometimes referred to as blocked forces.
If the boundary is infinitely stiff then there is no response at the interface $u_2 = 0$ and the following relation can be written:

$$
\begin{bmatrix}
  u_1 \\
  u_2 = 0
\end{bmatrix} =
\begin{bmatrix}
  Y^A_{11} & Y^A_{12} \\
  Y^A_{21} & Y^A_{22}
\end{bmatrix}
\begin{bmatrix}
  f_1 \\
  g^A_2 = -g^bl_2
\end{bmatrix}.
$$

(3.11)

Following the definition of equivalent forces (Eq. (3.10)), it can be observed that they are equal to the blocked forces $g^bl_2$.

$$
g^bl_2 = f^eq_2 = (Y^A_{22})^{-1}Y^A_{21}f_1.
$$

(3.12)

The main limitation of the block-force method is the assumption of an infinitely stiff boundary. In practice, achieving this condition in all directions, throughout the whole frequency range is almost impossible. If the rigidness is not satisfied at the interface the accuracy of estimated forces degrades.

In practice the blocked-force is expected to perform the best at a low frequency range, where the assumption of rigidness can be satisfied and where the rotational DoFs are less dominant when compared to the translational DoFs [54,55,66]. An additional limitation is the measurement of moment at the interface, since 6-DoF forces transducers are not commonly available.

Figure 3.3: Schematic representation of blocked force TPA [14].

### 3.3.2 Free velocity

With the free velocity TPA method the operational measurements are performed with the interfaces left free, as depicted in Fig. 3.4.

It is, in fact, an exact counterpart to the blocked-force method, as now the interface is completely free $u_2^{\text{free}}$.

$$
\begin{bmatrix}
  u_1 \\
  u_2 = u_2^{\text{free}}
\end{bmatrix} =
\begin{bmatrix}
  Y^A_{11} & Y^A_{12} \\
  Y^A_{21} & Y^A_{22}
\end{bmatrix}
\begin{bmatrix}
  f_1 \\
  g^A_2 = 0
\end{bmatrix}.
$$

(3.13)

The following relation can be derived from free displacements at the interface:

$$
\begin{align*}
  u_2^{\text{free}} &= Y^A_{21}f_1; \\
  f^eq_2 &= (Y^A_{22})^{-1}u_2^{\text{free}}.
\end{align*}
$$

(3.14)

It can be observed that the equivalent forces are indeed a property of the source alone.

---

2The name free-velocity (and not free-displacements) originates in acoustical engineering, where velocity is commonly referred to in combination with acoustic pressure.
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In theory the free boundary condition can be satisfied rather quickly. However, with experimental testing obtaining free-free conditions is not straightforward. One can commonly expect problems at low-frequency range due to the effects from the ropes, rubbers or foam on which the measurements are performed [56,57]. Additionally, the active components often needs to be connected to a certain load or mount for operating. There is also no friction and damping present at the interface during the operational measurements. These are always present in the assembled configuration. Given all the disadvantages, the practical applicability of the method is limited, especially at the lower frequency range.

![Figure 3.4: Schematic representation of free velocity TPA [14].](image)

### 3.3.3 Hybrid interface

The hybrid interface TPA method tries to leverage the blocked-force and the free-velocity method by accounting for the dynamics of the support structure [58]. Consider the source subsystem A connected to the testbench R as depicted in Fig. 3.5. The following system of equations can be written:

\[
\begin{bmatrix}
u_A^1 \\
u_A^2 \\
u_R^2
\end{bmatrix} =
\begin{bmatrix}
Y_{11}^A & Y_{12}^A & 0 \\
Y_{21}^A & Y_{22}^A & 0 \\
0 & 0 & Y_{22}^R
\end{bmatrix}
\begin{bmatrix}
f_1^A \\
g_A^2 \\
g_R^2
\end{bmatrix},
\]

(3.15)

where \(Y_{22}^R\) denotes the interface admittance of the support structure. By enforcing compatibility (\(u_A^2 = u_R^2\)) and equilibrium (\(g_A^2 = -g_R^2\)) on the interface, the interface forces \(g_R^2\) can be derived as:

\[
g_R^2 = (Y_{22}^A + Y_{22}^R)^{-1}Y_{21}^Af_1.
\]

(3.16)

By placing the interface forces back into the Eq. (3.15) the interface displacements \(u_A^2\) can be obtained:

\[
u_A^2 = \left[I - Y_{22}^A(Y_{22}^A + Y_{22}^R)^{-1}\right]Y_{21}^Af_1.
\]

(3.17)

By combining Eqs. (3.16) and (3.17) equivalent forces can be obtained in a way that eliminates the dynamics of the testbench \(Y_{22}^R\):

\[
f_2^{eq} = (Y_{22}^A)^{-1}Y_{21}^Af_1 = g_R^2 + (Y_{22}^A)^{-1}u_A^2.
\]

(3.18)

It can be observed that the derived Eq. (3.18) can in fact represent both the blocked force (\(Y^R = 0\), Eq. (3.12)) and the free velocity (\(Y^R = \infty\), Eq. (3.14)) [56].
The hybrid interface method was first proposed as a non-rigid test bench compensation for the blocked force method [55]. The method is, nonetheless, generally applicable. However, it should be noted that the method is, in practice, time-consuming, as the interface forces $g^R_2$ need to be measured.

The interface forces can be substituted by measuring the subsystem admittance measurements $Y^R_{22}$:

$$f_{eq}^2 = (Y^R_{22})^{-1}u^A_2 + (Y^A_{22})^{-1}u^A_2.$$  \hspace{1cm} (3.19)

### 3.3.4 In-situ

The in-situ TPA method tries to characterize the equivalent forces directly from the original assembled configuration. Consider Eq. (3.19) where the inverted admittance FRFs can be added, as they represent the dynamic stiffness:

$$f_{eq}^2 = (Z^R_{22} + Z^A_{22})u^A_2 = Z^{AR}_{22}u^A_2.$$ \hspace{1cm} (3.20)

By transforming back to admittance notation, the equivalent forces can be calculated from the inverse of the admittance of the assembly’s interface:

$$f_{eq}^2 = (Y^{AR}_{22})^{-1}u^A_2.$$ \hspace{1cm} (3.21)

Furthermore, the equivalent forces can also be obtained in an overdetermined fashion by a sufficient set of indicator DoFs $u_4$ on the passive receiving subsystem:

$$f_{eq}^2 = (Y^{AR}_{42})^+u_4.$$ \hspace{1cm} (3.22)

The location of indicator DoFs is to some extent arbitrary, but they should be placed around the interface at the passive side. The receiver can also be directly the target assembly AB (as depicted in Fig. 3.6):

$$f_{eq}^2 = (Y^{AB}_{22})^{-1}u^A_2 = (Y^{AB}_{42})^+u_4.$$ \hspace{1cm} (3.23)

The equivalent forces are a sole property of the component A, and, in fact, no disassembling of the active part is required for the equivalent force identification [59, 60]. An interesting characteristic of the in-situ TPA is the location of the indicator DoFs $u_4$, which can be positioned arbitrarily in the vicinity of the interface. One important limitation of the in-situ approach is that the operational excitation $f_1$ can only originate from the active subsystem [14].
3.3.5 Pseudo-forces

The pseudo-forces TPA method can be treated as a generalisation of the in-situ concept. With the in-situ TPA method the identified equivalent forces are bound to the domain of the interface. On the other hand the pseudo-forces method assumes a set of pseudo-forces acting on the outer surface of the active component, cancelling out the effect of the operational excitation on the passive side \[61\] \[63\].

Consider the target assembly AB, depicted in Fig. 3.7, with a set of \( s \) pseudo-force \( f_{ps} \) and \( n \geq s \) indicator DoFs placed near the interface on the passive side. The response \( u_4 \) can be written as a result of operational excitation \( f_1 \) as follows:

\[
 u_4 = Y_{AB}^{41} f_1 = Y_{42}^{AB} (Y_{22}^A + Y_{22}^B)^{-1} u_{2,free} \quad \text{where} \quad u_{2,f_1} \triangleq Y_{21}A f_1. \tag{3.24}
\]

The pseudo-forces \( f_{ps} \) have to be determined in such a way as to best recreate the operational responses at \( u_4 \), with the active component turned off. A similar relation can be written for pseudo-forces as with operational excitation (Eq. (3.24)):

\[
 u_4 = Y_{AB}^{4ps} f_{ps} = Y_{42}^{AB} (Y_{22}^A + Y_{22}^B)^{-1} u_{2,free}^{f_{ps}} \quad \text{where} \quad u_{2,f_{ps}} \triangleq Y_{21}^{AB} f_{ps}. \tag{3.25}
\]

Now the pseudo-forces can be determined by solving an overdetermined system with using the operational measurements \( u_4 \):

\[
 f_{ps} = (Y^{AB})^+ u_4. \tag{3.26}
\]
The pseudo-forces are, in fact, a property solely of the source component A, under the assumption that the free interface velocity \( \mathbf{u}_{2\text{free}} \) can be fully represented by a set of identified pseudo-forces. The pseudo-force can now be applied back to the FRFs of the target assembled system and the correct receiver response \( \mathbf{u}_3 \) is obtained:

\[
\mathbf{u}_3 = \mathbf{Y}^{AB}_{3ps} \mathbf{f}_{ps} = \mathbf{Y}^{AB}_{31} \mathbf{f}_1.
\]  

(3.27)
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Experimental modelling in dynamic substructuring has great potential, as the standard FRFs measurements enable the direct acquisition of global dynamic characteristics at the points of measurement. When compared to numerical modelling, the experimentally obtained model has, in fact, real dynamic properties. With the current state-of-the-art data acquisition modules and sensors accurate results can be obtained up to a several-kHz range. In practice, the application of dynamic substructuring in the frequency domain with experimental data can be quite challenging. To demonstrate the main issues associated with experimental substructuring, first consider a simple rigid connection between two beams (Fig. 4.1). With frequency-based substructuring, the measurement of the drive- and transfer-point FRFs is required at every connection DoF. If we consider that we have a perfectly rigid point connection, three translations and three rotations DoFs have to be measured. In a numerical case this is feasible and relatively easy to obtain.

\[ u_A^1, f_A^1, u_A^2, f_A^2, u_B^2, f_B^2, u_B^3, f_B^3 \]

Figure 4.1: A simplified substructuring problem.

Obtaining the same FRFs from experimental testing on a real structure (imagine a simple bolted connection between two beams depicted in Fig. 4.2) is practically impossible. A tri-axial accelerometer and an excitation that applies a pure moment excitation would be required to measure rotational DoFs around each axis; currently, there is no available hardware that would be able to reliably perform these measurements. Furthermore, the requirements for accurate measurements with frequency-based substructuring are much more demanding when compared to the measurements required for modal testing, where measurements performed with a higher degree of bias error can still be acceptable. Due to these strict conditions, the underlying limitations of experimental testing are amplified when used in dynamic substructuring.

The ever present noise and the bias error associated with experimental measurements make the inversion in FBS ill-conditioned. Even the most simple bolted connection (Fig. 4.3a) is not perfectly rigid at high frequency range; therefore, the connection
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Figure 4.2: A simple substructuring problem where two simple beams are joined by a bolted connection.

cannot be idealized as a point connection. Connections between substructures can be even more complex. An example of a typical continuous interface is depicted in Fig. 4.3b. With this type of connection, the assumption of a discrete interface cannot be made. Furthermore, any real-life bolted connection is not, in fact, perfectly "welded" together, and the interface stiffness and damping should be considered in the substructuring process.

Figure 4.3: Two distinct substructure connectivity examples: a) discrete interface point, b) continuous interface surface.

Even with all the disadvantages and limitations experimental frequency-based substructuring can be used on real-life complex dynamic systems. If the interface dynamics are properly defined, the measurements performed accurately, and the required post-processing is done correctly, then the experimental dynamic substructuring can be an invaluable engineering tool.

In this chapter, the theory behind the state-of-the-art experimental approaches for frequency based substructuring is introduced and thoroughly discussed. First section 4.1 introduces the virtual point transformation [9, 27] together with an extension to include directly measured rotational response. The extension is considered to be one of the scientific contributions of this thesis [37, 38]. Next section 4.2, the singular vector transformation is introduced, which was proposed in [40]. Additionally, the formulation of SVT is extended to include weighting matrices. The third section 4.3 introduces the theory behind the system equivalent model mixing [20]. Furthermore, possible limitations from coupling reduced models in SEMM are introduced, which is used in the mixing of two equivalent experimental models [39].
## 4.1 Virtual point transformation

In frequency-based substructuring, the imposition of both compatibility and equilibrium conditions at the interface at collocated DoFs (output and input) is required. With experimental modelling, that condition is not as straightforward as with the numerical modelling. The DoFs measured on both sides of the interface are usually not at matching locations with experimental testing (Fig. 4.4). Furthermore, even if collocation could be obtained, the lack of rotational DoFs would still impose a large problem.

![Figure 4.4: Schematic representation of experimental setup required for virtual point transformation. Tri-axial accelerometers (red cubes), impact excitations (red arrows) and virtual points (green coordinate systems).](image)

Both of the aforementioned problems (collocation of DoFs and rotational DoFs) can be resolved with the use of the Virtual Point Transformation (VPT)\(^{[9,27]}\). The VPT projects measured translational input and output signals on the Interface Deformation Modes (IDMs)\(^{[1]}\). Commonly, 6 rigid IDMs are used for the transformation (i.e. 3 translations and 3 rotations). With this projection, only the dynamics which load the interface in a purely rigid manner are retained. The flexible motion is, in fact, filtered out with the transformation, and the interface problem is weakened (section 2.5). The name virtual originates from the fact that the selected virtual point can be anywhere in the vicinity of the interface.

The transformation can be applied in a relatively simple manner with the following equation\(^{[9]}\):

\[
Y_{qm} = T_u Y_{22} T_f^T, \tag{4.1}
\]

where \(Y_{22}\) denotes admittance FRF of interface DoFs (i.e. \(u_2^A\) and \(u_2^B\)), \(T_u\) is the displacement transformation matrix derived in section 4.1.1 and \(T_f\) is the force transformation matrix derived in section 4.1.2. The \(Y_{qm}\) is the VP FRF matrix with perfectly

---

\(^{1}\)The abbreviation IDMs was first defined as Interface Deformation Modes\(^{[27]}\) and later as Interface Displacement Modes\(^{[9]}\) in the VPT, as commonly only the rigid interface behaviour is considered.
collocated output and input DoF. With the resulting VP FRF matrix, the substructures can be coupled or decoupled with the LM-FBS.

### 4.1.1 Interface displacement reduction

Interface displacement reduction is defined with a set of interface displacement modes (IDMs) contained in matrix $\mathbf{R}_u \in \mathbb{R}^{n_u \times m}$. Measured interface displacements $n_u$ are expressed with $m$ interface displacement modes or so called virtual point DoFs. Consider a simple point connection depicted in Fig. 4.5.

![Figure 4.5: Basic interface example with virtual point and tri-axial translational sensor](image)

If only the rigid IDMs are considered, then the virtual point has 6 DoFs. That are three translations $q^\nu = [q^\nu_X, q^\nu_Y, q^\nu_Z]$ and three rotations $q^\nu_\theta = [q^\nu_\theta_X, q^\nu_\theta_Y, q^\nu_\theta_Z]$ DoFs. The following kinematic relation can be written between virtual point DoFs $q^\nu$ and sensor displacement $u^k$, provided that the location and orientation of the sensor are known in advance:

$$
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix} =
\begin{bmatrix}
u_{X}^{k} \\
u_{Y}^{k} \\
u_{Z}^{k}
\end{bmatrix} + \begin{bmatrix}
u_{X}^{k} & \nu_{X}^{k} & \nu_{X}^{k} \\
u_{Y}^{k} & \nu_{Y}^{k} & \nu_{Y}^{k} \\
u_{Z}^{k} & \nu_{Z}^{k} & \nu_{Z}^{k}
\end{bmatrix} + \begin{bmatrix}
u_{X}^{k} \\
u_{Y}^{k} \\
u_{Z}^{k}
\end{bmatrix} + \begin{bmatrix}
u_{X}^{k} \\
u_{Y}^{k} \\
u_{Z}^{k}
\end{bmatrix},
\tag{4.2}
$$

where vector $\mu^k_u$ contains any residual motion, not included in the subspace of IDMs. If the rigid assumption of the interface is valid in the considered frequency range, then the residual motion in $\mu^k_u$ will most likely be negligible. Eq. (4.2) can be expanded to include all measured displacements:

$$
u = \mathbf{R}_u q + \nu_u.
\tag{4.3}$$
A symmetric weighting matrix \( W_u \) can be introduced to gain more control over the transformation, as different channels can be prioritised in the reduction. Finally, the equation is solved for \( q \) [9]:

\[
q = (R_u^T W_u R_u)^{-1} R_u^T W_u u.
\tag{4.4}
\]

Residual displacements are then equal to:

\[
R_u^T W_u u = 0.
\tag{4.5}
\]

Eq. (4.4) can further be simplified and the displacement transformation matrix \( T_u \) is defined as:

\[
q = T_u u \quad \text{where} \quad T_u \triangleq (R_u^T W_u R_u)^{-1} R_u^T W_u.
\tag{4.6}
\]

Defined interface displacement reduction can also be used to calculate the IDM-filtered response, which can afterwards be used to evaluate sensor consistency:

\[
\hat{u} = F_u u \quad \text{where} \quad F_u \triangleq R_u T_u.
\tag{4.7}
\]

### 4.1.2 Interface force reduction

For the interface-force reduction a similar matrix containing IDMs is constructed. From the interface example in Fig. 4.5, it is clear that the force \( f_h \) will result in a virtual point load \( m^\nu \). Therefore, the following relation can be written:

\[
\begin{bmatrix}
m^\nu_X \\
m^\nu_Y \\
m^\nu_Z \\
m_{\theta_X}^\nu \\
m_{\theta_Y}^\nu \\
m_{\theta_Z}^\nu
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -r^h_Z & r^h_Y \\
-r^h_Y & 0 & -r^h_X \\
r^h_X & -r^h_Y & 0
\end{bmatrix}
\begin{bmatrix}
e^h_X \\
e^h_Y \\
e^h_Z
\end{bmatrix} f_h.
\tag{4.8}
\]

The contribution from the \( n_f \) input forces can be combined, and, accordingly Eq. (4.8) can be expanded as follows:

\[
m = R_i^T f,
\tag{4.9}
\]

where \( R_i^T \in \mathbb{R}^{m \times n_f} \) is the matrix containing IDMs. In order to perform the virtual point transformation according to Eq. (4.1), a force transformation matrix \( T_i^T \) is needed. Eq. (4.9) is typically under-determined, since \( n_f \geq m \); therefore, the inversion is achieved with the weighted right inverse of \( R_i^T \) (i.e., finding a solution that has a minimum \( W_f \)-norm) [9] [2]

\[
\hat{f} = W_f R_i (R_i^T W_i R_i)^{-1} m.
\tag{4.10}
\]

The problem of finding virtual forces is fundamentally different from displacement transformation, due to the under-determined problem. With the displacement transformation Eq. (4.3), a solution is found by minimizing a (weighted) norm of residuals. Whereas for the force transformation, a solution is found through distribution of forces that create the required reduced resultants, with a minimum (weighted) norm.
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where \( W_f \) is a symmetrical weighting matrix. Eq. (4.10) can be further rewritten:

\[
\tilde{f} = T^T f \quad \text{where} \quad T^T f = W_f R_f (R^T f W_f R_f)^{-1}.
\] (4.11)

If we were to choose the sensor faces for the impact locations, the absolute values in the matrices \( R_u \) and \( R_f \) would be the same [45,67]. However, the obtained FRFs would be inconsistent, due to a high-probability of overloading the sensors and poor excitation for cross-directional FRFs [9]. Therefore, the use of sensor faces as impact locations should be discouraged.

Defined interface force reduction can also be used to calculate the IDM-filtered forces, which can afterwards be used to evaluate impact consistency:

\[
\tilde{f} = F_f f \quad \text{where} \quad F_f = R_f T_f. \] (4.12)

4.1.3 Expanded VPT with directly measured rotational response

Rotational DoFs at the interface are essential for the coupling and decoupling of substructures in the FBS. With the virtual point transformation, the rotational DoFs are estimated based on the prevailing assumption of local rigidity in the vicinity of the interface, provided that the locations of the sensors and impacts are known. However, both the estimation of the sensor and the impact positions and orientations can only be determined up to a certain measurement accuracy [68]. The high sensitivity of indirect methods led to the development of a direct, quartz-based, piezoelectric, rotational accelerometer [69].

The rotational FRFs obtained from the rotational accelerometer are accurate and the sensor itself has a relatively low cross-axis sensitivity [70]. The use of a rotational accelerometer in the FBS has already proven to be useful [44], which is one of the primary reasons behind the expansion of the virtual point transformation with a directly measured rotational response [37].

To expand VPT with the incorporation of directly measured rotational response first consider a tri-axial rotation sensor on a simple interface as depicted in Fig. 4.6.

The main idea behind the VP transformation stays the same: the measured rotational response is projected onto the interface deformation modes to obtain the collocated VP FRF Matrix. The following kinematic relation can be written between the virtual point DoF \( q^\nu \) and the sensor rotation \( \theta^k \) [37]:

\[
\begin{bmatrix}
\dot{\theta}_x^k \\
\dot{\theta}_y^k \\
\dot{\theta}_z^k
\end{bmatrix} =
\begin{bmatrix}
e_{x,X}^k & e_{x,Y}^k & e_{x,Z}^k \\
e_{y,X}^k & e_{y,Y}^k & e_{y,Z}^k \\
e_{z,X}^k & e_{z,Y}^k & e_{z,Z}^k
\end{bmatrix}
\begin{bmatrix}
q^\nu_X \\
q^\nu_Y \\
q^\nu_Z
\end{bmatrix}
+ \begin{bmatrix}
\mu_{\theta_x} \\
\mu_{\theta_y} \\
\mu_{\theta_z}
\end{bmatrix},
\] (4.13)
where $\mu^k_\theta$ contains any residual rotational motion. The same as for the translational response applies here: if the rigid assumption is valid, the residual motion is negligible. If Eq. (4.2) is compared with Eq. (4.13), the advantages of the proposed expansion are clear. The kinematic relation for the rotations is dependent only on the sensor orientation, whereas, with translations the relation is dependent on the sensor orientation and also the sensor position. Eq. (4.13) can be expanded to include all the directly measured rotations:

$$\theta = R_\theta q + \mu_\theta. \quad (4.14)$$

Eqs. (4.3) and (4.14) can be combined for all the measured displacements and rotations as follows:

$$\begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} R_u \\ R_\theta \end{bmatrix} q + \begin{bmatrix} \mu_u \\ \mu_\theta \end{bmatrix} = R_{u,\theta} q + \mu_{u,\theta}. \quad (4.15)$$

To solve Eq. (4.15) for $q$ in a minimal-quadratic sense, the norm of the weighted residuals on the displacements and rotations is minimized:

$$q = \text{argmin} \left( \begin{bmatrix} \mu_u \\ \mu_\theta \end{bmatrix}^T W_{u,\theta} \begin{bmatrix} \mu_u \\ \mu_\theta \end{bmatrix} \right) \quad \text{where} \quad W_{u,\theta} = \text{diag}[W_u, W_\theta]. \quad (4.16)$$

In this approach, the rotations of the VP are not determined solely by the rotational DoFs. Indeed, the VP rotations also influence the translation residual (Eq. (4.2)), and the degrees of freedom computed by Eq. (4.16) for the VP minimize $\mu^T_u W_u \mu_u + \mu^T_\theta W_\theta \mu_\theta$. Therefore, rotational DoFs can still be estimated even if less than 3 rotations are directly measured for a VP.

As explained above, the Eq. (4.16) minimizes the sum of residual on the translation and rotational sensors. Therefore, in order to evaluate both residuals in a comparable norm, a proper weighting matrix for the rotations should be used.
Consider that the rotational measurement around \( x \) axis at rotation sensor \( k \) (Fig. 4.6) has a residual \( \mu^k_{\theta} \). If the VP would be given that residual rotation, the sensor would undergo a displacement \( \mu^k_{\theta} I^k_x \), where \( I^k_x \) is the distance between the \( x \) axis across the VP and the sensor location and is equal to \( \sqrt{r_x^2 + r_z^2} \). A similar reasoning can be done for the rotational residuals around the other axes and for each additional rotational sensor. If the norm of overall displacements due to the rotational residuals at the VP is chosen to be minimised, the following rotational weighting matrix should be used [37]:

\[
W_\theta = \text{diag}\left[ (I^k_x)^2, (I^k_y)^2, (I^k_z)^2, \ldots \right].
\] (4.17)

With the proposed formulation of the rotational weighting matrix the Eq. (4.15) can be solved for \( q \), and the combined transformation matrix \( T_{u,\theta} \) is obtained:

\[
q = T_{u,\theta} \begin{bmatrix} u \\ \theta \end{bmatrix} \quad \text{where} \quad T_{u,\theta} \triangleq (R_{u,\theta}^T W_{u,\theta} R_{u,\theta})^{-1} R_{u,\theta}^T W_{u,\theta}.
\] (4.18)

The proposed formulation of the combined weighing matrix is minimizing the norm of the displacement residuals for each translation sensor and minimizing the norm of overall displacement for each rotational sensor due to the rotational residual at the virtual point [37]. Therefore, the residuals from the rotational and translation sensors can be compared, and Eq. (4.16) is consistent (assuming \( W_u \) is taken as identity).

Note that the positions of the rotational sensors are only used to define a proper norm for the least square problem: if the rotational measurements were all perfect, the rotation of the VP would be found exactly, notwithstanding any error in the position of those sensors (Eq. (4.13)). This is not the case when rotations are derived from the translation sensors (Eq. (4.2)).

A specific case for the proposed weighting matrix is the measuring of the rotations right at the VP. In such a case, the weighting matrix \( W_\theta \) (Eq. (4.17)) would become a zero matrix and the directly measured rotational response would be excluded from the transformation. Therefore, the expanded VPT cannot be used when the rotations are measured perfectly at the VP. However, the position of the VP can always be moved away from the position of the rotational accelerometer, since the position of the VP in the vicinity of the interface is arbitrary to some extent. This limit case shows however that the scaling proposed, based on pure geometrical reasoning, is probably not the ultimate choice and maybe different scaling strategies should be developed in the future.

### 4.1.4 Measurement quality indicators

One of the advantages of the VPT is also the ability to evaluate the quality of the transformation and performed measurements by utilizing the assumption of interface rigidity\(^3\). After the transformation, the virtual DoF can be expanded (or projected) back on the measured DoFs and with that the assumption of interface rigidity can

\(^3\text{Commonly, only rigid body IDMs are considered in the VP transformation, if additional deformation modes would be included also their contribution can be included.}\)
Experimental substructuring in frequency domain

be evaluated \[9, 26\]. Two different quality indicators can be calculated for the displacement/force transformation. These are the specific and the overall sensor/impact consistency indicators \[9\]. With the consistency indicators, bad sensor channels or impact locations can be identified and discarded from the final transformation.

A coherence criterion was introduced to evaluate the difference between a measured and a filtered response \[26\]. A spectral coherence function can be used to determine the variation between two complex numbers \(a\) and \(b\):

\[
\text{coh}(a, b) = \frac{(a + b)(\bar{a} + \bar{b})}{2|a + b|^2}.
\]

(4.19)

The function compares the complex number with respect to their amplitude as well as the phase and it is bounded between 0 \((a = -b)\) and 1 \((a = b)\).

4.1.4.1 Sensor consistency

The sensor consistency function evaluates the assumption of rigidness on the measured response with the respect to a certain load case. The load case, denoted by \(f_2\), is composed by one or more excitations not included in the transformation (reliably distant from the evaluated VP):

\[
u_{1,2} = Y_{12} f_2.
\]

(4.20)

The filtered response is then acquired by pre-multiplying with the filter matrix \[9\]:

\[
\tilde{u}_{1,2} = F_{11} Y_{12} f_2.
\]

(4.21)

Now both the measured response \(u_{1,2}\) and the filtered response \(\tilde{u}_{1,2}\) are compared, and the two consistency indicators can be defined.

Overall sensor consistency

In order to quantify all sensor channels, the norm of both vectors can be compared throughout the whole frequency range:

\[
\rho_{u_{1,2}}(\omega) = \frac{\|\tilde{u}_{1,2}(\omega)\|}{\|u_{1,2}(\omega)\|}.
\]

(4.22)

The indicator is thus a frequency dependent function valued between 0 and 1 (full consistency). An overall sensor consistency value of 1 throughout the whole frequency range means that the locations and orientations of all the sensor channels are correct and that the assumption of rigidness is valid. It could happen that at higher frequencies the value would drop. This would indicate the presence of flexible IDMs, the contribution of which is now filtered out by the transformation.

Specific sensor consistency

The specific sensor consistency is evaluated for the individual channel included in the transformation by using the predefined coherence criterion:

\[
\rho_{u_{i,2}}(\omega) = \text{coh}(\tilde{u}_{i,2}(\omega), u_{i,2}(\omega)) \quad \text{where} \quad u_i \in u_1.
\]

(4.23)
Experimental substructuring in frequency domain

the resulting indicator is a frequency dependent function that can be used to assess each sensor channel. A constant low value of the specific sensor consistency indicator would indicate either an incorrect orientation, location or calibration value of the sensors, which can be afterwards be identified from the specific sensor consistency.

4.1.4.2 Impact consistency

In order to evaluate the accuracy of the impact positions and directions an impact consistency indicator is defined in a way similar to the sensor consistency. First a linear combination of responses $y_2$ is defined, by introducing vector $w_2$:

$$ y_2 = w_2^T u_2. $$  \hspace{1cm} (4.24)

Now both the measured and the filtered response can be defined for each impact [9]:

$$ y_{2,1}(\omega) = w_2^T Y_{21}, $$ \hspace{1cm} (4.25)

$$ \tilde{y}_{2,1}(\omega) = w_2^T Y_{21} \mathbf{F}_{11}. $$ \hspace{1cm} (4.26)

Vector $y_{2,1}$ includes the sum of responses that results from a single impact, and the vector $\tilde{y}_{2,1}$ includes the same sum of responses as resulting from a filtered impact.

**Overall impact consistency**

By comparing the norm of both vectors, an indicator can be defined throughout the whole frequency range:

$$ \rho_{f_{1,2}}(\omega) = \frac{\| \tilde{y}_{2,1}(\omega) \|}{\| y_{2,1}(\omega) \|}. $$ \hspace{1cm} (4.27)

A high value of the overall impact consistency indicator implies that the impact forces $f_i$ can be fully represented by the rigid IDMs (3 forces and 3 moments).

**Specific impact consistency**

To ascertain the individual channel included in the transformation, a coherence criterion is used for the comparison:

$$ \rho_{f_{j,2}}(\omega) = \text{coh}(\tilde{y}_{2,j}(\omega), y_{2,j}(\omega)). $$ \hspace{1cm} (4.28)

A low specific impact consistency values can indicate an incorrect position, direction, double impact, or that the impact resulted in a signal overload at any channel.

4.1.4.3 Reciprocity and passivity

One of the more convenient properties of linear structures is their reciprocity. After transforming to the VP DoFs, the collocation between virtual responses and forces is obtained. An additional quality indicator can be defined using the reciprocity property
of the VP DoFs. A coherence criterion can be used to on the VP DoFs per frequency line \[26\]:

\[
\chi_{ij}(\omega) = \text{coh}(Y_{ij}(\omega), Y_{ji}(\omega))
\]  

(4.29)

where \(i\) and \(j\) denote the different DoFs in the VP FRF matrix. The only limitation of reciprocity is that it can only be used on the non-diagonal entries. On the diagonal entries, the coherence is \(\chi_{ii} = 1\) by definition.

For the diagonal FRFs, the passivity can be evaluated because the driving-point FRFs should always be minimum-phase functions. Therefore, the phase of the driving-point FRFs should always be bounded by \(\angle Y_{ii} \in [0^\circ, 180^\circ]\) for accelerance FRFs.

### 4.1.5 Evaluation of standard and expanded VPT

In this section standard and expanded VPT are compared. A numerical simulation is performed to simulate measurements on the simple beam-like structure depicted in Fig. 4.7. The dynamic response of the structure is obtained from a finite element analysis with a proposed free-free boundary condition. The geometrical and material properties of the analyzed structure are presented in Table 4.1.

![Schematic representation of beam-like structure with 2 virtual points and three sensor locations (red) for each virtual point][]{fig:4.7}

Figure 4.7: Schematic representation of beam-like structure with 2 virtual points and three sensor locations (red) for each virtual point \[37\].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
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</tr>
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<td>mm</td>
<td>190</td>
</tr>
<tr>
<td>(w)</td>
<td>mm</td>
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</tr>
<tr>
<td>(E)</td>
<td>GPa</td>
<td>210</td>
</tr>
</tbody>
</table>

Table 4.1: Beam-like structure’s geometrical and material properties.

For each virtual point, displacement and rotation admittance FRF were synthesised at three different locations (Fig. 4.7). Twelve different impact locations (Fig. 4.10a) per virtual point were simulated. Overall, 864 FRFs were generated \((Y_{\text{sim}}^{\text{AB}} \in \mathbb{C}^{24 \times 32})\) from the numerical model in the frequency bandwidth from zero to 3 kHz. Rotation and translation FRF were averaged from nodal displacements obtained from nodes on the surface over an 8 \(\times\) 8 mm area in order to replicate experimental measurement of FRF.
4.1.5.1 Transformation to the VP FRFs

Virtual point transformation was applied to transform the synthesised FRF matrix into a virtual point FRF matrix. Two different transformations were considered. The first one was a standard VPT with three tri-axial translation sensors. The second one was an expanded VPT with two tri-axial translation sensors and a single tri-axial rotational sensor per virtual point. Displacement and force weighting matrices were chosen to be the identity matrix for objective comparison between the two transformations.

The transformed FRF for the two cases of transformation and the numerically obtained validation driving- and transfer-point VP FRF obtained from numerical model are depicted in Fig. 4.8. It can be observed that, in most directions, both transformations are in a good agreement with the validation FRF as seen in Fig. 4.8a and Fig. 4.8b. But for rotations around the Z direction, expanded VP transformation out-performs the standard one as can be seen in Fig. 4.8c and Fig. 4.8d. This is related to the aforementioned averaging of the translation FRFs over the area.

![Figure 4.8: Comparison between expanded VPT with rotations, standard VPT and numerically obtained FRFs for VP [37]: a) $Y_{\theta X - M_Z}$; b) $Y_{\theta Y - M_X}$; c) $Y_{qX - F_Y}$; d) $Y_{qZ - M_Y}$. Expanded VPT with rotations (red line), standard VPT (dashed line), reference FRF (green line).](image)

Figure 4.9 shows the frequency-averaged reciprocity of the transformed VP admittance matrix for the two transformations. The average reciprocity of standard VPT over the whole frequency bandwidth is 75%, while expanded VPT with rotations has an average reciprocity of 83%. The largest deviations in reciprocity are observed for the Z direction and for rotations around the X and Y axis. This can be explained by the nature of the response of analysed structure. Excitation in the X or Y direction leads to a
negligible response in the $Z$ direction. Therefore even a small uncertainty in excitation or response can lead to a large reciprocity error.

Figure 4.9: Frequency-averaged reciprocity of the transformed virtual point FRF matrix [37]: a) standard VPT; b) expanded VPT with rotations.

4.1.5.2 Global sensitivity analysis

The overall quality of virtual point transformation primarily depends on sensor and impact positioning. Even a small uncertainty in sensor and impact orientation or location can lead to erroneous transformation. The effect of a small deviation in the direction of impacts on the quality of VPT has already been considered in [9]. Therefore, this effect will be omitted in the analysis (i.e. the locations and directions of impacts are assumed to be correct).

Small deviations in the sensor location are considered in terms of the overall quality of the transformation. The evaluation model is considered to:

$$\chi_{ij}(\mathbf{r}_e^{A1}, \mathbf{r}_e^{A2}, \mathbf{r}_e^{A3}, \mathbf{r}_e^{B1}, \mathbf{r}_e^{B2}, \mathbf{r}_e^{B3}) = \text{coh}(Y_{ij}, Y_{ji}) \quad Y_{ij}, Y_{ji} \in \mathbf{Y}_{qm},$$

(4.30)
Experimental substructuring in frequency domain

where \( \chi_{ij} \) is a scalar value of averaged coherence (Eq. (5.6)) over the whole frequency bandwidth and \( r^e_e = [r^e_x, r^e_y, r^e_z] \) is a vector of deviations in each sensor location.

Additionally, the effect of deviation in sensors sensitivity is analysed. The evaluation model for the second case is equal to:

\[
\chi_{ij}(s^{1A}_e, s^{2A}_e, s^{3A}_e, s^{1B}_e, s^{2B}_e, s^{3B}_e) = \text{coh}(Y_{ij}, Y_{ji}) \quad Y_{ij}, Y_{ji} \in Y_{qm}, \tag{4.31}
\]

where \( s^*_e = [s^*_x, s^*_y, s^*_z] \) is the vector of sensitivity deviations for each sensor. The location of sensors was assumed to be correct in this case.

**Sobol’s sensitivity analysis**

A sensitivity analysis can be performed in various ways with numerous different techniques. Due to its robustness, a Sobol’s sensitivity analysis is often used to estimate the global sensitivity. Method was originally developed by Sobol [71] in 2001 and then further improved by Saltelli et. al. [72,73]. First order Sobol’s sensitivity index of each input parameter is calculated as:

\[
S^1_{\chi_{ij}} = \frac{\mathbb{V}_{r_i}(\mathbb{E}_{r_{\sim i}}[\chi_{ij}|r_i])}{\mathbb{V}(\chi_{ij})}, \tag{4.32}
\]

where \( \mathbb{V}(\ast) \) is the variance operator, \( \mathbb{E}[\ast] \) is the expectation operator, \( r_i \) is the \( i \)th input parameter and \( r_{\sim i} \) is a set of all parameters except the \( r_i \). The first order index measures the main effect of parameter \( r_i \) alone. The total effect index measures the effect of parameter \( r_i \) and includes all higher order interactions with other input parameters. The total effect index is defined as:

\[
S^T_{\chi_{ij}} = 1 - \frac{\mathbb{V}_{r_{\sim i}}(\mathbb{E}_{r_i}[\chi_{ij}|r_{\sim i}])}{\mathbb{V}(\chi_{ij})}. \tag{4.33}
\]

The workflow of global sensitivity analysis for the two evaluation models (Eq. (4.30) and (4.31)) is depicted in Fig. 4.11. Both aforementioned VP transformations (standard and expanded) were analysed and compared to each other.

Firstly uncertainty in sensor location was analysed (Fig. 4.11a). The location of each sensor was deviated inside a 5 mm interval according to the Saltelli sample scheme. For each iteration, virtual point transformation was performed, and the coherence criterion (Eq. (5.6)) was calculated between reference FRF and transformed FRF. Reference FRF is calculated with the location of the sensors assumed as correct.

Secondly uncertainty in sensor sensitivity was analysed in a similar way (Fig. 4.11b). The sensitivity of each sensor axis was deviated inside a 5% interval according to the Saltelli sample scheme, and the coherence criterion was calculated for each iteration.

**Deviation in sensor location**

The first-order and total-order Sobol sensitivity indexes for the influence of sensor locations deviations on the FRF \( Y_{M^{B}_{q_{A}}} \) are shown in Fig. 4.12. A clear advantage of the expanded VPT with rotations is seen with fewer positions effecting the overall FRF.
Experimental substructuring in frequency domain

**Figure 4.11:** Workflow of the global sensitivity calculation [37]: a) deviation in the sensor location; b) deviation in the sensor sensitivity.

![Workflow Diagram](image)

**Figure 4.12:** Sobol-sensitivity index for deviations in the sensor location for FRF $Y_{\theta B,X-MB}$ [37]: a) standard VPT; b) expanded VPT with rotations.

![Sobol Sensitivity Index](image)

Figure 4.13 shows averaged total-order Sobol sensitivity indexes for the whole VP FRF matrix. A deviation of sensor location is less impactful for expanded VPT with rotations than for standard VPT. A deviation in sensor location can also be detected with overall sensor consistency [9]. However, a small deviation in sensor location or sensor sensitivity leads to the same poor sensor consistency over the whole frequency range. Therefore, it is difficult to distinguish between the two effects.
Deviation in sensor measurement sensitivity

First-order and total-order Sobol sensitivity indexes for how small deviations in sensor sensitivity effect the FRF $Y_{MB} - q_A^T$ are shown in Fig. 4.14. A similar conclusion can be drawn as before for deviations in sensor placement. Standard VPT is more susceptible to small deviations in sensor sensitivity.

Figure 4.13: Averaged total-order Sobol-sensitivity index of the coherence criterion for deviations in the sensor location [37]: a) standard VPT; b) expanded VPT with rotations.

Figure 4.14: Sobol-sensitivity index for deviations in the sensor measurement sensitivity for the FRF $Y_{AX} - M_{XY}$ [37]: a) standard VPT; b) expanded VPT with rotations.
In Fig. 4.15 averaged total-order Sobol sensitivity indexes for the whole VP FRF matrix are shown. It is evident that expanded VPT with rotations is less prone to uncertainties in sensor sensitivity.

Figure 4.15: Averaged total-order Sobol-sensitivity index of the coherence criterion for deviations in the sensor measurement sensitivity [37]: a) standard VPT; b) expanded VPT with rotations.

4.2 Singular Vector Transformation (SVT)

Singular Vector Transformation (SVT) is a reduction method that exploits the properties of the Singular Value Decomposition (SVD) to define a reduction basis [40]. The SVD is already extensively used in structural dynamics (section 4.2.1); however, the definition of a reduction subspace through SVD was currently reported only on a decoupling application [40]. The reduced subspace is defined directly from the measured dynamics. No geometrical or numerical model is required. Furthermore, the reduction is frequency-dependent and can include rigid as well as flexible interface behaviour, which is one of the primary advantages of the methodology. The SVT can be used in decoupling as well as in coupling applications.

The SVD can be considered as a generalization of the eigendecomposition of a square matrix to any rectangular matrix. An arbitrary complex matrix \( \mathbf{A} \in \mathbb{C}^{n \times m} \) can be factorized with SVD as follows:

\[
\mathbf{A} = \mathbf{U} \Sigma \mathbf{V}^H,
\]  

(4.34)

where the columns of \( \mathbf{U} \in \mathbb{C}^{n \times n} \) and \( \mathbf{V} \in \mathbb{C}^{m \times m} \) are left and right singular vectors, which are orthonormal. \(^4\) Matrix \( \Sigma \) is a diagonal matrix containing non-negative values, which are called singular values of the matrix \( \mathbf{A} \).

\(^4\) The notation \( \{\ast\}^H \) represents a conjugate transpose (\( \mathbf{A}^H = \overline{\mathbf{A}}^T = \mathbf{A}^\top \)).
Experimental substructuring in frequency domain

The singular values are sorted from the largest $\sigma_1$ to the smallest $\sigma_r$ singular value (where $r$ is the rank of the matrix $A$). The original matrix $A$ can be split into ordered rank-one components as follows:

$$A = \sum_{i=1}^{r} u_i \sigma_i v_i^H = u_1 \sigma_1 v_1^H + \cdots + u_r \sigma_r v_r^H,$$

(4.35)

where $u_i$ and $v_i$ are the $i$-th left and right singular vectors. If $n$ smallest singular values are discarded from the summation (leaving $k = r - n$ singular values), then we obtain the best rank $k$ approximation of the original $A$ matrix. This procedure is also known as a truncated SVD.

4.2.1 Applications of SVD in structural dynamics

In structural dynamics various techniques were developed for completely different application areas, which exploit the underlying properties of the SVD. As each technique and its application were developed independently, they have many different names, even though they share the same fundamentals. Some of the techniques that utilize the properties of the SVD are listed below.

**Complex Mode Indicator Function (CMIF)**

The CMIF is a graphical representation of a square of singular values, computed at each frequency line of the FRF matrix \[74,75\]:

$$Y(\omega) = U(\omega) \Sigma(\omega) V(\omega)^H,$$

(4.36)

where the left singular vectors $U$ represent response modes (approximate mode shapes) and the right singular vectors $V$ represent excitation modes (approximate modal participation factors) at each frequency line. The CMIF is defined as a square of singular values:

$$\text{CMIF}(\omega) = (\Sigma(\omega))^2.$$

(4.37)

A peak in the CMIF can indicate a natural frequency of the system; however, care should be taken as noise, spectral leakage or non-linearity can also cause a peak in the CMIF \[76,77\].

The CMIF can, in fact, be used as a modal parameter estimation method and it was one of the first mode indication functions developed \[74\]. Furthermore, the CMIF can also be used to evaluate the rigidness of the interface. For example, if the CMIF would display only 6 dominant modes, which are largely synchronous in phase, it can be presumed that the associated response modes are, in fact, rigid \[9\].

**Principal Response Functions (PRFs)**

The PRFs are in concept similar to the CMIF as they are calculated by performing SVD on the FRF matrix \[75,78\]. However, instead of performing the SVD on the frequency "cross-section" of the FRF matrix, the matrix is rearranged in two dimensions:

$$Y \in \mathbb{C}^{n_u \times n_f \times n_\omega} \rightarrow Y_{\text{PRF}} \in \mathbb{C}^{n_u n_f \times n_\omega}.$$  

(4.38)
The SVD can now be performed on the entire data as follows:

\[
Y_{\text{PRF}} = \begin{bmatrix} Y_{11}, & \ldots, & Y_{1n_f}, & \ldots, & Y_{n_u n_f} \end{bmatrix} = U \Sigma V^H,
\]  

(4.39)

The left singular vectors \(U\) contain the spectral characteristics and the singular values \(\Sigma\) the dominance of each associated vector. The PRFs are then defined as [75]:

\[
\text{PRF} = U \Sigma = Y_{\text{PRF}} V.
\]  

(4.40)

If the measured FRF matrix contains enough over determination of the measured system, then the PRFs reveal the dominant spectral information nature and the noise floor of the measurements [76]. By selecting only the \(r\) dominant singular values one can use the PRFs as a simple FRF filter in the frequency-domain. One of the benefits of PRFs, when compared to the CMIF, is also their spectral continuity, as the SVD is performed on the whole set of FRFs at once.

**Principal Component Analysis (PCA)**

The PCA is a dimensionality reduction method, which can be used to reduce the dimensionality of large data sets [79]. It provides a linear transformation from a set of physical variables to a set of new virtual or principal variables. The application of PCA extends far from structural dynamics and its an invaluable tool in big data and machine learning. There are, in fact, various approaches on how to calculate the principal components. However, SVD is most commonly used in cases of rectangular data input [76].

There are in fact numerous research topics in structural dynamics where the PCA is used to identify various features or where large datasets are reduced to fewer dominant degrees of freedom. However, the PCA is primarily used in condition monitoring [80–82].

**Independent Component Analysis (ICA)**

The ICA is considered as an extension to the PCA and can be used in a similar manner, for feature extraction and dimensionality reduction [83]. The main difference between the two methods is that the PCA removes correlations, but not higher order dependence. Meanwhile the ICA removes correlations as well as the higher order dependence. Additionally, the vectors associated with each degree of freedom are orthogonal for the PCA, while for the ICA, they are not orthogonal.

The applications of ICA in structural dynamics are similar to applications of PCA, primarily in condition monitoring [84–86].

### 4.2.2 Singular vectors as a reduction space

The main idea behind the SVT is to define a reduction basis directly from the measured dynamics. The reduction should involve rigid motion at the interface as well as the most dominant flexible motion. Consider the interface between two substructures AB and B depicted in Fig. 4.16 for decoupling application. If only rigid IDMs were be used...
Experimental substructuring in frequency domain

Figure 4.16: An example of a collocated set of inputs/outputs required for the SVT in a decoupling application [40]: a) assembled configuration AB; b) substructure B.

...to define the reduction basis, the reduced compatibility and equilibrium conditions wouldn’t be sufficient to decouple the two substructures.

In order to define the reduction basis directly from the measured dynamics by using the SVD, outputs should be placed in a collocated manner on both substructures and also the inputs should also be collocated (but not necessary collocated between each other). Furthermore, they should be homogeneously placed on the interface so that all the dominant interface modes of interest are controlled and observed in the frequency range of interest.

The next step in the reduction is to simply apply the SVD at each frequency line (the same as for the CMIF) on one of the substructures as follows:

$$Y_A(\omega) = U_A(\omega)\Sigma_A(\omega)V_A^H(\omega),$$

both left and right singular vectors $U_A(\omega)$ and $V_A^H(\omega)$ are already orthonormal at each frequency line and, therefore, can directly be used as a reduction basis. By limiting the number of singular DoFs at the interface (retaining $r$ singular values), the truncated bases $U_r(\omega)$ and $V_r(\omega)$ can be obtained and compatibility and equilibrium conditions can be weakened. The same reduction basis can now be used on the substructure B (assuming that there is the required collocation between inputs and separately outputs). Because the reduction is, in fact, similar to the VPT, so a similar notation is adopted to mathematically derive the reduction. The frequency dependence $\star(\omega)$ is omitted from the derivation for the sake of simplicity.

### 4.2.2.1 Interface displacement reduction

In a similar manner as for the VPT, the $n$ interface displacements $\mathbf{u}$ are expressed by $r$ generalized SV displacements $\mathbf{\zeta}$. The number of truncated DoFs is smaller than...
the number of interface DoFs \((r < n_u)\). Therefore, since we have an overdetermined problem, a residual displacement \(\mu\) will always be present after the reduction:

\[
u = U_r \zeta + \mu.
\] (4.42)

In order to gain the ability to give some outputs more or less weight, a symmetric weighting matrix \(W_u\) is introduced. With this matrix, the minimisation of the weighted norm of residuals is achieved:

\[
U_r^H W_u \mu = 0.
\] (4.43)

The reduction from \(u\) to \(\zeta\) is in fact a standard weighted least-square solution:

\[
\zeta = \left(U_r^H W_u U_r\right)^{-1} U_r^H W_u u.
\] (4.44)

The Eq. (4.44) can be further simplified by defining a displacement transformation matrix \(T_u\):

\[
\zeta = T_u u \quad \text{where} \quad T_u \triangleq \left(U_r^H W_u U_r\right)^{-1} U_r^H W_u.
\] (4.45)

By placing \(\zeta\) back into Eq. (4.42), a filter matrix \(F_u\) can be defined (reduction and a back projection):

\[
\tilde{u} = F_u u \quad \text{where} \quad F_u \triangleq U_r T_u,
\] (4.46)

where \(\tilde{u}\) are filtered displacements, from which the measurement quality indicators can be defined. Both the overall and the specific sensor consistency can be used with the SVT in the same way as with the VPT (section 4.1.4.1).

The frequency-dependent reduction basis \(U_r\) contains the \(r\) most dominant response modes; therefore, also the generalized SV displacements \(\zeta\) are also sorted by dominance at each frequency line.

### 4.2.2.2 Interface force reduction

An analogous derivation of the reduction can be applied to the measured forces \(f\). However, now the reduced set of right singular vectors \(V_r\) is used as a reduction basis:

\[
\eta = V_r^H f,
\] (4.47)

where the \(\eta\) are the generalized SV forces, whose number of it is ordinarily larger than the number of interface forces \(f\) \((r < n_f)\). Therefore, the Eq. (4.47) is in fact underdetermined and the solution is achieved with the weighted right inverse of \(V_r^H\) (i.e. finding a solution that has a minimal \(W_f\)-norm):

\[
\hat{f} = W_f V_r (V_r^H W_f V_r)^{-1} \eta.
\] (4.48)

The \(W_f\) is a symmetric weighting matrix, which can be used to give some of the inputs more or less weight. The Eq. (4.48) can be rewritten, by defining a force transformation matrix \(T_f\):

\[
\hat{f} = T_f^H \eta \quad \text{where} \quad T_f^H \triangleq W_f V_r (V_r^H W_f V_r)^{-1}.
\] (4.49)
Finally, a filter matrix $F_f$ can be defined (reduction and a back projection): 

$$\tilde{f} = F_f f \quad \text{where} \quad F_f \triangleq V_r T_f,$$

(4.50)

where $\tilde{f}$ are the filtered interface forces that are used to define measurement quality indicators. Both the overall and the specific impact consistency can be used with the SVT in the same way as with the VPT (section 4.1.4.2).

The frequency-dependent reduction basis $V_r$ contains the $r$ most dominant excitation modes (similar as to the $U_r$). Therefore, also the generalized SV displacements $\zeta$ are also sorted by dominance at each frequency line.

### 4.2.3 Physical interpretation

The main concept behind the SVT is fairly simple: project the measured dynamics on the subspace composed by the dominant singular components, which are defined directly from the measurements \(^{[40]}\). However, the SVT can only be applied on a decoupling or coupling application if the primary assumption of collocation is satisfied.

Consider an application of the SVT on a decoupling application between two substructures $AB$ and $B$ depicted in Fig. 4.16. First, a reduction basis is defined on the basis of measurements of one of the substructures. Consider the reduction ($U_r^A$ and $V_r^A$) based on the measurements of the structure $A$ (which assumes that both weighting matrices are an identity matrix): \(^{[5]}\)

$$T_u^A = ((U_r^A)^H W_u U_r^A)^{-1} (U_r^A)^H W_u = (U_r^A)^H,$$

(4.51)

and, similarly, for the force reduction:

$$(T_f^A)^H = W_f^A V_r^A ((V_r^A)^H W_f^A V_r^A)^{-1} = V_r^A.$$

(4.52)

The transformation to the reduced generalized set of singular DoFs can be obtained by matrix multiplication for both substructures $A$ and $AB$:

$$Y_{\zeta A}^A = T_u^A Y_{\zeta A} (T_f^A)^H \quad \text{and} \quad Y_{\zeta A}^A = T_u^A Y_{A} (T_f^A)^H.$$

(4.53)

The reduced admittance for the substructure $A$ $Y_{\zeta A}^A$ is exactly the diagonal matrix of the $r$ retained singular values $\Sigma_A$ of $Y_{A}$ ($Y_{\zeta A}^A = \Sigma_A$). The reduced admittance for the substructure $AB$ $Y_{\zeta A}^{AB}$ is the non-diagonal complex matrix ($Y_{\zeta A}^{AB} \neq \Sigma_{AB}$).

By filtering $Y_{\zeta A}^A$ (reduction and a back projection) it can be determined how well the reduced basis acquires the interface dynamics. On the other hand by filtering $Y_{\zeta A}^{AB}$ with the principal subspaces of $Y_{A}$ it can be determined how the interface dynamics of the structure $AB$ are represented by the reduced subspace of the structure $A$.

The main underlying assumptions, that have to be satisfied for a successful application of the SVT are listed bellow \(^{[40]}\):

\(^{[5]}\)Both left and right singular vectors are orthonormal. The following holds: $U_r^{-1} = U_r^H$ and $V_r^{-1} = V_r^H$. Therefore, no matrix inversion is required for the transformation, if the weighting matrices $W_u$ and $W_f$ are identity.
1. **Collocation on inputs and outputs:**
   Inputs and outputs should be located and oriented in the same way on both structures in the transformation. With this the reduced basis defined from the dynamics of one substructure can be used on the other one. For the outputs, this is not problematic, as the sensor can stay in place between different measurements campaigns. However, there will always be a bias on inputs, as it is practically impossible to satisfy exactly the same location and orientation of impact excitation.

2. **Controllability and observability:**
   The location and orientation of outputs and inputs should be able to control and observe approximately the same interface dynamics. Since collocation is not required between inputs and outputs separately, the measured substructure admittance matrix is a non-symmetric complex matrix. The left and right singular vectors are not the same and in theory span different spaces. However, truncated subsets of both vectors can control and observe the same dynamic information. From a practical standpoint, the user should ensure a homogeneous spatial distribution of inputs and outputs on the interface.

3. **Similar interface dynamics between structures:**
   The chosen reduced subspace from one structure should be able to map the observed/controlled dynamics in the other structure. For the rigid interface motion, this is satisfied without any limitations; however, for higher complex modes it can happen that the interface modes between two substructures are completely different. This problem can be addressed by using a transmission simulator at the interface [30].

4.2.3.1 **Advantages**

One of the advantages is that a geometrical model is not required, as the information about the location and the orientation of the outputs/inputs is not necessary (if the collocation is satisfied). In practice, this is highly advantageous with a high number of outputs/inputs, complex geometries and challenging test scenarios (impact testing on hard to reach places).

With the SVT, flexible interfaces can also be coupled or decoupled, without a numerical or analytical model of the interface. An alternative way to couple a flexible interface in FBS is with the VPT by adding flexible IDMs to the transformation. However, the flexible IDMs used for extension can only be simple deformations (e.g. extension, torsion) which can be defined by an analytical relation. If the most dominant mode is known in advance the method has shown itself to be effective. However, on the other hand with the SVT the most dominant flexible interface motion is already directly included in the transformation.

4.2.3.2 **Limitations**

The main limitation of the SVT is the procedure of determination of how many reduced DoFs are used in the transformation. The reduction basis is frequency-dependent, and
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it would be beneficial if the process could be automated. Commonly, only rigid interface motion is dominant in the low-frequency range, and with the increase in the frequency, also the flexible motion also becomes influential. Therefore, a lower number of reduced DoFs should be used in the low-frequency range relative to the high frequency range. Currently, the selection process is based on the judgement of the person using the CMIF plot, or on the determination of a threshold value based on the CMIF or PRFs.

4.2.4 Comparing SVT with Transmission Simulator method

An alternative method for coupling/decoupling continuous interfaces that exhibit flexible IDMs is the transmission simulator method. The TS method employs a temporary fixture for measurement and solves the decoupling and coupling step in the modal domain. The method was first proposed by Allen et. al. [30] as Modal Constraints for Fixtures and Subsystems (MCFS). The method has gained a lot of popularity in the recent years, as it enables the inclusion of rotations and the bending modes of the interface.

To some extent, the SVT is similar to the TS method. Both approaches involve a weakening of compatibility and equilibrium with a reduced set of characteristic motion of the interface. The TS method uses a set of TS physical mode shapes (most commonly obtained from a separate numerical model) for the weakening, whereas the SVT obtains a reduced basis directly from the measured dynamics by means of left and right singular vectors, which means that no additional numerical model is required for the transformation.

4.3 System Equivalent Model Mixing (SEMM)

With System Equivalent Model Mixing (SEMM) [20], multiple equivalent-response models can be mixed into a hybrid model, using Lagrange Multiplier Frequency-Based Substructuring (LM FBS) [11]. With SEMM, two equivalent models are coupled on the overlapping DoFs, and the unwanted dynamics are removed by decoupling a reduced unwanted model. A schematic representation of SEMM is depicted in Fig. 4.17.

\[
\mathbf{Y}_{\text{par}} + \mathbf{Y}_{\text{ov}} = \mathbf{Y}_{\text{rem}} = \mathbf{Y}_{\text{SEMM}}
\]

Figure 4.17: Schematic representation of System Equivalent Model Mixing (SEMM) [20].
A brief explanation of each model used within SEMM is provided below [20]:

**Parent model** \( Y_{\text{par}} \) provides a DoF structure, which the hybrid model inherits.

**Overlay model** \( Y_{\text{ov}} \) provides the dynamics by coupling them with the parent model. The overlay model DoFs are a subset of the parent model DoFs.

**Removed model** \( Y_{\text{rem}} \) is the condensed parent model, which gets decoupled from the parent model to remove the parent model dynamics.

**Hybrid model** \( Y_{\text{SEMM}} \) is the final model after the mixing of the parent model and the overlay model.

The parent model can be split into the internal \( i \) and boundary \( b \) DoFs. The overlapping DoFs between the parent and the overlay model are the boundary DoFs and the internal DoFs are unique to the parent model:

\[
Y_{\text{par}} \triangleq \begin{bmatrix} Y_{ii} & Y_{ib} \\ Y_{bi} & Y_{bb} \end{bmatrix}; \quad Y_{\text{ov}} \triangleq \begin{bmatrix} Y_{bb} \end{bmatrix}. \tag{4.54}
\]

A numerical model is commonly used as a parent model, which provides the full DoF set and an experimental model is used as an overlay model, which provides the dynamic properties. The equation of motion for the uncoupled model can be formulated as follows:

\[
u = Y (f + g); \text{ where } Y = \begin{bmatrix} Y_{\text{par}} & 0 & 0 \\ 0 & -Y_{\text{rem}} & 0 \\ 0 & 0 & Y_{\text{ov}} \end{bmatrix}, \quad f = \begin{bmatrix} f_{\text{par}} \\ f_{\text{rem}} \\ f_{\text{ov}} \end{bmatrix}, \quad g = \begin{bmatrix} g_{\text{par}} \\ g_{\text{rem}} \\ g_{\text{ov}} \end{bmatrix}, \tag{4.55}
\]

where \( Y_{\text{par}} \) denotes the admittance FRF matrix of the parent model (numerical model), \( Y_{\text{rem}} \) the removed model and \( Y_{\text{ov}} \) the overlay model (experimental model). Vector \( f \) represents the forces applied to the system and vector \( g \) the interface forces of each model. Note that the admittance of the removed model needs to be negative to decouple the dynamics of the parent model.

The parent model can be decoupled as a whole or as a reduced model only at the overlay DoFs. However, if the dynamics of the parent model are decoupled only at the overlay DoFs, spurious peaks can occur in the hybrid model [20]. This is due to the condensed removed interface \( Y_{\text{rem}} \), as not all the dynamic properties of the parent model are removed in the decoupling step. The problem with the conflicting dynamics can be resolved by expanding the size of the removed interface and decoupling the dynamics of the parent model as a whole (\( Y_{\text{rem}} = Y_{\text{par}} \)). Various sizes of the removed model are depicted in Fig. 4.18. Here only the use of the extended interface is derived. For other cases, the interested reader is refereed to [20].

In order to couple or uncouple the different models using the LM FBS [11] the two conditions must be satisfied at the interface. The first is the condition of compatibility:

\[
u_{\text{par}}^i - u_{\text{rem}}^i = 0; \quad u_{\text{ov}}^i - u_{\text{ov}}^i = 0, \tag{4.56}
\]

where the subscript \( g \) denotes the global DoF set combining the boundary and the internal DoF set. The second condition is the force equilibrium:

\[
g_{\text{par}}^b + g_{\text{rem}}^b = 0; \quad g_{\text{ov}}^b + g_{\text{par}}^i + g_{\text{rem}}^i + g_{\text{ov}}^i = 0. \tag{4.57}
\]
Both conditions can be written in matrix notation using a signed Boolean matrix $B$ and a set of Lagrange multipliers $\lambda$ as follows:

$$Bu = 0, \quad g = -B^T \lambda; \quad \text{where} \quad B = \begin{bmatrix} I & 0 & -I & 0 & 0 \\ 0 & I & 0 & -I & 0 \\ 0 & 0 & I & 0 & -I \end{bmatrix}. \quad (4.58)$$

The equations can be solved using the LM FBS method \[11\]. By utilizing a primal admittance notation, a single-line formulation can be derived for the hybrid SEMM model \[20\]:

$$Y_{SEMM} = Y_{gg} - Y_{gg}^{par}(Y_{rem}^{bg})^+(Y_{rem}^{bb} - Y_{ov}^{bb})(Y_{rem}^{gb})^+Y_{gg}^{par}. \quad (4.59)$$

The SEMM can also be regarded as an expansion technique, in which a limited experimental DoF set (overlay model) is projected onto the larger numerical DoF set (parent model). One of the primary advantages of the SEMM is that even though a DoF set is expanded, the final hybrid model remains a full-rank model. This is the main difference between the SEMM and other popular expansion techniques, such as the physical expansion methods \[22, 87\] or modal expansion \[24, 88, 89\] techniques, where the full-rank model is lost in the condensation process \[90\].

### 4.3.1 Singular-value based filter

Experimental measurements are always influenced by measurement uncertainty and bias. Therefore, the overlay model used in SEMM will to some extent always contain measurement error. This error is then projected on the parent model and it will always
be present in the hybrid model. Based on the conditioning of the parent model, these
expanded errors may be much larger than the original error [91].

Recall the final single-line formulation of the hybrid model, based on the fully-extended
SEMM method:

\[
Y_{SEMM} = Y_{gg} - Y_{par}^{bg} (Y_{rem}^{bg} + Y_{cov}^{bg}) + Y_{rem}^{gb} Y_{par}^{gg}.
\] (4.60)

The first inverse defines the projection of response and the second inverse defines the
projection of forces. By truncating both inverses at each frequency line a singular-
value based filter can be introduced to SEMM [91]. A general inverse can be found by
utilizing a property of the SVD such as follows:

\[
(Y_{rem}^{bg})^+ = \sum_{i=1}^{N=n_u} U_i \Sigma_i^{-1} V_i^H; \quad (Y_{rem}^{gb})^+ = \sum_{j=1}^{N=n_f} U_j \Sigma_j^{-1} V_j^H.
\] (4.61)

The number of singular values is equal to the overlay model size (the number of output
\(n_u\) and input \(n_f\) locations). By choosing a reduced number of singular values a SVD-
truncated inverse can be obtained:

\[
(Y_{rem}^{bg})^+ = \sum_{i=1}^{N=r_u} U_i \Sigma_i^{-1} V_i^H; \quad (Y_{rem}^{gb})^+ = \sum_{j=1}^{N=r_f} U_j \Sigma_j^{-1} V_j^H.
\] (4.62)

With this reduction (\(r_u < n_u\) and \(r_f < n_f\)) the projection is reduced in a least-square
manner and smaller inconsistencies within the overlay model are reduced. One of the
beneficial characteristics of the SV-based filter is that the hybrid model will remain
full-rank regardless of the number of singular values discarded [91]. Since the reduction
can be defined for each frequency line a user can define a different reduction for the
low- and the high-frequency range.

After applying the SV-based filter, the hybrid model interface FRFs (denoted by \(b\)) are
no longer identical to the overlay model FRFs. It was shown on a simulated numerical
case, that the SV-based filter can reduce the impact of measurement uncertainties and
bias [91]. The main limitation of the SV-based filter is choosing the number of singular
values to be truncated, as the optimal procedure still undefined and the chosen number
is left to the user.

### 4.3.2 Reduced size of the parent model

SEMM is generally used to mix a numerical model (the parent model provides the DoF
set) with an experimental model (the overlay model provides the dynamic properties).
Therefore, a full-response model for the parent model is always accessible from the

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6The SV-based filter in SEMM can be regarded in a way similar as to removing a mode shape from
a projection step in SEREP.
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Due to the use of an experimental model as the parent model, the full-response model cannot be obtained in practice. Commonly, a much smaller number of input DoFs will be present with an experimental parent model. A reduced example of a parent model is depicted in Fig. 4.19. A high number of output DoFs can be obtained even with an experiment by utilizing for example full-field high-speed camera displacement identification.

![Figure 4.19: Different size of the parent model](image)

Figure 4.19: Different size of the parent model [39]: a) basic SEMM, b) SEMM with the reduced size of the parent model.

The reduced size of the parent model and with that also the reduced size of the removed model imposes certain limitations on the decoupling step in the SEMM. If a full-response model is used for the parent model and subsequently also for the removed model, the hybrid model on the interface DoF only includes the dynamic properties of the overlay model (the dynamic properties of the parent model are removed completely). This is not the case with the use of a reduced parent model, where at the interface DoF the dynamic properties of the parent model will still be present, even after the decoupling step [39]. The reason for this is the relatively poor observability and controllability of the reduced interface. If the leftover parent dynamics were to impose a problem, an objective criterion, such as the Interface Completeness Criterion (ICC), should be used to evaluate the quality of the decoupling step [92].

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When referring to a full-response model a full-FRF admittance matrix is considered. The FRFs for each response DoF are determined for the excitation at each separate DoF; therefore, the admittance matrix for each frequency line is in fact a square matrix.
5 Application of DS and TPA on complex systems

Dynamic substructuring can be an invaluable experimental tool for analysing complex dynamic systems. However, the experimental application of frequency based substructuring, is often considered to be challenging in practice, due to the several limitations and possible problems. The required experimental measurements of FRFs matrices and of operational response have to be performed accurately, where careful consideration should be taken to minimize random error (noisy cables) and systematic bias (deviations in impact location and orientation) as much as possible. However, even if the experiment is performed with sufficient accuracy, the consistency of the predicted results is not certain. Additionally, proper controllability and observability of all interfaces between considered substructures have to be guaranteed.

Even with all the limitations, the advantages of dynamic substructuring prevail. If properly applied the dynamic substructuring can give an invaluable insight into complex dynamic systems. Access to the ability to assemble and disassemble substructures in a FEM-compatible way offers great potential. Both numerically and experimentally modelled substructures can be connected. Furthermore, by utilizing the transfer path analysis methodology the various transfer paths through the dynamic system can be evaluated. By considering the partial contribution of each transmission path, the most dominant one can be identified and, if required, addressed.

In this chapter, four applications of frequency based substructuring are shown. First, an application of coupling and decoupling is performed on an automotive test structure (Section 5.1) [38]. Next, an application of a hybrid approach for estimating accurate full-field FRFs from high-speed camera data is shown (Section 5.2), which was reported in [39] and is considered one of the scientific contributions of this thesis. The third section shows an application of singular vector transformation in both coupling and decoupling application (Section 5.3). The SVT application on a decoupling application was reported in [40], here the method was extended to a coupling application. The final section shows an application of in-situ TPA to identify equivalent forces of a BPM electric motor (Section 5.4).
5.1 Experimental substructuring on automotive test-structure

This section demonstrates a practical application of the LM FBS using the standard and expanded virtual point transformation on the test structure \cite{98}. The test structure is designed to mimic the dynamics of an engine unit flexibly mounted on a chassis in a real car (Fig. 5.3). Rubber mounts from the automotive industry are used to suspend a steel plate holding the excitation source, which is an electrodynamic shaker. The structure is intended as a laboratory test bench for an application of different concepts of dynamic substructuring and transfer-path analysis \cite{93}.

![Schematic representation of the automotive laboratory test-structure](image)

Figure 5.1: Schematic representation of the automotive laboratory test-structure \cite{93}.

The coupling is performed between the source $Y^A$ and the receiver structure $Y^B$, which is the frame with the rubber bushings included. However, the VPT measurements cannot be performed directly on the receiver side due to insufficient space for positioning sensors and impacts. A Transmission Simulator (TS) $Y^{TS}$ (Fig. 5.3b) is used to enable measurements around the rubber bushings \cite{94}. A schematic representation of the whole substructuring process is depicted in Fig. 5.2. Moreover, the numerical condition

\[
Y_{AB} = Y_A + Y_{BTS} - Y_{TS}
\]

Figure 5.2: Schematic representation of the whole substructuring process.

of the coupling procedure is increased, since the TS used has dynamic properties that
are similar to the source structure. Accordingly, the receiver structure in the decoupled state, $Y_{BTS}$, behaves in a way that is similar to that of the final assembled state, $Y_{AB}$ (Fig. 5.3a [94]). Usually, a numerical model of the TS is used to increase the quality of the decoupling; however, with this setup, an experimental model of the TS was used. The final DS process can be represented with a simple equation, where the circled plus/minus sign denotes the coupling/decoupling procedure:

$$Y_{AB} = Y_{BTS} \ominus Y_{TS} \oplus Y_{A},$$

where $Y_{BTS}$ denotes the assembly of substructures $Y_{B}$ and $Y_{TS}$.

### 5.1.1 Application of the VPT

The virtual point transformation was applied to all three experimental models $Y_{BTS}$, $Y_{TS}$, and $Y_{A}$. One virtual point for each rubber bushing was used, and altogether three VPTs were performed (Fig. 5.3). The rubber bushing at VP$_1$ represents the transmission mount, VP$_2$ is the rubber bushing at the roll mount and VP$_3$ is at the engine mount’s rubber bushing.

The final assembly structure $Y_{AB}$ (used as a reference) and $Y_{BTS}$ were rigidly fixed on a stiff and vibration-free test table during the measurements. The source $Y_{A}$ and transmission simulator $Y_{TS}$ were measured in a free-free condition (Fig. 5.5b). All the FRFs were measured by impact testing using a modal hammer with a vinyl tip. For the standard VPT, three tri-axial accelerometers (Kistler Type 8688A) were used and
nine different impact locations for each VP. Altogether, 27 translation channels with 27 impact locations were used for the standard VPT.

A rotational accelerometer (Kistler Type 8840) was used to measure the rotational response for the extended VPT. The rotational accelerometer used measures only the rotation around one axis, and only one rotational accelerometer was used for the whole measurement setup. Therefore, the rotational accelerometer was used as a roving sensor to acquire three rotational FRFs for each VP (as depicted in Fig. 4.9 for two directions).

Figure 5.4: Positions of the rotational accelerometer [38]: a) VP1 direction -RX; b) VP3 direction +RX.

Figure 5.5: Transmission simulator structure [38]: a) schematic representation of VPs together with visible impact locations depicted with arrows in different color for each VP; b) photograph of the experimental setup.
Quality of the VP transformation

As outlined before, the VP transformation was performed on all three experimental models; however, the results shown for the VPT quality in this section are only for the receiver structure with the TS $Y^{BTS}$. The quality of both transformations can be estimated from the frequency-averaged coherence criterion for reciprocity depicted in Fig. 5.6. The average reciprocity of the standard VPT is 58%, while the expanded VPT with rotations has an average reciprocity of 61%.

![Expanded VPT](image1)

![Standard VPT](image2)

Figure 5.6: Frequency-averaged reciprocity of the VPT for all three virtual points of the receiver structure with the transmission simulator $Y^{BTS}$: a) standard VPT; b) expanded VPT.

The expanded VPT slightly outperforms the standard VPT, as can be seen from the average reciprocity criterion. A similar observation can be made if we compare the separate FRFs: in Fig. 5.7, the reciprocal transformed FRFs on $Y^{BTS}$ $VP_1$-$y$/$VP_1$-$\theta_x$ (2,4) and $VP_1$-$\theta_x$/$VP_1$-$y$ (4,2) are shown. The expanded VPT outperforms the standard in the higher frequency range, and in the low-frequency range the FRFs obtained with the standard VPT are more reciprocal.

---

1The first number in the bracket (2,4) refers to the output position and the second to the input position in the FRFs matrix i.e. (output,input).
Figure 5.7: Comparison of specific virtual point FRFs $\text{VP}_{1-y}/\text{VP}_{1-\theta_z} (2,4)$ and $\text{VP}_{1-\theta_z}/\text{VP}_{1-y} (4,2)$ for the receiver structure with the transmission simulator $\text{Y}^{\text{BTS}}$ [38]: a) expanded VPT; b) standard VPT.

Figure 5.8: Comparison of the FRFs $\text{VP}_{2-x}/\text{VP}_{3-\theta_z} (7,18)$ and $\text{VP}_{3-\theta_z}/\text{VP}_{2-x} (18,7)$ for the receiver structure with the transmission simulator $\text{Y}^{\text{BTS}}$ [38]: a) expanded VPT; b) standard VPT.

For both transformations a relatively high sensor and impact consistency were obtained (on average above 90% for the specific as well as for the overall consistency). Therefore, the positions and orientations of the impacts and sensors can be assumed to be accurate, and the assumption of the interface rigidness in the frequency range of interest is satisfied. However, a high-quality VPT does not necessarily guarantee consistent coupling results. In the following section the final coupling results are presented, and both transformations are compared.
5.1.2 Dynamic substructuring results

The final assembled system and the positions of the reference sensors and impact positions are depicted in Fig. 5.9. The reference FRFs were measured in the assembled system to gain a reference with which the whole coupling procedure can be validated.

Figure 5.9: A view of the assembled system $Y^{AB}$ together with the reference sensors (denoted with letter S) and impact positions (denoted with letter I) [38].

The coupling FRFs are obtained using the LM FBS notation. A signed Boolean matrix $B$ was used to define the interface DoF for the coupling and decoupling procedure. SVD truncation was used on the interface flexibility matrix $Y_{int}$ to remove the six smallest singular values before applying the matrix inversion in LM-FBS (Eq. (2.28)) to decrease the errors of the coupling procedure [95].

In Fig. 5.10 the FRF $S_{15z}/I_{10}$ is shown for both VP transformations, together with a reference. It is clear that with the expanded VPT a more consistent FRF is obtained. Similar results can be observed for the FRF $S_{15x}/I_{23}$ shown in Fig. 5.11. The coupling results from the rotational accelerometer are in agreement with the reference, especially in the low-frequency range. At higher frequencies the results from standard and expanded VPT are similar. One of the possible reasons is that the assumption of the local rigidity is not completely valid any more, and the residual motion (Eq. (4.15)) after the transformation will contain not only the measurement error (which should be lower for the expanded VPT) but also the flexible motion around the interface. Therefore, both transformations will yield similar results as the primary source of error is the flexibility of the interface. In order to show the overall quality of the coupling results for both

2With the three VPs on the TS, a certain level of over-determination can be expected at the interface, since the whole block is in fact rigid in the low-frequency range. Therefore, even a small error in one of the VPTs can lead to erroneous coupling results. Applying a truncated SVD on the interface flexibility $Y_{int} = BY^{AB}B^T$ can be interpreted as a weakening of the interface compatibility, which is advantageous in this configuration.
Figure 5.10: Comparison of the final assembly FRF $S_{15z/I0}$ obtained with a standard and an expanded VPT, together with a reference measurement \[38\].

Figure 5.11: Comparison of the final assembly FRF $S_{15z/I23}$ obtained with a standard and an expanded VPT, together with a reference measurement \[38\].

transformations, a coherence criterion (Eq. (4.19)) is used to compare the coupling and reference results. The frequency-averaged value of the coherence criterion is shown in Fig. 5.12. It is clear that the expansion of the VPT with a rotational response increases the consistency of the coupling procedure for all the FRFs.
5.1.3 Discussion

Extending the virtual point transformation with a directly measured rotational response can improve the consistency of the coupled FRFs. With the introduction of a rotational weighting matrix in the VPT, the residuals on the rotation and translation sensors are minimized in a comparable norm. With the inclusion of a rotational response, the virtual point transformation can become less sensitive to small deviations in the positions and measurement sensitivity of the sensors.

An application of the extended VPT is presented on a test structure, mimicking the dynamics of an engine unit flexibly mounted on a chassis. The coupling results are obtained with the expanded VPT and are compared with the standard VPT. The additional information on the rotational FRFs of the interface yields a more accurate coupling result, which can be observed on all the reference FRFs. Currently, applying the proposed expansion of the VPT is hindered by the poor availability of rotational accelerometers, as only uni-axial rotational accelerometers are available on the market. Additionally, due to the limited number of rotational accelerometers, the measurements usually cannot be performed within a single measurement campaign, which can be time consuming if a large number of VPs are present in a complex structure.
5.2 Using SEMM to couple equivalent experimental models

It is not uncommon to perform experimental measurements with various measurements systems to obtain equivalent experimental models, which are to some extent, overlapping. For example, with the use of high-speed camera, a dense full-field response can be identified, as each separate pixel can be used for displacement identification. However, the displacement identification from optical measurement systems is hindered by a relatively high levels of noise. To resolve the problems associated with high levels of noise, separate, more accurate measurements have to be performed.

A typical application is a full-field mode shape identification. The displacements caused by vibrations are generally within or even below the noise level [96, 97], especially in the high-frequency range. Nevertheless, modal identification is still possible if a combination of accelerometer and high-speed camera data is used in the LSCF/LSFD identification procedure [98]. However, the real and imaginary part of the reconstructed FRFs after mode shape identification are not consistent with a reference measurement in the range where the noise is predominant.

System equivalent model mixing can be used to resolve the erroneously identified full-field FRFs, as two different response models are already used for combined accelerometer/camera mode-shape identification [98]. The whole concept of hybrid estimation was introduced in [39], where two equivalent experimental models are mixed (section 4.3.2). In this section, only the main results are shown and the main advantages and disadvantages are outlined. A schematic depiction of the whole hybrid-identification approach for the full-field FRFs estimation is shown in Fig. 5.13. First, the displacement is identified from the high-speed-camera measurements. Simultaneously, an accurate response model from the accelerometer measurements can be obtained. In the next step, the mode shapes are estimated based on the combined accelerometer/camera identification [98]. Finally, the reconstructed FRFs from the estimated mode shapes are used as a parent model and the accelerometer measurements are used as the overlay model in SEMM.

![Figure 5.13: Schematic depiction of the identification of full-field hybrid FRFs from high-speed-camera measurements using SEMM [39].](image)

The SEMM is generally used to mix a numerical model (the parent model provides
the DoF set) with an experimental model (the overlay model provides the dynamic properties). Therefore, a full-response model for the parent model is always accessible from the numerical model. On the other hand, if the experimental model is used as the parent model, the full-response model cannot be obtained practically.

The reduced size of the parent model and with that also the reduced size of the removed model imposes certain limitations on the decoupling step in the SEMM. If a full-response model is used for the parent model and subsequently also for the removed model, the hybrid model on the interface DoF only includes the dynamic properties of the overlay model (the dynamic properties of the parent model are removed completely). This is not the case with the use of a reduced parent model, where the dynamic properties of the parent model will still be present at the interface DoF, even after the decoupling step. The reason for this is the relatively poor observability and controllability of the reduced interface. If the leftover parent dynamics were to impose a problem, an objective criterion, such as the Interface Completeness Criterion (ICC), should be used to evaluate the quality of the decoupling step.

The proposed hybrid estimation of the full-field FRFs can be summarized as follows:

1. Displacement identification from high-speed-camera measurements.
2. Combined accelerometer/camera mode-shape estimation from system complex eigenvalues identified from accelerometer data.
3. Reconstruction of the full-field FRFs based on the identified mode shapes.
4. Calculation of a hybrid model from the reconstructed FRFs with the accelerometer measurements using SEMM.

5.2.1 Mode shape identification on optical flow data

High-speed-camera-based measurements produce a sequence of images that carry information about the motion of objects. Each pixel has an intensity value that changes over time, as the object changes its position on the image. Simplified Gradient-Based Optical Flow (SGBOF) can be used for the displacement identification on each separate image pixel. If the local intensity gradient is assumed to be constant, a linear relation between the change in pixel intensity $\Delta I$ and object displacement $\Delta s$ can be obtained as follows:

$$\Delta s(x, y, t) = \frac{I(x, y, t) - I(x, y, t + \Delta t)}{|\nabla I_0|} \pm \sqrt{\Delta x_L^2 + \Delta y_L^2},$$

(5.2)

where $\Delta x_L$ and $\Delta y_L$ are the integer displacements used when the object’s motion exceeds half of a pixel in the $x$ or $y$ direction. $|\nabla I_0|$ is the scalar value of the intensity gradient and $I_0$ is the intensity gradient of the reference image. Since $\Delta s$ represents
the absolute displacements, the direction correction needs to be used if the directional displacements are required:

$$\Delta x(x, y, t) = \frac{\partial I}{\partial x} \left| \nabla I_0 \right| \Delta s(x, y, t).$$  \hspace{1cm} (5.3)

A high-intensity gradient is needed for a consistent displacement identification, since the sensitivity of the SGBOF increases with the intensity gradient (Eq. (5.2)). A speckle pattern or a similar pattern design is used on the measured surface to increase the intensity gradient. With a single high-speed camera in one set-up only 2D displacements can be measured. However, if a stereoscopic set-up is used, 3D measurements can also be performed \cite{99}. Recently, frequency-domain triangulation was introduced for 3D operating-deflection-shape identification \cite{100}.

Even with a proper pattern the measured data is generally relatively noisy. The noise imposes a problem on modal identification, especially in the frequency range where the amplitude of the measured response is close to the overall noise level. The use of advanced modal identification techniques on the identified displacements with the SGBOF, such as the Least-Squares Complex Frequency (LSCF) method \cite{101}, are commonly inconsistent, or the modal parameters cannot even be identified. However, this problem can be resolved by using a combined accelerometer/camera identification \cite{98}. First, the system’s complex eigenvalues $\text{acc} \lambda_r$ are obtained from the accelerometer measurements. These contain the natural frequencies $\omega_r$ and the damping ratios $\zeta_r$:

$$\text{acc} \lambda_r = -\zeta_r \omega_r \pm i \omega_r \sqrt{1 - \zeta_r^2}. \hspace{1cm} (5.4)$$

The identified $\text{acc} \lambda_r$ are then applied to the Least-Squares Frequency Domain (LSFD) method \cite{102} to determine the modal constants $rA_j$ together with the upper and lower residuals $A_U$ and $A_L$:

$$\text{cam} Y_j(\omega) = \sum_{r=1}^{N} \left( \frac{rA_j}{\omega - \text{acc} \lambda_r} + \frac{rA_j^*}{\omega - \text{acc} \lambda_r^*} \right) - \frac{A_L}{\omega^2} + A_U, \hspace{1cm} (5.5)$$

where * denotes a complex conjugate and the subscript $j$ the location of each separate pixel. For each location, all the frequency points are taken into account to construct an over-determined set of equations from Eq. (5.5). Each equation set is then solved for the modal constants, upper and lower residuals using the least-squares method.

### 5.2.2 Hybrid model from accelerometer and high-speed-camera data

An experiment was performed on a solid steel beam with dimensions $12 \times 40 \times 600$ mm. The beam was supported on polyurethane-foam blocks representing approximately free-free boundary conditions. The experimental setup is shown in Figure 5.14. Two different experimental models were acquired, the first one from six equally spaced accelerometers and the second one from the high-speed camera. An additional accelerometer was used as a reference, where the proposed methodology was validated.

\footnote{The LSCF/LSFD modal parameter estimation is supported in open source Python package pyEMA \cite{103}.}
Dytran 3097A2T uni-axial accelerometers weighing 4.3 g were used to measure the response and a PCB 086C03 modal hammer with a hard metal tip was used to excite the structure. A Fastcam SA-Z high-speed camera was used to measure the response. A pattern with horizontal lines was added to the front face of the beam in order to maximize the pixel intensity gradient in the vertical direction. The camera was set to have a resolution of 1024×48 pixels and was recording at 200000 frames per second. The displacements were identified at 1000 points along the length of the beam with the SGBOF (Eq. (5.3)).

![Experimental setup with high-speed camera](image)

Figure 5.14: Photograph of the experimental setup with the high-speed camera [39].

### 5.2.2.1 Identification of the modal parameters

A combined accelerometer/camera mode-shape identification [98] was used to identify the mode shapes of the steel beam. System complex eigenvalues from the accelerometer model were determined from a stabilization diagram using the LSCF. Accelerometer eigenvalues were then used in the LSFD identification (Eq. (5.5)), and, altogether, eight mode shapes were determined in the frequency range up to 6 kHz.

All the identified mode shapes are shown in Fig. 5.15. The advantage of a full-field identification of the mode shapes can be clearly seen in the increased spatial resolution. The limitations of discrete measurements using accelerometers are also observable also from the AutoMAC criterion depicted in Fig. 5.16. Up to the 4th mode shape, the spatial resolution with the discrete accelerometers measurements is sufficient to identify the corresponding mode shapes. However, in the high frequency range, the spatial resolution of accelerometer is not adequate to deduce the corresponding mode shapes. This can be interpreted as a spatial aliasing and consequently the real mode shape cannot be identified. Similar observation can be made when comparing mode

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5Modal Assurance Criterion (MAC) [39] is defined as a scalar constant relating to the degree of consistency (linearity) between two mode shapes. The AutoMAC compares a set of mode shapes to itself using a MAC criterion.
shapes from the accelerometer and the high-speed camera using the MAC criterion (Fig. 5.16).

Figure 5.15: Identified mode shapes from high-speed camera and accelerometers using the LSCF/LSFD identification [39].

Figure 5.16: Comparison of identified mode shapes from high-speed camera and accelerometers using an AutoMAC and MAC criterion [39]: a) AutoMAC high-speed camera; b) AutoMAC accelerometers c) MAC high-speed camera/accelerometers.
5.2.2.2 Hybrid full-field FRFs estimation

In the final step a hybrid model is formed from the high-speed-camera and accelerometer measurements. With the proposed methodology, an experimental model is used as the parent model. Therefore, a full-response model for the parent model is not available. In the current configuration, one would need to measure at 1000 different impact locations to acquire the full-response model. That is not achievable in practice, and for this reason a smaller response model was used for the construction of the hybrid model. In this research only one impact location was used. The dimensions of each separate model are therefore equal to:

\[ Y_{\text{par}}^{\text{cam}} \in \mathbb{C}^{1000 \times 1}, \quad Y_{\text{ov}}^{\text{acc}} \in \mathbb{C}^{6 \times 1}, \quad Y_{\text{hybrid}}^{\text{SEMM}} \in \mathbb{C}^{1000 \times 1}. \]  

Equation (5.6)

A schematic representation of each used model is shown in Fig. 5.17, together with the location of the impact.

![Figure 5.17: Schematic representation of the different models for the hybrid estimation of full-field FRFs](image)

The final hybrid FRF model at the reference position is shown in Fig. 5.18 for the real part of the FRF and in Fig. 5.19 for the imaginary part. Additionally, the reference measurement, the reconstructed FRFs and the identified displacements from the high-speed camera are shown for a side-by-side comparison. Below the 2-kHz range, where the noise on the identified displacements from the high-speed camera is not dominant, the overall shape of the reconstructed FRFs from the combined accelerometer/camera

![Figure 5.18: Comparison of the real part of different FRFs at the reference position](image)
approach agrees well with reference measurement. Above 2 kHz, the noise becomes dominant and the deviation in the displacement amplitude as well as an inconsistency in the overall shape of the FRF can be observed more clearly. In contrast, the proposed hybrid model accurately predicts the shape and the amplitude of both the real and the imaginary parts of the reconstructed FRFs in the low- as well as in the high-frequency range. The inconsistent shape of the real and imaginary parts in the high-frequency range (combined accelerometer/camera identification) indicates the discrepancies in the overall phase of the FRFs. These deviations can be clearly seen from a Nyquist diagram in the region around 6th and 8th natural frequencies (Fig. 5.20). However, the alignment of the reconstructed FRFs from the hybrid model correlates well with the FRFs from the reference acceleration measurement. This indicates that the hybrid model correctly predicts the phase angle of the dynamic response.

A coherence criterion [26] was used to objectively evaluate the performance and accuracy of the hybrid approach. In the lower-frequency range (for the first three natural frequencies), high coherence values of the FRFs can be observed, both for the combined accelerometer/camera and the hybrid approach (Fig. 5.21). In the higher-frequency range, the coherence values of the FRFs obtained using the combined accelerometer/camera approach gradually decrease. However, the hybrid approach retains the high coherence values of the FRFs also in this frequency region as well. The hybrid approach, in which two different experimental models are mixed using a dynamic substructuring approach, can evidently increase the reliability and consistency of the full-field FRFs estimated from the high-speed-camera data.

5.2.2.3 Size of the overlay model

The required numbers of DoFs used in the overlay model depends on the dynamic properties of the observed structure. Nevertheless, even an overlay model with a relatively small size of DoFs can be used to form a consistent hybrid model [90]. Therefore, the required size of the overlay model is case specific. The overlay model should be able to
observe the relevant dynamic properties of the system (even one DoF may be sufficient in certain limit cases).

For the displayed experimental case, the number of DoFs used in the overlay model can be reduced. In Fig. 5.22, a comparison of two different hybrid models is depicted. The first hybrid model is where all 6 DoFs measured with the accelerometers are included. The second hybrid model is formed from a reduced overlay model where only 2 DoFs are used (the locations used for both hybrid models are graphically depicted in Figure 5.22 legend). It can be observed that even with a reduced overlay model the inconsistencies in real and imaginary part of the FRFs in the high frequency range are resolved.

5.2.3 Discussion

The proposed methodology relies on the identified DoFs of the high speed camera which are used as a parent model in SEMM (i.e. no numerical or analytical model is required for the expansion). Therefore, only the identified DoFs are available in the full-field FRF estimation. If only a single-camera system is used, the out-of-plane motion cannot be identified from a single set-up. If a stereo pair is used, also the out-of-plane DoF can be identified [104]. In addition, if a large part of the structure is hidden from the camera field of view, those DoF cannot be identified. However, a
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Figure 5.21: Comparison of mean coherence values around each natural frequency for both hybrid and combined accelerometer/camera FRFs evaluated with the reference measurement [39].

Figure 5.22: Comparison of different FRFs at the reference position displayed in 3D [39]. Reference accelerometer (---), hybrid model (- - - -), hybrid model - reduced (-- -- --).

multi-view system [103] or frequency domain triangulation [100] can be used to acquire spatial measurements that would otherwise not be available.

Furthermore, SEMM could also be used to expand the identified full-field FRFs on DoF that are outside the field of view. This can be achieved by using a numerical model or analytical as a parent model. A similar methodology was implemented in the modal domain with System Equivalent Reduction Expansion Process (SEREP) [106], where strain shape expansion was applied on optical measurements.
5.3 Applications of SVT in DS

Singular Vector Transformation (SVT) was introduced in the theory section 4.2. The SVT projects the measured datasets into a subspace composed by dominant singular vectors. The reduced orthonormal frequency dependent basis is defined directly from the measured FRFs. For defining the reduction, no geometrical or analytical model is required, which can be highly advantageous. If proper observability and controllability is guaranteed the reduction can include not only rigid, but also the most dominant flexible modes of interest over the whole frequency range of interest.

In this section the SVT is applied on a substructuring test structure, which is intended to benchmark various substructuring methodologies. First, an application of SVT is shown on a decoupling process, where both numerical and experimental examples are evaluated. Second, an application of SVT is shown on a coupling process, where a TS structure is introduced to enable the required collocation between inputs and outputs.

5.3.1 AM substructuring test structure

Typically various methodologies in dynamic substructuring are tested and validated on a benchmark structures. Benchmark structures are commonly simple structures that are easy to manufacture and are designed to exhibit some of the characteristics of a complex real-life products.

The "AM" benchmark structure consists of two different subsystems:
- A - shaped component / substructure A (Fig. 5.23a),
- S - shaped component / substructure B (Fig. 5.23b).

![Figure 5.23: AM benchmark structure: a) substructure A; b) substructure B.](image)

Both substructures are machined from a single aluminium block and provide multiple connection options, with which various boundary conditions can be achieved (free-free, or fixed-free, or fixed-fixed). The substructures can also be connected in various configurations, such as point interface (Fig. 5.24a) and continuous interface (Fig. 5.24b).

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*The design concept is similar to the benchmark structure presented in [107]*.
The main idea behind the AM benchmark structure is to have a lightly damped system which is easily manufactured and numerically modelled. Both substructures are designed to display a considerable number of well-separated flexible vibration modes in all directions within the frequency range of interest, i.e. 0-2000 Hz. Furthermore, the first natural frequencies of both structures are above 100 Hz to avoid problems with experimental rigid body modes with the free-free boundary condition configuration.

![Figure 5.24: Two different configurations of AM benchmark structure: a) point interface; b) continuous interface.](image)

### 5.3.2 A decoupling application with a point interface

First the SVT application is shown on a decoupling application with a point interface between the two substructures A and B. Therefore, only one relatively small contact area is shared between the two elements. The final result obtained by decoupling substructure B from the assembled configuration AB, as depicted in Fig. 5.25.

![Figure 5.25: A decoupling application with a point interface.](image)

The decoupling process is first performed on a numerical test case and afterwards the same configuration is applied to an experimental test case. Both in numerical and experimental examples, the SVT is compared with some of the most common state-of-the-art decoupling strategies.
5.3.2.1 Numerical test case

For the numerical example, the FRFs are synthesized with mode superposition by using a 100 modes and a modal damping of 0.3% is added to all modes (considered to be lightly damped system). In Fig. 5.26 two mode shapes of the assembled configuration are depicted. The substructure A is fixed to the ground far away from the point interface, the fixed-support is depicted with two black cylinders. In order to simulate real-life measurements a Gaussian random distributed noise is added to both real and imaginary parts of the FRFs. The random noise represents the noise on input and output piezo transducers and is assumed to be uncorrelated due to the simplicity\[108]\[7\]

\[\begin{align*}
\text{Real part: } \sigma_r &= a|Y_{ij}| + b \\
\text{Imaginary part: } \sigma_i &= c|Y_{ij}| + d
\end{align*}\] (5.7)

Figure 5.26: Numerical mode shapes in the assembled configuration AB: a) 1\textsuperscript{st} mode shape; b) 4\textsuperscript{th} mode shape.

The configuration of inputs and outputs in the assembled AM configuration is depicted in Fig. 5.27. The assembly is fixed on two points where black cylindrical supports are depicted. The outputs, as well as are the inputs are collocated in each configurations, to account for the underlying assumption of the SVT. The inputs and outputs around the interface are positioned in the close proximity, to account for the local rigidity of the VPT. Additionally, inputs and outputs are uniformly distributed throughout the substructure B to observe and control as many of the modes as possible.

For the reference comparison the FRFs were synthesised on the A substructure. Altogether 6 reference inputs and outputs were simulated. For visualization purposes only two different FRFs (Fig. 5.28) are used to depict the comparison. However, similar conclusions could also be observed on all the other FRFs.

The decoupling step can be performed only at the interface or it can in fact be extended to include the internal DoFs (Section 2.4). This is, in fact, one of the ways increase the consistency of the decoupling results. In Fig. 5.29 the CMIF parameter of substructure B is shown for the interface DoFs and for the extended interface with internal DoFs. The first 6 singular values are highlighted, and it can be observed that the motion of the interface is in fact rigid in the low-frequency region.

7The following values were used in the FRF synthetization: \[a = 1 \cdot 10^{-3}, \, b = 8 \cdot 10^{-4}, \, c = 1 \cdot 10^{-3}, \, d = 7 \cdot 10^{-4}\].
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Figure 5.27: Overview of the assembled system AB with inputs (red), sensors (gray cubes) and associated output channels (blue).

Figure 5.28: A view of the structure A together with the reference channel and impact positions: a) location of FRF $Y_{14}$; b) location of FRF $Y_{25}$.

Figure 5.29: The Complex Mode Indicator Function (CMIF) of substructure B: a) only at the interface, b) interface and internal DoFs.
Standard interface

One of the key features of the SVT is also the ability to use different number of singular DoFs at the interface. In Figure 5.30, the effect of the number of used singular DoFs with standard interface decoupling is shown on the two reference FRFs. It can be observed that the higher number of singular DoFs does not necessary increase the quality of the decoupling results. It can be observed that with a higher number of singular DoFs the final decoupling results are exhibiting increased stiffness. However, as more DoFs are included in the decoupling process, also the errors are amplified, which directly explains the increased spuriosity.

![Graphs showing the effect of selected number of singular DoFs at the interface.](image)

Figure 5.30: The effect of selected number of singular DoFs at the interface: a) FRF $Y_{14}$; b) FRF $Y_{25}$. 

Amplitude [m/s²/N]
Extended interface

Better decoupling results are obtained with an extended interface, where the internal DoFs are included in the compatibility and equilibrium conditions. However, a similar observation can be made as with the interface decoupling. When the number of singular DoFs increases, the stiffness of the decoupled system in fact increases and also higher levels of spuriosity can be observed. The main problem remains how many singular DoFs should be taken for the optimal decoupling results. Nevertheless, the optimal number of singular DoFs should be frequency-dependent, as at a low-frequency range a lower number of singular DoFs yields better results. This can also be explained using the CMIF graph (Fig. 5.29), where it can be observed that at a higher frequency range a higher number of singular values is required to obtain the same ratio.

Figure 5.31: The effect of selected number of singular DoFs with extended interface: a) FRF $Y_{14}$; b) FRF $Y_{25}$. 
5.3.2.2 Experimental test case

For the experimental test case the same input/output configuration is used as for the numerical example. The experimental setup is shown in Fig. 5.32. Similarly to the numerical example three different configurations were measured. First the assembled configuration AB and then the substructures A and B separately. The final decoupling results are compared with the reference measurements performed directly on the substructure A.

The assembled configuration and the substructure B are fixed to a vibration-free table with cylindrical supports and bolted joints with a M10 hex screw with a tightening torque of 20 Nm. The free-free boundary condition on the substructure B is approximated with thin rubber ropes. The measurements were performed with impact excitation (SIMO), using a modal hammer with a vinyl tip. Standard triaxial acceleration piezo transducers with approximate sensitivity of 100 mV/g were used. For the FRF estimation, the H1-estimator was used.

![Figure 5.32: Experimental setup of the assembled configuration AB](image)

For performing the SVT only the measured FRF data is required. On the other hand, the VPT requires the positional information relative to the virtual point. The virtual point was placed in the center of the interface area, and only rigid IDMs are considered in the transformation.

Comparison of different state-of-the art approaches

Within the LM-FBS framework, there are various ways to decouple two substructures. In this section, the most commonly used methods are compared with reference to the presented experimental case. These are virtual point transformation (standard and with extended interface), direct decoupling (with extended interface) and singular vector transformation (standard and with extended interface)

For the visualization purposes, only the best results are selected and showed for all possible variations. In Fig. 5.33 the direct and SVT are compared with the reference measurements, and in Fig. 5.34 different VPT decoupling results are shown.
Some observations on each decoupling approach are provided below:

**Direct decoupling**
A direct decoupling approach is extended to internal DoFs and a truncated SVD is applied on the interface flexibility matrix \( Y_{\text{int}} \). The application of truncated SVD reduces the overall noise levels in the decoupling results. However, most resonance peaks are clipped, and the overall fit with the reference is rather poor.

**VPT decoupling**
While the overall decoupling results from the VPT are not accurate, they are more consistent than the direct decoupling. The VPT reduction spaces containing only the rigid IDMs seem to limit the controlled and the observed dynamics too much. The extended compatibility and equilibrium increases the overall consistency of the results. However, the problem of the propagation of random error and bias seems to be an additional problem of the VPT decoupling approach.

**SVT decoupling**
The SVT is performed either only at the interface DoFs or with extended internal DoFs. Applying the SVT only at the interface yields results that are similar to the standard VPT. The SVT with extend interface outperforms all other approaches and it seems to be less sensitive to measurements errors than the other approaches. In this application, the number of singular values used in the reduction was constant throughout the whole frequency range. The SVT results could be further improved by using a frequency-dependent number of singular DoFs.

![Comparison of direct and SVT decoupling approaches with the reference measurements on \( Y_B \): direct decoupling with extended interface and truncated SVD on the \( Y_{\text{int}} \) with a rank 3 truncation (---), SVT with only internal DoFs and rank 6 reduction spaces (----), SVT with extended interface and rank 6 reduction space (-----) and a reference FRF (-----).](image-url)
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Figure 5.34: Comparison of various VPT decoupling approaches with the reference measurements on $Y_B$ [40]: standard VPT (---), VPT with extended compatibility (——), VPT with extended interface and truncated SVD on the $Y_{int}$ with a rank 5 truncation (----) and a reference FRF (---).

**Condition number of the interface flexibility matrix**

Condition number is an indicator of how much the output value of a function can change for a small errors on the input. With the LM-FBS a well-conditioned of the interface flexibility matrix $Y_{int}$ is vital for consistent DS results. In Fig. 5.35 the condition number of all decoupling approaches is shown. It can be observed that both standard and extended SVT yield the lowest condition number, which means that the pseudo-
inverse is well-conditioned when compared to the rest of the decoupling approaches. Therefore, an increased robustness to standard measurements errors can be expected.

**Overall best decoupling results**

The extended SVT outperforms all other approaches throughout the whole frequency range in the current decoupling application. For the final validation the results are compared with the VPT interface decoupling which was the second-best overall (Fig. 5.36). The same comparison on the FRF $Y_{11}$ is depicted in Fig. 5.37. The SVT exhibits some spuriosity, which could be addressed by including a higher number of singular DoFs in the transformation.

Figure 5.36: Comparison of two best decoupling approaches on FRF $Y_{14}$: standard VPT (--), SVT with extended interface with rank 6 reduction space (---) and reference FRF (-----).

**5.3.3 A coupling application with a continuous interface**

Here a coupling application with the SVT is shown with a continuous interface, as depicted in Fig. 5.38. The underlying assumption of the SVT is the separate collocation of inputs/outputs. For a coupling case, the collocation is hard to obtain in practice, even with a continuous interface. However, if a transmission simulator structure is used in the coupling procedure then the required collocation can be achieved.

The whole coupling process is performed only on a numerical test case. The SVT is compared both with the VPT and the direct coupling approach. With the VPT only rigid IDMs are considered at the interface.

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8With the VPT the rigid IDMs could be extended with flexible IDMs as proposed in [28]. However, due to the complexity of the structure it would be hard to determine the analytical relation required for the extended deformation modes.
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Figure 5.37: Comparison of two best decoupling approaches FRF $Y_{25}$: standard VPT (---), SVT with extended interface with rank 6 reduction space (---) and reference FRF (---).

Figure 5.38: A coupling application with a continuous interface and a transmission simulator structure.

**Numerical test case**

The FRFs were synthesized with the same procedure as with the point interface example. A 100 modes were used in the mode superposition and a modal damping of 0.3% was added to all modes. Furthermore, the same randomly distributed noise was added to the FRFs to approximate real measurements. The configuration of inputs and outputs in the assembled configuration with a continuous interface is depicted in Fig. 5.41. The inputs and outputs are uniformly distributed throughout the transmission simulator to observe and control as much of the modes as possible. The outputs are collocated in each configurations, as well as are the inputs, to satisfy for the underlying assumption of the SVT.

The CMIF parameter of all three configurations is depicted in Fig. 5.40. The first 6 singular values throughout the whole frequency range are highlighted. It can be observed that the interface rigidness can only be assumed for the low frequency range (well bellow 500Hz). Therefore, it can be expected that the SVT will outperform the VPT throughout the whole frequency range.
The final coupling results were compared with the FRFs synthesised on the assembled configuration AB. 6 reference inputs and outputs were simulated, similar to the case of the point interface. However, the results are shown only on two different FRFs (Fig. 5.41) as the observations could be made on all FRFs.
Coupling results on FRF $Y_{25}$

In Fig. 5.42 the coupling results on FRF $Y_{25}$ are shown for different number of singular DoFs. It can be observed that with the higher number of DoFs we can observe an unwanted stiffening in the lower frequency range (bellow 500 Hz). This can be directly explained from the CMIF parameter, as in this range the interface is fairly rigid. Therefore, an increased number of DoFs at the interface has a negative effect in the low-frequency range.

In Fig. 5.43, the best overall results are shown for SVT, VPT and the direct coupling. It can be observed, that apart from some spuriousity below 250Hz range, the SVT outperforms both direct and VPT coupling. The direct coupling is highly affected by random error, which can be addressed, to some extent with the truncated SVD on the $Y_{int}$.

![Figure 5.42: Comparison of how different number of singular values effect the coupling results (FRF $Y_{25}$).](image)

![Figure 5.43: Comparison of coupling results with the reference for SVT, VPT and direct coupling approach (FRF $Y_{25}$).](image)
**Coupling results on FRF $Y_{12}$**

Similar stiffening with an increased number of singular DoFs can also be made on FRF $Y_{12}$ (Fig. 5.44). At higher frequencies (above 1kHz), a relatively high number of singular DoFs is required for consistent coupling results. Even by looking at the CMIF parameter (Fig. 5.41), the choice on the number of required singular DoFs is not straightforward. Nevertheless, the SVT outperforms both direct and the VPT coupling also on FRF $Y_{12}$ (Fig. 5.45).

![Figure 5.44](image1.png)  
Figure 5.44: Comparison of how different number of singular values effect the coupling results (FRF $Y_{12}$).

![Figure 5.45](image2.png)  
Figure 5.45: Comparison of coupling results with the reference for SVT, VPT and direct coupling approach (FRF $Y_{12}$).
5.3.4 Discussion

An experimental application of frequency based coupling and decoupling on a lightly damped system remains a challenge both in academia and industry. One of the main problems is the unyielding presence of errors in the data acquisition process (random error and bias). In this section an application of a novel promising technique called Singular Vector Transformation (SVT) was shown both on experimental and numerical examples. The SVT projects the acquired data into subspaces composed by dominant singular vector. The main assumption of the transformation is that the inputs/outputs are separately collocated. The frequency-dependent basis can, in fact, include both rigid and flexible deformation modes of the interface. Since the reduction basis is acquired directly from the measured dynamics, no geometrical or analytical model is required. Furthermore, due to the underlying properties of the singular value decomposition, the majority of random errors can be filtered out of the transformation.

In this section, a successful coupling and decoupling application with an SVT is displayed on a lightly damped system with both point and continuous interface. The SVT approach was compared with current state-of-the-art approaches in frequency-based substructuring. For the presented case of the coupling and decoupling application, the SVT outperformed all of the other approaches. Additionally, the SVT is in fact the most robust in terms of the sensitivity to measurements errors. Additionally, the condition number of the interface flexibility matrix is significantly reduced. The main drawback of the SVT is the underlying need for the separate collocation between inputs/outputs. For the coupling example, an additional structure called a transmission simulator is required to satisfy the collocation.

The choice for the number of singular DoFs used at the interface was made by evaluating the CMIF parameter at the interface. Also, a constant number of singular DoFs was used throughout the whole frequency range. The application of the SVT can in fact be improved by optimally varying the number of singular DoFs at each frequency line. A combination of the CMIF parameter and PRFs could be used to estimate the optimal number of singular DoFs; however, finding an optimal selection process was out of the scope of the current research.
5.4 In-situ source characterization for washing machine BPM motor

In this section, an in-situ TPA is applied on a washing machine drum and tub test-bench assembly. The test bench was designed to evaluate the performance of various Brushless Permanent Magnet (BMP) motors, which are commonly used in washing machines (Fig. 5.46). Various electric motors can be exchanged with ease and are easily accessible, which is ideal for impact testing. The test bench also enables the measurement of torque applied to the WM drum; however, torque measurements were not required for the current in-situ identification. The main goal was to characterize the equivalent forces from the electric motor at different speed settings subjected to various loads. The equivalent forces are used to evaluate electric motors from various suppliers. Here only the characterization of a single electric motor is presented.

Figure 5.46: Washing machine drum and tub test bench assembly.

Characterizing the source independently from the receiver dynamics can be highly beneficial. With the in-situ TPA, the equivalent forces can be obtained only from one experimental setup, directly from the original assembly. This can help with the realisation of the predefined vibro-acoustic target settings, which would otherwise become cumbersome if, for example, one were to test various electric motors on each product separately. Since the equivalent forces are independent of the receiver dynamics, the electric motor can be evaluated for the whole product line.

9The equivalent forces are often referred to as blocked forces, both in literature and in industrial practice. The name blocked forces is used as the blocked-force was the first commonly used method for source characterization.

10The FRFs of the assembled configuration have to be known in advance. However, the assembled FRFs can be estimated from a numerical model or by using a coupling approach.
5.4.1 Application of the VPT

The electric motor is connected through four rubber mounts to the drum and tub assembly. The coupling points can be considered as point interfaces, and the local rigidity required for the VPT can be assumed. For applying the in-situ TPA, the VPT is commonly used to define a set of virtual forces (3 forces and 3 moments per virtual point). In this setup, four VPs were used to characterize the equivalent forces. The locations of the VPs and also the reference locations on the receiver are depicted in Fig. 5.47.

For each virtual point 9 impact locations around each VP were used to excite the structure and three tri-axial accelerometers (indicator sensors $u_i$, Fig. 3.6) were used for each VP. The measured forces $f_2$ are transformed to virtual point forces $m_2$, based on the relative position to the VP:

$$Y_{u_i m_2} = Y_{u_i f_2} T_{f_2}^T,$$

(5.8)

where the $T_{f_2}^T$ is the force transformation matrix (Eq. (4.11)). A close-up view of one of the virtual point is shown in Fig. 5.48. The impact locations were placed directly on the motor and the indicator sensors were placed on the receiver structure. In this way, the vibro-isolations are regarded as a part of the receiver structure. To maximize the characterization quality each set of indicator sensors was placed in the proximity of the interface.

The overall quality of the VP transformation can be evaluated by looking at the impact consistency indicator. In Fig. 5.49 the overall impact consistency for each VP.
Figure 5.48: A close-up view of virtual point 1: a) experimental setup; b) 3D display of accelerometers, channels and impact positions.

Figure 5.49: Overall impact consistency indicator for each virtual point.

locations is shown. Due to the relatively high values of the consistency indicator, the measurements that relate virtual point transformation can be considered to be correct.

On the receiver structure, 2 tri-axial accelerometers are used as a reference. (target sensors $u_3$) Before the operational measurements, an additional impact location on the electric motor was used as an operational excitation. With this, the on-board validation can be evaluated throughout the whole frequency region, since, when the motor is operating, only the main harmonics are dominant in the frequency spectrum.
5.4.2 Equivalent force characterization

Characterization of equivalent forces is a two-step process. Firstly the FRFs are measured in the assembled configuration $Y_{u^4u^2}$ with the source deactivated. Secondly the operational measurements are performed under various loads to obtain the response $u_4$. After obtaining the transformed FRFs $Y_{u^4m^2}$ the equivalent forces can be characterised by means of a simple pseudo-inverse:

$$m_{eq}^2 = (Y_{u^4m^2})^+ u_{op}^4. \tag{5.9}$$

The problem of characterizing is overdetermined (9 channels for 6 virtual forces) and a solution is obtained in a least-square sense. The in-situ determination finds a set of virtual forces that best reproduce the response at $u_4$. Therefore, the presence of any measurement error in the response $u_4$ will also be present in the equivalent forces. This problem can be resolved, to some extent, by applying a truncated SVD to determine the pseudoinverse. With this, the measurement error can be filtered out to some extent.

**On-board validation**

The quality of the identified equivalent forces can be assessed by on-board validation. The on-board validation compares the operational measurements at the target sensors $u_{op}^3$ with the predicted response $u_{TPA}^3$ based on the identified equivalent forces. Response prediction can be calculated as follows:

$$u_{TPA}^3 = Y_{u^4m^2} f_{eq}. \tag{5.10}$$

If the predicted response $u_{TPA}^3$ is consistent with measured response $u_{op}^3$ during the operational excitation, then the equivalent forces are valid to recreate the response at the target sensors. Therefore, the controllability (predefined virtual forces $m_2$) and observability (location of indicator DoFs $u_4$) of the current interface is valid. If any deviation is detected between the two responses, either the predefined forces are not correct or the target sensors are too far from the interface.

**Equivalent force noise**

The operational excitation can sometimes be relatively low, and the sensor noise could affect the operational measurements. The measured response will always contain sensor noise:

$$u_{meas}^4 = u_{op}^4 + u_{noise}^4. \tag{5.11}$$

Therefore, the sensor noise directly affects the measured equivalent forces as follows

$$f_{meas}^2 = (Y_{u^4})^{-1}(u_{op}^4 + u_{noise}^4) = \underbrace{(Y_{u^4})^{-1}u_{op}^4}_{f_{meas}^2} + \underbrace{(Y_{u^4})^{-1}u_{noise}^4}_{f_{noise}^2}. \tag{5.12}$$

The term $f_{noise}^2$ is referred to as equivalent or blocked force noise. The equivalent force noise can be estimated by using noise measurements on the sensors $u_4$ (the source deactivated). By comparing the equivalent force noise $f_{noise}^2$ with the measured equivalent forces $f_{meas}^2$ the effect of sensor noise can be estimated. If the difference between the two is relatively small, the measured equivalent forces $f_{meas}^2$ should be interpreted with care.

---

11 The expansion can be made under the assumption of a linear system.
5.4.2.1 Impact excitation

The operational response $u_{4op}$ has to be measured under various operational conditions. However, to first validate the whole in-situ approach, an additional impact location on the source is used to simulate the operational excitation.

The identified equivalent forces from the impact excitation are shown in Fig. 5.50. It can be observed that the equivalent forces and moments are lower at VP 1 and 2 than at VP 3 and 4. This can be explained by the different design of vibroisolations mounting. At VP 1 and 2 the rubber mounts are not preloaded (Fig. 5.48), while at VP 3 and 4 a relatively high deformation of rubber can be observed after mounting the electric motor (tightening torque of 15 Nm was applied to mount the electric motor).

![Identified equivalent force: a) virtual forces; b) virtual moments.](image)

The on-board validation on two reference locations is depicted in Fig. 5.51 and Fig. 5.52. The predicted response $u_{3TPA}$ is consistent with the measured response $u_{3op}$ throughout the whole frequency range. A similar consistency was observed on all reference target channels.

![An on-board validation at the reference location $u_{3z}$. Measured response $u_{3z}^{op}$ (---) and predicted response $u_{3z}^{TPA}$ (---).](image)
Figure 5.52: An on-board validation at the reference location $u_{31x}$. Measured response $u_{31x}^{op}$ (---) and predicted response $u_{31x}^{TPA}$ (—).

With the equivalent forces known, a partial response contribution can be evaluated for each separate component of excitation. The partial responses can then be displayed as a heatmap, as shown in Fig. 5.53 for the response $u_{31x}$. With this display, the dominant transfer path can quickly be distinguished. As it can be observed with the current impact excitation the dominant transfer path is through the VP 3. This is due to the selection of the location for the impact excitation, as it is located in the proximity of the VP 3.

Figure 5.53: Partial responses at one reference location for each separate identified equivalent forces.

5.4.2.2 Operational excitation

A typical rotation speed of a WM drum can be anywhere in the range of 1200-1800 rpm, depending on the WM model. The load exerted on a motor during operation can be divided into two main parts. The first load is the total mass of wet clothes which is always present. The second load is due to the unbalanced load within the WM drum, which is different for each WM cycle as the distribution of clothes is random for each washing cycle.
Two different operating conditions were tested at different rotation speeds between 1200-1800 rpm. First the electric motor spinning freely without the belt installed was evaluated, and, second, a belt was connected and a distributed load of 10 kg was installed in the WM drum.

In Fig. 5.54 an on-board validation is depicted for the operational case without any load on the electric motor. The on-board validation can be validated; however, the dominant response is only around the main harmonics associated with the rotating speed of electric motor. It can be observed that the amplitude of the first harmonic increases with the higher rotational speed, which could also be observed from the equivalent forces.

In Fig. 5.55 an on-board validation for the operational case with a load exerted on the electric motor and with a load in the drum of WM is depicted. Here the amplitude of response also increases with higher rotational speed. Under the load a higher amplitude of equivalent forces could be observed than in the case without any load. The identified equivalent forces under load and with the belt installed should be cross-validated. Since the WM drum is spinning with the belt installed, the entire operational excitation is not only a characteristic of the source alone, which is the underlying assumption of the in-situ TPA. For a better in-situ characterization, a separate test-bench would be required where the simulated load would be completely decoupled from the electric motor.
Figure 5.55: An on-board validation at different speed settings with a constant load on electric motor: a) measured response $u_{31x}^{op}$, b) predicted response $u_{31x}^{TPA}$.

5.4.3 Discussion

A practical application of an in-situ TPA was presented to characterize the equivalent forces of a BPM electric motor on a WM drum and tub test bench assembly. The equivalent forces were validated with the on-board validation. Two different operational excitations were used to evaluate the in-situ approach. The first was the use of an impact location on the source structure to simulate an operational response. The second was the operation of electric motor at different speed settings. The impact excitation is used to validate the force characterization throughout the whole frequency range, since with operational excitation the dominant response is only around the main harmonics. The identified equivalent forces were consistent with the impact excitation. However, for the operational excitation, some of the limitations of the current test-bench setup were problematic.

The operational measurements at different speed settings were made both without any load on the electric motor (spinning completely free without the belt) and with a small load within the drum of a WM. For the setup with the belt installed the WM drum is also rotating and can be regarded as an additional operational excitation. The determined equivalent forces are, therefore, not entirely correct. With the current WM drum and tub test bench assembly, this problem cannot addressed, as the electric motor
should be separated from the assembly. An additional test bench would be required, with which various loads could be simulated without the additional operational excitation on the passive side. Furthermore, the influence of the Gyroscopic effects should be evaluated for both the drum and the rotor. The investigation of the full effect was out of the scope of the current research; however, the preliminary investigation into the Gyroscopic effect from electric motor rotor spinning, found that the influence is negligible for this particular case.

With the current in-situ application only an on-board validation was used to evaluate the equivalent forces. The next step should be a cross-validation where the source is characterized in another assembly and compared with the current equivalent forces. Afterwards, the identified equivalent forces can afterwards be used to determine the best overall BPM motor for various WM models. After an optimal electric motor is selected, a structural optimization of the receiver side can be made, since the equivalent forces are a property of the source alone.
6 Conclusions

The ever increasing demand for an increased vibro-acoustic comfort in our lives has led to numerous advancements in Noise, Vibration and Harshness engineering in the past decade. NVH engineering is most commonly and extensively used in the aerospace and the automotive industries. However, the tendency to decrease the noise is also present in the development of all household appliances, as regards both the manufacturer’s and the customer’s perspective. Home appliances are essentially a part of our living space, and since a silent and peaceful living environment is often associated with high quality of life, the NVH characteristics of appliances are vital.

Nowadays complex mechanical systems, such as home appliances, are designed in an increasingly modular fashion. The whole product is divided into subcomponents. While the main core parts are developed by the manufacturers themselves, the majority of smaller components are commonly outsourced to a sub-suppliers. Each separate sub-system has to follow an extensive list of predefined requirements so that the final product will achieve the requested target settings. Achieving the specified target setting in regards to vibro-acoustic performance can be especially challenging, especially due to the shear amount of subsystems and their complexity.

The main topic of this thesis is the frequency based substructuring and transfer path analysis performed directly with experimental modelling. First, a brief overview of the theory behind the structural dynamics in the frequency domain is provided, together with an overview of dynamic substructuring in the frequency domain (Chapter 2). Chapter 3 introduces a transfer path analysis from a substructuring perspective. An emphasis is on component-based TPA, which is commonly used in combination with other frequency based substructuring approaches introduced in the next chapter. Chapter 4 introduces current state-of-the-art experimental approaches in frequency based substructuring. The focus is primarily on a proper interface modelling, where virtual point transformation and singular vector transformation are thoroughly explored and validated. In Chapter 5, four applications of dynamic substructuring are explored. First, an application of standard and extended virtual point transformation is shown on a complex automotive test-bench. Both a coupling and a decoupling procedure were used to acquire final assembled results. Second, a use of the system equivalent model mixing methodology is shown with the determination of full-field FRFs from noisy high-speed camera data. Third, the singular vector transformation is applied on both numerical and experimental application on the assembly and disassembly of a test structure. Finally, the characterization of a washing machine motor through
Conclusions

equivalent forces is performed on a drum and tub test-bench assembly by utilizing the in-situ transfer path analysis.

The thesis confirms all of the scientific hypotheses given in Section 1.4. Throughout the research, multiple additional questions were opened, and new ideas were presented for future work. The frequency based substructuring methods discussed throughout the thesis, as well as the 3D display used for depicting the experimental setups, were integrated into a newly developed Python package pyFBS. The pyFBS provides a combination of open-source tools that enable the user to efficiently utilize state-of-the-art FBS methodologies (Appendix A.2 for more details). Overall, the experimental approach in frequency-based substructuring greatly benefits NVH engineering and should be adopted whenever complex dynamic systems are consider with respect to their vibro-acoustic performance.

6.1 Scientific contributions

The main scientific contribution of the thesis can be summarized with the two main developments:

**Extended VPT with directly measured rotational response:**

Virtual point transformation is extended to include directly measure rotational response. Consistency of the extended VPT is achieved by using a rotational weighting matrix, which is formulated to minimize the norm of the overall displacements due to the rotational residual at the location of the VP for each rotational response. The extended VPT is less susceptible to deviations in the impact locations and orientation, as well as to deviations in the sensor’s sensitivity. The rotational response is measured using a direct piezoelectric rotational accelerometer. The extended VPT was applied both to a numerical example as well as to two experimental examples. The second experimental example was a complex automotive test structure used to mimic the dynamics of an engine unit flexibly mounted on a chassis in a real car. Both numerical and experimental examples showed that the extended VPT can be used to acquire more consistent results.

**Full-field FRFs estimation from noisy high-speed camera data:**

The use of a high-speed camera for dynamic measurements is becoming a compelling alternative to accelerometers and laser vibrometers. The estimated displacement response from a high-speed camera generally exhibits relatively high levels of noise. The noise is especially problematic in high-frequency ranges, where the amplitudes of displacements are typically very small. If only the mode shapes of the analyzed structure are required then the identification with Least-Square Complex Frequency in combination with single accelerometer measurements can be used to obtain mode shapes even below the noise level. However, if the mode shapes are used to reconstruct full-field FRFs, the reconstructed FRFs are not consistent in the high-frequency range. A consistent set of full-field FRFs can be obtained by adding an additional step after the reconstruction. This step consists of using System Equivalent Model Mixing to form a hybrid model from two experimental models of the
same system. The first model is the reconstructed full-field FRFs that contribute the full-field DoF set. The second model is the accelerometer measurements that provide accurate dynamic characteristics. The hybrid model has been shown to have an increased accuracy, especially in the high-frequency range.

**Singular vector transformation applied on a coupling example:**

Singular vector transformation enables a user to define a reduction basis directly from the available FRF datasets. No geometrical or analytical model is required for the transformation. If the basic assumption of the transformation are met (separate collocation on outputs and inputs) the reduction basis is able to control and observe most of the rigid and flexible modes of the interface, over a broad frequency range. The reduction can also be frequency dependent, where a fewer modes can be included in the lower frequency range compared to the high frequency range. In this thesis an application of SVT is performed on a coupling example, where the method outperforms commonly used current state-of-the-art methods. The method also shows a decreased sensitivity to measurements errors and decreases the condition number of the interface problem.
Bibliography


Bibliography


A Appendix

A.1 Sobol sensitivity analysis

The method of global sensitivity indexes developed by Sobol' is based on the analysis of variance (ANOVA) decomposition \[71\]. Consider that the evaluation model is described by a square integrable function \( f(x) \) defined in the unit hypercube \( I^n = [0,1]^n \):

\[
u = f(x) = f(x_1, \ldots, x_n).
\]

(1.1)

where \( u \) is a scalar output and \( x \) an input inside the unit hypercube. The first step to Sobol’s method is the decomposition of the evaluation function in the following form:

\[
u = f(x_1, \ldots, x_n) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j} f_{ij}(x_i,x_j) + \cdots + f_{1\ldots n}(x_1, \ldots, x_n),
\]

(1.2)

where \( f_0 = \int f(x)dx \). The decomposition is unique if the integral of each term \( f_{i_1\ldots i_s}(x_{i_1}, \ldots, x_{i_s}) \) over any independent variable is zero:

\[
\int f_{i_1\ldots i_s}(x_{i_1}, \ldots, x_{i_s})dx_k = 0 \quad \text{for} \quad k = i_1, \ldots, i_s.
\]

(1.3)

It follows that all the terms in (1.2) are mutually orthogonal and can be expressed as integrals of \( f(x) \). As assumed, the function \( f(x) \) is square integrable:

\[
\left[ \int f^2(x)dx - f_0^2 \right] = \sum_{i=1}^{n} \int_{D_i} f_i^2(x_i)dx_i + \sum_{i<j} \int_{D_{ij}} f_{ij}^2(x_i,x_j)dx_i dx_j + \cdots + \int_{D_{1\ldots n}} f_{1\ldots n}^2(x_1, \ldots, x_n)dx_1 \cdots dx_n.
\]

(1.4)

where \( D \) is the total variance and \( D_i, D_{ij} \) and \( D_{1\ldots n} \) are partial variances. Dividing both sides of the equation (1.4) by the total variance \( D \) we obtain the definition of the global sensitivity indexes:

\[
\sum_{i=1}^{n} \frac{D_i}{D} + \sum_{i<j} \frac{D_{ij}}{D} + \cdots + \frac{D_{1\ldots n}}{D} = 1.
\]

(1.5)
Appendix

The most commonly used sensitivity indexes are the so-called first-order and total-order indexes. The first-order sensitivity index $S_i$ defines the first-order effect of $x_i$ on the model output and the total-order sensitivity index $S_i^T$ the total effect, i.e., the first and all the higher-order effects of the factor $x_i$.

The main advantage of the Sobol sensitivity analysis is the computation algorithm that allows an estimation of the global sensitivity indexes using only the output values of $f(x)$. Monte Carlo sampling-based methods have been developed for first-order and interaction indexes by Sobol [71] and additionally for a total-order index by Saltelli [72].

A.2 pyFBS: A Python package for Frequency Based Substructuring

pyFBS is a Python package for Frequency Based Substructuring. The package implements an object-oriented approach for dynamic substructuring.

Current state-of-the-art methodologies in frequency based substructuring are available in pyFBS. Each method can be used interchangeably with others or as a standalone method. Also, a 3D display is available, allowing the user to simply place and orient associated input/outputs used with each method. Furthermore, basic and application examples are provided with the package, together with real experimental and numerical data.

pyFBS enables the user to use state-of-the-art dynamic substructuring methodologies in an intuitive manner. Currently implemented features are listed below.

**3D display**
Structures and positions of impacts, sensors and channels can be visualized in 3D. The 3D display is built on top of PyVista [110] and offers an intuitive way to displaying the relevant data. Sensors and impacts can be interactively positioned on structures and the updated positions can be directly used within the pyFBS. With this feature the experimental setup can be prepared in advance, to avoid possible mistakes in experimental modelling. Furthermore, various animations can be performed directly in the 3D display, such as the animation of mode shapes or operational deflection shapes.

**FRF synthetization**
Frequency Response Functions can be synthetized for predefined positions of channels and impacts in a numerical model. Currently, mode superposition FRF synthetization is supported, where mass and stiffness matrices are imported from FEM software. Damping can be introduced as modal damping for each mode shape. Additionally, noise can be introduced to the response. Thus, a realistic set of FRFs representing experimental measurements can be obtained.

**Virtual Point Transformation (VPT)**
VPT projects measured dynamics on the predefined interface displacement modes
The interface is usually considered to be rigid. Therefore, only 6 rigid IDMs are used in the transformation. After applying the transformation, a collocated set of FRFs is obtained, which can afterwards be used directly in DS. Expanded VPT is also supported, which means that directly measured rotational response can also be included in the transformation.

**Singular Vector Transformation (SVT)**

SVT projects measured dynamics into subspaces composed of dominant singular vectors [40]. The singular vectors are extracted directly from the measured interface dynamics by using Singular Value Decomposition (SVD). Since the reduction space is defined directly from the measured dynamics, no numerical or geometrical model is required.

**System Equivalent Model Mixing (SEMM)**

SEMM enables mixing of two equivalent frequency-based models into a hybrid model [90]. The models used can either be of numerical or experimental nature. One of the models provides the dynamic properties (overlay model) and the second model provides a set of degrees of freedom. A numerical model is commonly used as a parent model and an experimental model is used as an overlay model.

![Figure A2.1: An example of an AM benchmark structure depicted in the pyFBS 3D display](image)

The development of the pyFBS is an ongoing effort. The package is currently used as a research tool. In the future, more examples on the topic of DS and the applications of Transfer Path Analysis (TPA) are going to be introduced into the documentation. Furthermore, implementation of Operational Source Identification (OSI) is going to be integrated into the pyFBS in the near future.
B Slovenski povzetek

1 Uvod

Vibro-akustično udobje v gospodinjstvih postaja vse bolj pomembno. Mirno in tiho bivalno okolje je neposredno povezano z visoko kakovostjo življenja. Del našega bivalnega okolja predstavljajo tudi gospodinjski aparat, zato je tiho delovanje aparatov ključno. Dodatno pa vibro-akustične karakteristike sistema, skupaj z vizualnim izgledom, prispevajo k skupni interpretaciji izdelka višje kvalitete in s tem višje vrednosti na trgu [1]. Kompleksni izdelki, kot so gospodinjski aparat, so danes zasnovani modularno, pri čemer je posamezna komponenta razvita pri proizvajalcu ali pa pri zunanjem dobavitelju. Ravno zato je koristno okarakterizirati dinamske lastnosti vsakega podsistema posamezno. Metode dinamike podstrukturiranja nam omogočajo, da povezemo dinamske lastnosti podsistemov in napovemo odziv celotnega sistema [2], kot je shematsko prikazano na sliki 4.3.

Slika 1: Shematski prikaz metodologije dinamike podstruktur, kjer lahko s karakterizacijo posameznega podsklopa napovemo odziv sestavljenega sistema.
1.1 Motivacija

Eksperimentalno in numerično modeliranje sta ključna pri sodobnem razvoju mehan- 
skih komponent. Po opredelitvi ciljnih specifikacij, ki so določene na podlagi tržnih 
raziskav in primerjalnih analiz, se celoten sistem razdeli na podsisteme. Razvoj vsakega 
podsistema se nato prenese ali na interni oddelek ali na zunanjega dobavitelja. Primer 
sodobnega razvoja je prikazan na V-modelu na sliki 1.2 [3, 4]. Kot je razvidno iz she-
matskega prikaza, se numerične metode uporabljajo primarno v zgodnji fazi razvoja, 
medtem ko se eksperimentalne metode uporabljajo za končno validacijo po sami real-
izaciji komponent.

Čas, ki je na voljo za razvoj produkta, se iz leta v leto krajša. Ravno zato je potrebno 
vibro-akustične lastnosti produkta določiti vnaprej, Že kar v samem predrazvoj. V 
kolikor želimo obvladovati zastavljene lastnosti vnaprej, je potrebno vključiti metode 
vibro-akustične karakterizacije že v zgodnji fazi razvoja. Modularen pristop je pri 
obrahnavi kompleksnejših izdelkov ključen, saj se veliko podkomponent izdela pri zu-
nanjih dobaviteljih. Zato je pravilno numerično modeliranje zahtevno in velikokrat 
praktično nemogoče zaradi pomanjkanja kritičnih informacij. V tem primeru se lahko 
posamezen podsistem eksperimentalno modelira. Vibro-akustični odziv sestavljenega 
sistema se lahko napove z uporabo metod dinamike podstruktr [2].

1.2 Obravnavano znanstveno področje

Skozi zgodovino se je razvilo več različnih pristopov modeliranja dinamike posameznega 
podsistema. Sodobne formulacije lahko razdelimo v pet različnih skupin. Prva skupina 
omogoče modeliranje v fizikalni domeni, kjer je dinamika posamezne podstrukture 
definirana s pomikom v posamezni točki. Diskretna reprezentacija strukture je naj-
ječkrat izračunana z uporabo metode končnih elementov (ang. Finite Element Mod-
eling) [5]. Druga skupina omogoča modeliranje v modalni domeni, kjer se pod-
strukturna definira z vektorji podprostora z uporabo modalne superpozicije. Najbolj


Slika 3: Dinamika podstrukturn v petih domenah in medsebojni prehodi [9].

1.3 Opredelitev raziskovalnega problema


Kljub vsem prednostim, pa hibridno modeliranje ni vedno dostopno, saj je lahko numerično modeliranje podistema nemogoče bodi zaradi premalo časa bodi zaradi pomanjkanja kritičnih informacij. Obetavna metoda, s katero lahko modeliramo pod sisteme le z eksperimentalnim pristopom, se imenuje transformacija virtualne točke (ang. *Virtual Point Transformation*) [26]. Transformacija predstavlja nadgradnjo EMP metode (ang. *Equivalent Multiple Point Connection*) [111]. Pri transformaciji virtualne točke uporabimo geometrijsko transformacijo za projekcijo izmerjenih translacijskih prostostnih stopenj na predpostavljene deformacijske oblike povezave (ang. *Interface Deformation Modes*). Povezano lahko modelira toko ali kot fleksibilno, kjer so toge deformacijske oblike povezave razširjene s fleksibilnimi [28]. Celotna transformacija se lahko interpretira tudi kot minimizacijski postopek [29].

Glavna pomanjkljivost transformacije virtualne točke je možnost modeliranja le točkovnih povezav, ki se v bližini kontakta obnašajo relativno toko. Za modeliranje linijskih oz. površinskih kontaktov se najpogosteje uporablja metoda simulatorja prenosnosti

Vse eksperimentalne metode podstrukturiranja imajo skupen problem, kljub njihovi izraziti raznolikiosti. Že majhna napaka pri izvajanju eksperimenta lahko pri sklopljenem modelu privede do popolnoma napačnih in brezpredmetnih rezultatov [33,34]. V primeru lahko dušenih struktur je problem z merilnimi napakami še večji [35,36]. Kljub vsem tem pomanjkljivostim, povezanim z eksperimentalnimi modeli, njihov ogromen potencial odtehta sedanje omejitve.

1.4 Hipoteze dela

Glavne hipoteze doktorskega dela so navedene spodaj.

1. **Hipoteza:** Uporaba rotacijskega pospeškomera v transformaciji virtualne točke lahko zniža negotovost transformacije, povezane s pozicijo ter občutljivostjo senzorjev.

2. **Hipoteza:** Razširjena oblika transformacije virtualne točke izboljša konsistentnost sklopljenih frekvenčno prenosnih funkcij.

3. **Hipoteza:** Frekvenčno prenosne funkcije, izmerjene s hitro kamero, se lahko izboljšajo z uporabo metod dinamike podstruktur v frekvenčni domeni.

V doktorskem delu bo poudarek na karakterizaciji, razvoju ter analizi metod dinamičnega sklapljanja v frekvenčni domeni. Obrahnavana bosta tako numerični kot eksperimentalni pristop k modeliranju dinamike posamezne podstruktura. Predlagane metode sklapljanja bodo eksperimentalno validirane na realnih strukturah. Prispevek dela se bo odražal na naslednjih področjih:

1. **Transformacija virtualne točke:** raziskan bo vpliv merilne negotovosti lokacije in občutljivosti senzorjev, vključenih v transformacijo virtualne točke. Predstavljena bo razširitev transformacije virtualne točke, s katero se lahko merilna negotovost same transformacije zniža.

2. **Identifikacija polnega polja FPF:** uporaba hitre kamere v kombinaciji z metodami dinamičnega sklapljanja v frekvenčni domeni bo prikazana za izboljšanje izmerjenih frekvenčno prenosnih funkcij.

3. **Karakterizacija izvora:** uporabnost metod dinamičnega sklapljanja in analiza prenosnosti poti bodo predstavljene na primeru vibroakustične karakterizacije kompleksnega produkta. Glavne prednosti ter omejitve najpomembnejših metod bodo predstavljene na realnih primerih.
2 Povzetek

2.1 Eksperimentalno podstrukturiranje v frekvenčni domeni

Eksperimentalno modeliranje v dinamiki podstruktur ima izjemni potencial, saj s frekvenčno prenosnimi funkcijami zajamemo globalne karakteristike sistema neposredno na lokacijah meritev. V primerjavi z numeričnim modeliranjem v teoriji ni omejitev glede realnosti samega eksperimentalnega modela. S sodobno eksperimentalno opremo in metodami lahko pridobimo verodostojne podatke v frekvenčnem območju nekaj kHz.

V realizaciji se izkaže, da je uspešna eksperimentalna aplikacija metod dinamike podstruktur relativno zahtevna. Za prikaz glavnih omejitev eksperimentalnega pristopa si prizadevamo enostavno aplikacijo toge povezave med dvema nosilcema, kot je prikazano na sliki 4.1. Za uspešno sklopitev toge povezave je potrebno na vsaki podstrukturi izmeriti translacijske in rotacijske frekvenčno prenosne funkcije na sami povezavi. Na numeričnem modelu lahko te FPF tudi brez večjih težav modeliramo. V primeru eksperimentalnih meritev pa so enake FPF praktično nedostopne, kot na primer na realno togi povezavi, prikazani na sliki 4.2. Za merjenje rotacijskih prostostnih stopnij bi potrebovali tri-osi rotacijski pospeškomer, prav tako bi potrebovali zanesljiv izvor vzbujanja strukture samo z momentom. Trenutno ne obstaja eksperimentalna oprema, ki bi nam to omogočala. Dodatno pa je potrebna natančnost meritev za frekvenčno podstrukturiranje veliko večja, kot v primerjavi z meritvami za modalno analizo. Nedostopnost in strogi pogoj glede kvalitete meritev so glavni razlog pri eksperimentalni aplikaciji metod dinamike podstruktur.

Kljub vsem negotovostim eksperimentalnih metod dinamike podstruktur, pa so še vedno izjemno uporabne pri obravnavi kompleksnejših mehanskih sistemov. Glavno pozornost moramo nameniti pravilnemu modeliranju povezav med posameznimi podstrukturami, hkrati pa se morajo eksperimentalne meritve FPF izvesti z visoko stopnjo natančnosti.

2.2 Transformacija virtualne točke

Diskretna povezave s togi kontaktom (primer na sliki 4.2) so v praksi velikokrat prisotne. Ker so rotacijske prostostne stopnje v praksi nedostopne, so se razvile metodologije, ki implicitno zajamejo rotacijske prostostne stopnje. Ena izmed takšnih metod je tako imenovana ekvivalentna večtočkovna povezava EMPC (ang. Equivalent Multi-Point Connection). Z EMPC metodo popisemo rotacije posamezne toge povezave s skupno najmanj 9-imi prostostnimi stopnjami. V kolikor sklopimo vseh 9 translacijskih prostostnih stopnij, se izkaže, da je takšen pogoj preveč rigorozen in že najmanjša napaka vodi v nerealen sklopljen odziv.

Iz pomankljivosti EMPC metode se je razvila metoda transformacije virtualne točke VPT (ang. Virtual Point Transformation). Metoda VPT transformira izbrane izmerjene translacijske prostostne stopnje v virtualno točko pod predpostavko lokalne togosti (ime virtualna izhaja iz dejstva, da v tej točki v resnici ni izvedenih meritev). Transformirana virtualna točka ima le 6 prostostnih stopnij (3 translacije in 3 rotacije).
transformacijo VP tako zadostimo pogoju soležnosti prostostnih stopnj v kontaktu, kot tudi rotacijskim prostostnim stopnjam. V praksi se izkaže transformacija virtualne točka za zelo uporabno metodo, saj se s transformacijo zmanjša vpliv morebitne napak pri eksperimentalnih meritvah.

Transformacija virtualne točke se lahko razširí z direktno izmerjenim rotacijskim odzivom, ki se ga lahko izmeri z uporabo piezoelektričnega rotacijskega pospeškomera. Konsistentna transformacija translacijskega in rotacijskega odziva se doseže z uporabo rotacijske utežnostne matrike, ki je formulirana tako, da minimizira translacijski odziv na lokaciji rotacijskega senzorja, kot posledica ostanka rotacij pri transformaciji. Izkaže se, da je razširjena transformacija manj občutljiva na majhne napake v lokaciji in orientaciji udarcev, kot tudi na napake v občutljivosti uporabljenih senzorjev.

2.3 Transformacija singularnih vektorjev

Glavna pomanjkljivost transformacije virtualne točke je modeliranje povezav, ki ne izpolnjuje pogoja idealne tožnosti. Transformacija virtualne točke sicer omogoča razširitev z relativno enostavnimi deformacijami povezav, kot so upogib ali torzija, vendar so v praksi deformacijske oblike povezav mnogo bolj zapletene.

Metoda, ki omogoča modeliranje fleksibilnih povezav v okviru frekvenčnega podstrukturiranja, je transformacija singularnih vektorjev SVT (ang. singular vector transformation). Pri SVT transformaciji se uporabi lastnost ortogonalnosti levih in desnih singularnih vektorjev pri razcepu singularne vrednosti za definicijo redukcijske baze. Prostostne stopnje redukcijske baze so razvrščene glede na njihovo dominantnost, kar ima izreden pomen. Izbira števila prostostnih stopnij v kontaktu je odvisna od fleksibilnosti povezave kot tudi suma pri eksperimentalnih meritvah.

Pri praktičnih aplikacijah se izkaže, da se lahko SVT uporablja tako za sklapanje kot tudi za razsklapljanje struktur. Glavna omejitev metode SVT je v predpostavki soležnih prostostnih stopnij posamezno na lokacijah odziva in lokacijah vzbujanja na povezavi obeh struktur. Pri postopku razsklapljanja to ne predstavlja večjih omejitev, saj lahko soležnost praktično vedno zagotovimo. V primeru sklapanja pa je potrebno zaradi zagotavljanja soležnosti uporabiti dodatno strukturo v kontaktu, tako imenovan simulator prenosnosti.

2.4 Analiza prenosnih poti

Z uporabo metodologije dinamike podstruktur pa lahko obravnavamo prenosne poti vibracij med posameznimi podstrukturni. S tem lahko okarakteriziramo najbolj kritične poti prenosa v sestavu. Z analizo prenosnih poti lahko tudi okarakteriziramo aktiven izvor v sestavu, z naborom povezovalnih sil na kontaktu med aktivnim in pasivnim delom strukture. Ko je enkrat identificirana kritična pot prenosa v sestavu, se lahko začne z optimiranjem tako aktivnega kot pasivnega dela sestava.

V grobem delimo metode analize prenosnih poti v 3 skupine. V prvo skupino spadajo klasične metode, ki se primarno uporabljajo za identifikacijo kritičnih poti v že obstoječih produktih. Identificirane sile aktivnega dela sestava ob obratovanju so lastnost
dinamike celotnega sestava. Glavna slabost teh metod je, da je potrebno ob vsaki mo-
difikaciji ponoviti celoten nabor meritev. Druga skupina obsega metode prenosnih poti na ravni podkomponent. Identificirane povezovalne sile z metodami iz te skupine so lastnost samo aktivnega dela strukture in so popolnoma neodvisne od pasivnega dela strukture. Ravno zaradi te lastnosti so izredno uporabne pri raznih optimizacijskih postopkih, saj ni potrebno ponavljati meritev pri obratovanju za vsako spremembo. Zadnja skupina metod prenosnih poti pa se ne uporablja za identificiranje realnih povezovalnih sil, vendar le določi stopnjo prenosnosti med posameznimi povezovalnimi točkami. Zadnja skupina je uporabna v primerih, kadar nas zanima le prenosnost med posameznimi povezavami in ne realno modeliranje povezav.

3. Uporaba metod podstrukturiranja na kompleksnih sistemih

V zadnjem poglavju so prikazane štiri aplikacije frekvenčnega podstrukturiranja. Sprva je prikazana aplikacija sklapanja na testni strukturi, ki ponazarja sestav avtomobilskega motorja (Poglavje 5.1). Nato je prikazana aplikacija hibridnega pristopa k identifikaciji polnega polja FPF na šumnih podatkih hitre kamere (Poglavje 5.2). V tretjem poglavju je prikazana aplikacija transformacije singularnih vektorjev na postopku sklapanja in razsklanja na numeričnih in eksperimentalnih modelih (Poglavje 5.3). V zadnjem poglavju pa je prikazana karakterizacija ekvivalentnih vzbujevalnih sil na elektromotorju (Poglavje 5.4).

3.1 Aplikacija sklapanja na kompleksnih strukturah

Praktična aplikacija frekvenčnega podstrukturiranja je prikazana na kompleksni struk-
turi, ki ponazarja vpetje avtomobilskega motorja. Stresalnik ponazarja avtomobilski motor, vpet na fleksibilnih nosilcih, ki so pritrjeni na poenostavljeno šasijo. Testna struktura je zasnovana za testiranje različnih aplikacij frekvenčnega podstrukturiranja in analize prenosnih poti.

V dotični aplikaciji je bila opravljena primerjava aplikacije standardne in razširjene
transformacije virtualne točke v procesu sklapanja. Zaradi nedostopnosti do merilnih
mest je bil uporabljen še simulator prenosnosti, ki pa je bil na koncu odklopljen od
celotnega sestava. Izkaže se, da razširjena transformacija virtualne točke izkazuje bolj
konsistente rezultate sklapanja v primerjavi s standardno transformacijo. Vključitev
direktno izmerjenega rotacijskega odziva neposredno zniža občutljivost na manjše na-
pake pri izvajanju eksperimentalnih meritev.

3.2 Karakterizacija ekvivalentnih vzbujevalnih sil na elektromotorju

Aplikacija karakterizacije ekvivalentnih vzbujevalnih sil pri delovanju elektromotorja je
prikazana na testnem sestavu pralnega stroja. Karakterizirane ekvivalentne sile so bile
potrjene s povratno validacijo. Vpetje elektromotorja je bilo testirano v dveh različnih
stanjih. Sprva elektromotor ni deloval in je bilo kot vir vzbujanja uporabljeno kar
modalno kladivo. S tem se lahko okarakterizira kvaliteta transformacije čez celotno
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frekvenčno območje. V drugi postavitvi pa je elektromotor deloval sprva brez obremenitve in nato z manjšo obremenitvijo.

Pri delovanju elektromotorja so bile okarakterizirane sile z in brez nameščenega jermenja. V prvem primeru je elektromotor deloval brez obremenitve, v tem primeru se je karakterizacija izkazala za popolnoma pravilno. V drugem primeru, ko pa je elektromotor deloval pod obremenitvijo, pa predpostavke, potrebne za ekvivalentne sile, niso v celoti izpolnjene, saj vrtenje bobna skupaj z elektromotorjem predstavlja dodatno vzbujanje v celotnem sestavu. Pri trenutni konfiguraciji testnega sestava ni mogoče obremeniti elektromotorja in bi bila potrebna dodatna konstrukcija, kjer bi bilo mogoče okarakterizirati elektromotor pod različnimi obremenitvami.

4. Zaključki


Dandanes so kompleksni produkti, kot so gospodinjski aparati, zasnovani modularno. Celoten produkt je razdeljen na podsisteme, kjer so glavni podsklopi proizvedeni s strani proizvajalcev, glavni del manjših podsistemov pa so ponavadi razviti in proizvedeni s strani podizvajalcev. Vsak pod sistem mora tako izpolniti dolg list vnaprej določenih specifikacij, zato da končen produkt doseže vnaprej zastavljene ciljne zahteve. Doseganje ciljnih zahtev s področja vibro-akustičnih lastnosti je še posebej zahtevno, navzo razmed samega števila pod sistemov in kompleksnih interakcij med njimi.

sklapljanja in razsklapljanja testne strukture. Nazadnje je prikazana aplikacija karakterizacije izvora na elektro motorju na podlagi ekvivalentnih sil. Karakterizacija je izvedena na testnem sestavu bobna in kadi s pomočjo in-situ analize prenosnih poti.

Naloga potrdi vse zadane znanstvene hipoteze, podane v poglavju 1.4. Skozi celoten proces raziskave pa so se odprla mnoga dodatna raziskovalna vprašanja, kot tudi nove raziskovalne ideje za nadaljnje delo. Vse predstavljene metode frekvenčnega podstrukturiranja ter uporabljen 3D prikaz so integrirani v odprtorkodnem programskem paketu pyFBS. Paket pyFBS zajema kombinacijo prosto dostopnih orodij, ki omogočajo uporabniku hitro in učinkovito uporabo najnovejših metod frekvenčnega podstrukturiranja (več podrobnosti v prilogi A.2). Eksperimentalni pristop v dinamiki podstrukturiranja ima izjemen potencial pri optimizaciji NVH parametrov in bi se morale metode vključiti že v zgodnji razvojni fazi vseh kompleksnejših produktov.

4.1 Znanstveni doprinos

Glavni znanstveni doprinos naloge lahko povzamemo v dveh točkah:

**Razširjena transformacija virtualne točke z rotacijskim odzivom:**


**Predikcija polnega polja FPF iz šumnih meritev hitre kamere:**

Transformacija singularnih vektorjev aplicirana na primeru sklapljanja:

Transformacija singularnih vektorjev omogoča uporabniku definirati redukcijsko bazo direktno s izmerjenih FPF. Za samo transformacijo ni potrebe po geometrijskem ali analitičnem modelu. V kolikor se izpolne glavno predpostavko metode, ki je solemnost lokacije senzorjev ter posebej lokacije udarcev, lahko s transformacijo vključimo glavne toge in fleksibilne oblike gibanja povezave čez široko frekvenčno področje. Redukcija je lahko tudi frekvenčno odvisna, kjer se lahko v nižje frekvenčnem področju uporabi manjše število prostorstnih stopenj za samo transformacijo. V tej nalogi je predstavljena aplikacija SVT metode na primeru sklapljanja, kjer metoda izkazuje boljše rezultate, kot trenutno najpogosteje uporabljene metode. Metoda dodatno tudi izkazuje manjšo občutljivost na morebitne napake pri izvajanju eksperimentalnih meritev.
Curriculum Vitae

Tomaž Bregar was born in Novo Mesto on 7th of August 1993. He attended the primary school Jožeta Gorjupa in Kostanjevica na Krki and continued on a technical gymnasium at School center Novo Mesto. After finishing matura exam in 2012 he enrolled in the same year to the Faculty of Mechanical Engineering at the University of Ljubljana. He successfully graduated in the year of 2014 with a bachelor thesis titled *Analysis and specification of accelerated vibration testing*. The same year he enrolled on a master program at the same faculty and he finished his master’s thesis in 2017 with the title *Vibroacoustic characterization of ball bearings with increased clearance*. The same year he pursued a PhD on the research topic of frequency based substructuring and transfer path analysis. The same year he also started working at Gorenje d.o.o. in acoustics department. In the second year of his PhD studies he spent three months at the Chair of Applied Mechanics at the Technical University of Munich. The results of his research were published in three scientific papers and presented at an international scientific conference.

**Journal publications:**


**Conference publications:**


Izvirni znanstveni članki:


Objavljeni znanstveni prispevki na konferencah:


