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DREVESNE METODE ZA VREDNOTENJE OPCIJ

Magistrsko delo

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TREE METHODS FOR OPTION PRICING

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Adviser: prof. dr. Tomaz Košir

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Program dela

V delu opišite metode za vrednotenje opcij osnovane na binomskih in trinomskih drevesih in njihovo konvergenco k Black-Scholesovem modelu.

Work plan

In the thesis present the methods for option pricing based on binomial and trinomial trees and their convergence to the Black-Scholes model.

Osnovna literatura (Basic references)


Podpis mentorja (adviser):
Drevesne metode za vrednotenje opcij

Povzetek

Prvi drevesni model določanja cen opcij so Cox, Ross in Rubinstein predstavili nekaj let kasneje po revolucionarni formuli Blacka in Scholesa. Ponuja preprosto in intuitivno metodo določanja cen, uporablja pa se lahko tudi za odločanje o zgodnji izvršitvi opcij. V tem delu je prikazano, kako zgraditi ta drevesni model in izpeljati cenovno formulo.

Ko se število časovnih korakov \( n \) poveča, se cena, dobljena z drevesno metodo, približa ceni iz Black-Scholesovega modela. Ta konvergenca je dokazana in raziskovana v tem delu. Stopnja konvergence je počasna in nihajna, zato razpravljamo o tem, kako to konvergenco pospešiti.


Tree methods for option pricing

Abstract

The first tree model for option pricing was introduced by Cox, Ross and Rubinstein a few years after the revolutionary Black-Scholes formula. It provides a simple and intuitive pricing method, and it can also be used for decision making about early-exercise of options. We show how to construct this tree model and derive the pricing formula.

When the number of time steps \( n \) increases, the price obtained by the tree method converges to the Black-Scholes price. We prove this convergence and investigate further its behaviour. The convergence rate is slow and oscillatory, and thus we discuss how to accelerate this convergence.

Many other tree models have been constructed through the years in order to improve efficiency. We present both binomial and trinomial tree models and various choices of their parameters. Our focus is on European and American put and call options. However, it remains a challenge to decide on the optimal parametrization of the tree.


Ključne besede: vrednotenje opcij, Cox-Ross-Rubinstein model, drevesne metode

Keywords: option pricing, Cox-Ross-Rubinstein model, tree methods
1 Introduction

Derivatives play an important role in modern finance. Options are amongst the most popular financial derivatives. The option concept exists for centuries. However, they have been traded more extensively since the seventeenth century. Back then, the options contracts were not standardized, and the market was unregulated [17]. During the financial crisis of the 1930s, they got a bad reputation and were even considered illegal for some time. Options were mostly used as a hedge against risk. However, obtaining a profit without any risk exposure would be an arbitrage opportunity.

The main question is, how to price options in order to avoid arbitrage opportunities? This is why the mathematics behind option pricing was very challenging. It took many years of study until a fair pricing formula was derived.

A revolutionary change both in trading and scientific study of options happened in the early 1970s. In 1973, the Chicago Board Options Exchange (CBOE) was established, and trading options contracts that were properly standardized started modestly. At the same year, academics Fischer Black and Myron Scholes published the first completely satisfactory option pricing formula [4]. Later that year, Robert Merton also contributed by extending their model in some important ways [39]. Earlier attempts to derive a pricing formula before Black-Scholes-Merton were not widely used for trading (e.g. Bachelier [2], Bronzin [11], Boness [5], Thorpe-Kassouf [52]). Their formula was a major breakthrough in finance and for that Myron Scholes and Robert Merton were awarded the Nobel Prize in Economics in 1997. Unfortunately, Fischer Black passed away in 1995; otherwise, he would also be awarded. Their result had a huge impact on option pricing theory.

Thus, different options have been developed since 1973, and therefore, different pricing methods were presented. The mathematics behind the Black-Scholes formula was considered quite advanced. The option pricing theory in continuous time is based on complex stochastic calculus.

In 1975, during a conference in Israel, Mark Rubinstein and William Sharpe, himself Nobel Price laureate (1990), had a discussion on the Black-Scholes formula, and the idea of a two-state model for pricing grew out of this. This would be later known as the binomial pricing model.

The binomial model was presented in 1979 by J.C. Cox, S.A. Ross and M. Rubinstein in their influential paper [16]. They found an intuitive way to price options by using simple algebra, which would justify the continuous Black-Scholes model and the necessary economic concepts. Also, they realized that with the use of the Central Limit Theorem, the binomial model converges to the Black-Scholes in the limit. Thus, the binomial model provides a discrete-time approximation of the Black-Scholes model.

The binomial model became a widely used method for pricing options due to its intuitive approach and easy implementation. Most importantly, it can be used to price options with different payoff structures for which there isn’t a closed-form pricing formula, including American options.

After the Cox-Ross-Rubinstein tree model, many other tree models have been
constructed by academics. They provide a different choice of parameters. The construction of the new trees has been in different ways and mostly with the intention to improve the convergence. Although the binomial model appears to be a simple and intuitive model, it can be challenging to decide which parameterization to choose.

This thesis aims to provide a general introduction to the tree method as used for pricing options; presenting the most famous tree models and their assumptions as well as their convergence in the limit.

After the introduction, in Chapter 2, we study the option pricing theory. We begin by defining option contracts and types. Next, we give market assumptions that are needed when modelling a pricing formula. The most important one is the no-arbitrage assumptions, and therefore we define it mathematically. Moreover, we give the Fundamental Asset Pricing Theorem. We also give the definitions of a stochastic process and Brownian motion. Finally, we present the Black-Scholes model.

Chapter 3 is reserved for the Tree Method. We follow the work of Cox, Ross and Rubinstein. First, we present the one-period binomial tree, and then we extend it to a multi-period binomial tree. We derive the risk-neutral pricing formula. We show how a tree model can be calibrated. Furthermore, we give the price for American options. The trinomial tree model is also presented in the last section.

Chapter 4 is of great importance. We will show the convergence of the binomial tree to the Black-Scholes model. Unfortunately, the CRR model has a slow convergence rate. Thus, other scientists have tried to improve the convergence rate. Many alternative binomial and trinomial models have been proposed in the literature. We will explain a few of them and compare their performance. In the last section of Chapter 4, we introduce some techniques that are used to accelerate the convergence of the tree models.
2 Option Theory

Options have been traded and studied for centuries, but their importance in finance has only increased over the last decades. In 1973, option contracts were standardized and started modestly trading at Chicago Board Options Exchange (CBOE). In that year, also an important model for option evaluation was published by academics F. Black, M. Scholes, and later extended by R.C. Merton [4] [39]. Since then, many different options have been developed and therefore demanded new models and techniques for pricing.

In this chapter, we will present the basic concepts of option theory.

2.1 Option Types

Derivatives are financial instruments whose value depends or derives from the price of the underlying assets. Options are standard examples of derivatives. Options contracts can be written on stocks, stock indices, foreign currencies, futures, or are embedded in callable securities, mortgage prepayments, and portfolio insurance. The asset to which the option refers is called the underlying asset or the underlying [25]. In option contracts, two parties are involved, option holder (buyer) and option writer (seller). To initialize the contract, the buyer of the option has to pay a premium to the writer (seller).

There are two basic types of options.

Definition 2.1. A call option is a contract which gives its holder the right but not the obligation to buy a certain fixed amount of the underlying asset for a predetermined price on or before a certain date. A put option is a contract which gives its holder the right but not the obligation to sell a certain fixed amount of the underlying asset for a predetermined price on or before a certain date.

The predetermined price in the contract is called the strike price or exercising price and is denoted by $K$; the date in the contract is called maturity or expiry date and is denoted by $T$. The act of making the transaction is referred to as exercising the option; otherwise the option is abandoned.

There are three main groups of options based on contract specifications: European options, American options, and Exotic options. The terms have nothing to do with the geographical position of the options, but rather with the structure of the options.

2.1.1 European Options

European options are exercisable only at the expiry date. They can be evaluated by the Black-Scholes model, which offers an equation with a closed-form solution.

European call option

Let us assume that the holder has a call option. The option gives him the right to buy an underlying stock for the strike price $K$ at maturity time $T$. 
Let us denote by $S_T$ the stock price at maturity time $T$. At time $t = 0$, we only know the strike price $K$, but we don’t know the stock price $S_T$, which gives uncertainty to our model. From the perspective of the option holder, the payoff $C$ at maturity time $T$ from a European call option is given by the formula [40]:

$$C(S_T) = (S_T - K)^+ = max\{S_T - K, 0\}$$

that means

$$C(S_T) = \begin{cases} 
0 & \text{if } S_T \leq K \ (\text{option is abandoned}), \\
S_T - K & \text{if } S_T > K \ (\text{option is exercised}). 
\end{cases} \quad (2.1)$$

If at maturity time $T$ the stock price is lower than the strike price, a rational holder would not exercise the option. He could buy the underlying stock directly on the market, paying less than $K$. If at maturity time $T$ the stock price is greater than the strike price, the holder should exercise the right to buy the underlying stock at the strike price $K$. By selling the stock immediately on the market, the holder would gain a profit $S_T - K$.

![Figure 1: Payoff of a European call option with strike price $K$.](image)

**European put option**

Let us assume that the holder has a put option. The option gives him the right to sell an underlying stock for the strike price $K$ at maturity time $T$. We use the same notations as for the call option. Now the payoff $P$ for the holder of the put option at maturity time $T$ is given by the formula:

$$P(S_T) = (K - S_T)^+ = max\{K - S_T, 0\}$$

that means
\[ P(S_T) = \begin{cases} 
0 & \text{if } S_T \geq K \text{ (option is abandoned)}, \\
K - S_T & \text{if } S_T < K \text{ (option is exercised)}. 
\end{cases} \] (2.2)

If at maturity time \( T \) the stock price is lower than the strike price, the holder of the put option would exercise the option and sell it for the strike price \( K \). Otherwise, if at maturity time \( T \) the stock price is greater than the strike price \( K \), the holder would abandon the option and sell it directly on the market.

![Figure 2: Payoff of a European put option with strike price \( K \)](image)

2.1.2 American Options

American options can be exercised at any time before maturity. There exist American put and call options. American options give the holder more rights than their European equivalent and are therefore be more valuable. The holder of an American option has not only to decide whether to exercise the option or not, but also when to exercise it. The main challenge is in finding the optimal exercise time. Calculating prices for American options is more complicated, and there does not exist a closed-form solution. Therefore, several numerical methods are used to price them. In chapter 3, we will introduce a tree method to price American options.

European and American put and call options are known as plain vanilla products. They have standard well-defined properties and trade actively. There exist also non-standard products in the over-the-counter market that are known as Exotic options or exotics.

2.1.3 Exotic Options

Exotic options have contracts with different structures and features from plain vanilla options. They differ from them in expiration dates, exercise prices, payoffs, and underlying assets. Exotics are more sophisticated and generally much more profitable than plain vanilla options.
The most common types of Exotic options are:

- **Asian options** in which the payoff depends on the average price of the underlying asset over a certain period of time.

- **Bermudan options** which can be exercised at the date of their expiration but also at some predetermined dates in the contract.

- **Barrier options** where the payoff depends on whether the underlying asset’s price reaches a predetermined level B (barrier) during a certain period of time.

- **Lookback options** which do not have a specified exercise price but on the maturity date, the holder has the right to select the most favorable strike price among all the prices during the lifetime of the options.

- **Basket options** which are based on several underlying assets and their payoff is the weighted average of all underlying assets.

There are more types of options (see Chapter 6, [40]), but in this thesis, we are going to focus only on European and American options.

### 2.2 Modeling Assumptions

The fundamental problem of financial mathematics is pricing. Financial markets are very complicated. In order to develop pricing theory some simplifying assumptions must be considered. The simplifying assumptions used are those of [3]:

- **No market frictions** which means no transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.

- **No default risk** which means the same interest rate for borrowing and lending.

- **Competitive markets** where market participants act as price takers not price makers.

- **Rational agents**

- **No arbitrage.**

Real markets do involve frictions and many of the assumptions that we have mentioned above fail. These assumptions help us to ignore some complications when pricing. The use of these assumptions does not mean that the actual models
are a long way from reality, because actually they do focus on the most essential market features and provide a very reasonable pricing.

Pricing models are not perfect, but there is ongoing research to build more accurate models and remove their assumptions. The simplifying assumptions might differ in models, and we will introduce them when necessary.

Now we will focus on the no-arbitrage assumption. This assumption is a general approach when pricing options since there can’t be a market equilibrium otherwise [36]. From the assumption of the no-arbitrage market, we obtain bounds for the option prices.

Let us assume we have European options, a call option $C$ and a put option $P$ having the same underlying $S$ with strike price $K$ and maturity $T$. Furthermore, we assume a risk-free asset (bond) with a risk-free interest rate $r$ during the time interval $[0, T]$.

Wegiveafundamentalrelationshipthatestablishesthepricesofputandcalloptions.

Corollary 2.2 (Put-Call parity). Under the above assumptions we have the following put-call parity for non-dividend-paying stocks:

$$C_t - P_t = S_t - Ke^{-r(T-t)}, \quad t \in [0, T]$$

(2.3)

Having established Put-Call parity, the following bounds hold for European options.

Corollary 2.3 (Bounds from above and below for European options). For every $t \in [0, T]$

$$(S_t - Ke^{-r(T-t)})^+ < C_t < S_t$$

$$Ke^{-r(T-t)} - S_t^+ < P_t < Ke^{-r(T-t)}$$

(2.4)

The proofs for Corollary 2.2 and Corollary 2.3 can be found in ([43], p.6).

Now, let us construct a portfolio in a discrete-market on a time interval $[0, T]$ which consists of holding a number $\beta$ of riskless asset $B$ (bond) and a number $\alpha$ of risky assets $S$ (stock) that are stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$. Recall from probability theory that the triplet $(\Omega, \mathcal{F}, P)$ is a probability space where $\Omega$ is a set of outcomes, $\mathcal{F} = \{\mathcal{F}_n\}_{0 \leq n \leq N}$ is a filtration of $\sigma$-algebras and $P$ is a probability measure.

Definition 2.4. The value of the portfolio $(\alpha, \beta)$ at time $t_n$ is [43]:

$$V^{(\alpha, \beta)}_n = \alpha_n S_n + \beta_n B_n, \quad n = 1, \ldots, N.$$  

(2.5)

Definition 2.5. A portfolio (strategy) $(\alpha, \beta)$ is self-financing if it holds [43]:

$$V^{(\alpha, \beta)}_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1}, \quad n = 1, \ldots, N.$$  

(2.6)

In a self-financing strategy it is natural to assume that $(\alpha, \beta)$ is predictable due to the fact that strategies are based only upon the information available at the moment.
Definition 2.6. A strategy \((\alpha, \beta)\) is predictable if \((\alpha_n, \beta_n)\) is \(\mathcal{F}_{n-1}\) – measurable for every \(n = 1, \ldots, N\).

We have mentioned that arbitrage is a strategy that begins with zero initial investment, that has zero probability of losing money and has a positive probability of making a profit. Now we will define an arbitrage mathematically.

Definition 2.7. A self-financing and predictable strategy \((\alpha, \beta)\) in the market \(\mathcal{M} = (S, B)\) is an arbitrage strategy if the value \(V = V^{(\alpha, \beta)}\) is such that:

1. \(V_0 = 0\);
   and there exists \(n \geq 1\) such that
2. \(V_n \geq 0, \ P - a.s.;\)
3. \(P(\{V_n > 0\}) > 0\)

We say that the market \(\mathcal{M}\) is arbitrage-free if it does not contain any arbitrage strategies [43].

2.3 Fundamental Pricing Theorems

After defining the fundamental market assumption of no-arbitrage in the previous section, we will now extend it in terms of the existence of an equivalent martingale measure.

We consider a discrete market \(\mathcal{M}\) on the probability space \((\Omega, \mathcal{F}, P)\) and a numéraire \((Y_t)_{t=0}^T\), which is a price process (strictly positive sequence for all \(t \in \{0, 1, \ldots, T\}\)). The numéraire choice is not unique in general, and it can be changed for computational convenience. We choose as numéraire a riskless asset process \(B\).

Therefore, we define the discounted price by \(\tilde{S}_n = \frac{S_n}{B_n}\).

For more on change of measure and Girsanov’s theorem see ([43], p. 329-334).

Next, we define the equivalent martingale measure, which gives the concept of risk-neutrality and hence the most important results in mathematical finance such as the Black-Scholes formula, fundamental theorem of asset pricing, etc. First, let us recall from measure theory the following definition on the equivalence of two probability measures.

Definition 2.8 (Equivalence of Probability Measures). We say that two probability measures \(P\) and \(Q\) on a discrete probability space \((\Omega, \mathcal{F})\) are equivalent and we denote \(P \sim Q\) if:

\[
\forall \omega \in \Omega : \quad P(\omega) = 0 \iff Q(\omega) = 0 \quad (2.7)
\]

Definition 2.9. An equivalent martingale measure is a probability measure \(Q\) on \((\Omega, \mathcal{F})\) such that:

i) \(Q \sim P;\)

ii) the discounted price \(\tilde{S}\) is a \(Q\)-martingale.
By the martingale property we have
\[
\tilde{S}_k = E^Q \left[ \tilde{S}_n \mid \mathcal{F}_k \right], \quad 0 \leq k \leq n \leq N, \quad (2.8)
\]
therefore,
\[
E^Q \left[ \tilde{S}_n \right] = E^Q \left[ E^Q \left[ \tilde{S}_n \mid \mathcal{F}_0 \right] \right] = \tilde{S}_0, \quad n \leq N. \quad (2.9)
\]

Formula (6.2) has great importance in finance because it shows that the future expectation is equal to the current value of the discounted price process. Therefore (6.2) is a risk-neutral pricing formula and \( Q \) is a risk-neutral measure.

**Theorem 2.10 (The First Fundamental Theorem of Asset Pricing).** A market \( M \) is arbitrage-free if and only if there exists at least one equivalent martingale measure.

This theorem satisfies an economic requirement of the market since the absence of arbitrage is guaranteed by the existence of a martingale measure. The first proof of the above theorem was given by Harrison and Kreps (1979, [22]); another proof can be found in ([43], p.31).

Next, we’re going to settle the uniqueness of the martingale measure.

Let us denote by \( X \) a derivative security (contingent claim) that is a \( \mathcal{F}_n \)-measurable random variable on \( (\Omega, \mathcal{F}, P) \). So far, we have only mentioned the pricing problem of derivatives.

Another essential problem when studying claims \( X \) is the replication problem, which means to determine a strategy \((\alpha, \beta)\) (if it exists) such that it assumes the same value of the claim at maturity [43]:

\[
V_N^{(\alpha, \beta)} = X \quad a.s. \quad (2.10)
\]

If such a strategy exists, \( X \) is called replicable and \((\alpha, \beta)\) is called the replicating strategy [43].

We should expect that the value process \( V_N^{(\alpha, \beta)} \) associated with replicating strategy is given uniquely; since the existence of two admissible strategies \((\alpha, \beta)\) and \((\alpha', \beta')\) with \( V_N^{(\alpha, \beta)} \neq V_N^{(\alpha', \beta')} \) would allow a riskless profit and violate the no-arbitrage condition [21].

**Theorem 2.11.** Let \( X \) be an replicable derivative in an arbitrage-free market. For every replicating strategy \((\alpha, \beta)\) and for every equivalent martingale measure \( Q \) with numéraire \( B \), we have:

\[
E^Q \left[ \frac{X}{B_N} \mid \mathcal{F}_n \right] = \frac{V_n^{(\alpha, \beta)}}{B_n}, \quad n = 0, \ldots, N. \quad (2.11)
\]

The process \( \pi_X = V^{(\alpha, \beta)} \) is called the arbitrage price or the risk-neutral price of \( X \) [43].
Definition 2.12. A market $\mathcal{M}$ is complete if every derivative is replicable.

In a complete market, every contingent claim has a unique arbitrage price; therefore, we will generalize the market’s completeness in the following theorem.

Theorem 2.13 (Completeness Theorem). An arbitrage-free market $M$ is complete if and only if there exists a unique probability measure $Q$ equivalent to $P$ under which the discounted prices are martingales [3].

The results we have presented so far are summarized in the following theorem, which is of great importance since it establishes an economic-mathematical connection between the market’s completeness and the uniqueness of an equivalent martingale measure.

Theorem 2.14 (Second Fundamental Theorem of Asset Pricing). In an arbitrage-free complete market $M$, there exists a unique equivalent martingale measure $Q$.

There are several results and proofs of the theorems presented in this section. For more, please refer to the literature ([3], [43], [20]).

2.4 Stock Price Model

Before introducing pricing models, we have to show how the price is modelled. Stock prices can be analyzed with discrete models or continuous models. In this thesis, we will analyze them with the tree method, which is a discrete model of price movements. Before introducing trees in Chapter 3, we will give the main definitions of the continuous pricing model of Black-Scholes.

Randomness is crucial in any model, and prices follow a stochastic process.

Definition 2.15 (Stochastic process). A stochastic process is a collection of random variables indexed by time $\{X_t\}_{t \in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ where $T$ is an ordered set. The set $T$ can be either discrete, for example, $T = \mathbb{N}$ or continuous $T = [0, +\infty)$.

However, not every stochastic process is a reasonable model for prices. Considering the no-arbitrage principle, we would like that the price process is modelled to some extent as a fair game. For this reason, we use the concept of a martingale.

Definition 2.16. A real-valued stochastic process $\{X_t\}_{t \in [0,T]}$ adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ is a martingale if:

- $E[X_t] < \infty, \quad \forall t$
- $E[X_t | \mathcal{F}_s] = X_s$ a.s. $\forall 0 < s \leq t$.

Definition 2.17. A martingale $X$ is called continuous if almost surely, the function $t \mapsto X_t$ is continuous.

In reality, stock prices can change at any instant of time rather than just at some fixed time. In 1900 L. Bachelier [2] considered that stock prices can be modelled with Brownian motion. Later in 1923, N. Wiener constructed Brownian motion rigorously for the first time.
2.4.1 Brownian Motion

The Brownian motion was named after botanist Robert Brown which in 1827 observed the irregular and random motion of pollen particles suspended in fluid. Today, Brownian motion forms the basis of many models in financial markets, including the famous Black-Scholes model, which we are going to introduce in section 2.5.

**Definition 2.18 (Brownian motion).** A Brownian motion is a stochastic process $B_t, t \geq 0$ which satisfies:

1. $B_0 = 0$
2. $B_t$ has stationary, independent increments
3. $B_t$ is continuous in $t$;
4. The increments $B_t - B_s$ are normally distributed with mean 0 and variance $|t - s|$, $B_t - B_s \sim N(0, |t - s|)$. \hspace{1cm} (2.12)

From condition 4. we get that $B_t$ is normally distributed with mean $E[B_t] = 0$ and $Var[B_t] = t$ $B_t \sim N(0, t)$. \hspace{1cm} (2.13)

It is worth mentioning that $B_t$ is nowhere differentiable even though it is continuous.

**Corollary 2.19.** A Brownian motion process $B_t$ is a continuous martingale with respect to the filtration $\mathcal{F}_t$ (for $s \leq t$).

Wiener later introduced a similar process like Brownian motion.

**Definition 2.20.** A Wiener process $W_t$ is a process adapted to a filtration $\mathcal{F}_t$ such that

1. $W_0 = 0$
2. $W_t$ is an $\mathcal{F}_t$-martingale with $E[W_t^2] < \infty$ for all $t \geq 0$ and $E[(W_t - W_s)^2] = t - s$, $s \leq t$;
3. $W_t$ is continuous in $t$.

From the definition, it follows that $E[W_t - W_s] = 0$, in particular $E[W_t] = 0$ and $Var[W_t] = t$. For processes $B_t$ and $W_t$, we have the following result.

**Theorem 2.21 (Lévy).** A Wiener process is a Brownian motion process.

Stock prices follow a Markov process. A Markov process is a stochastic process where only the present value of a variable is relevant for predicting the future.

**Definition 2.22.** Let $\{X_t\}_{t \in [0,T]}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$. The process is said to be Markov if:
i) The stochastic process \( \{X_t\} \) is adapted to the filtration \( \mathcal{F}_t \), and

ii) *(The Markov property)*. For each \( t = 0, 1, ..., T - 1 \) the distribution \( X_{t+1} \) conditioned on \( \mathcal{F}_t \) is the same as the distribution of the \( X_{t+1} \) conditioned on \( X_t \).

The process defined above for \( t = 0, 1, ..., T - 1 \) is called a discrete-time stochastic process. Similarly, we can define for \( t = [0, \infty) \) a continuous-time stochastic process. Markov property is important for the market’s efficiency since the predictions of the future prices should be unaffected by the previous prices. A Wiener process is a particular type of Markov process.

**Theorem 2.23.** Let \( W_t, \ t \geq 0 \), be a Brownian motion and let \( \mathcal{F}_t, \ t \geq 0 \), be a filtration for this Brownian motion. Then \( W_t, \ t \geq 0 \), is a Markov process.

### 2.4.2 Itô Formula for Brownian Motion

Brownian motion paths are irregular and nowhere differentiable. Therefore, Lebesgue-Stieltjes integral does not apply to the paths of Brownian motion. In 1944, Kiyosi Itô introduced stochastic integration (Itô’s calculus) which is an extension of the classical Lebesgue-Stieltjes integral for stochastic processes [27]. Let \( W \) be a Brownian motion on a probability space \( (\Omega, \mathcal{F}, P) \).

**Definition 2.24 (Itô process).** An Itô process is a stochastic process \( X \) of the form

\[
X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \quad t \in [0, T]
\]  
(2.14)
where $X_0$ is a $\mathcal{F}_0$-measurable random variable.

The processes $\mu$ and $\sigma$ are called drift and volatility coefficients, respectively.  

The differential form of the above equation is:

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

(2.15)

**Theorem 2.25 (Itô formula for Brownian motion).** Let $f \in C^2$ and let $W$ be a real Brownian motion. Then $f(W)$ is an Itô process and we have:

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$$

(2.16)

**Theorem 2.26 (Itô’s Lemma).** Let $X$ be the Itô process in Equation (2.15) and $f(t, x) \in C^{1,2}$. Then the stochastic process $f(t, X_t)$ is an Itô process and we have:

$$df(t, X_t) = \partial_t f(t, X_t) dt + \partial_x f(t, X_t) dX_t + \frac{1}{2} \partial_{xx} f(t, X_t) d\langle X \rangle_t.$$  

(2.17)

For the proofs of the above theorems see [43], Chapter 5.

More on Stochastic Calculus can be found in [35], [41].

### 2.4.3 Geometric Brownian Motion

A standard Brownian motion as defined in Equation (2.18) can take negative values, which makes it questionable to use it for modelling stock prices. Therefore, a geometric Brownian motion was used by Samuelson (1965, [48]) and of course, Black and Scholes (1973, [4]) to model stock prices. Geometric Brownian motion describes the evolution of the stock price process $(S_t)$ and satisfies the following **stochastic differential equation** (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0$$

(2.18)

where $\mu, \sigma \in \mathbb{R}$ and $S_0$ is the initial stock price.

The SDE in (2.18) is a shorthand notation for the following **stochastic integral equation**:

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dW_s, \quad t \in [0, T]$$

(2.19)

The unique solution of the stochastic differential equation (2.18), or equivalently, the stochastic integral equation (2.19) is the process $S_t$ given by the formula:

$$S_t = S_0 \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t), \quad t \in [0, T]$$

(2.20)

### 2.5 The Black-Scholes Model

In 1973, Fischer Black and Myron Scholes introduced their model in the Journal of Political Economy [4], which was later extended by Robert Merton [39].

Previous to their publication, other formulas were also given, but they were not
considered complete or did not influence markets. Their model was a breakthrough in option pricing theory and had a huge impact on the market of derivatives. The model is known as the Black-Scholes (Black-Scholes-Merton) model, and it was intended to price European options. We have already introduced the main tools for deriving the Black-Scholes formula in the previous sections. The first assumption of the Black-Scholes model, is that the stock price follows a geometric Brownian motion as in Equation (2.18).

To derive the Black-Scholes formula, the following assumptions are considered [4]:

- The stock price follows the geometric Brownian motion:
  \[ dS_t = \mu S_t dt + \sigma S_t dW_t \]
- There are no transaction costs.
- There are no dividends or other contributions during the life of the option.
- The risk-free rate \( r \) is known and is constant through time.
- There are no arbitrage possibilities.
- Trading of the asset can take place continuously.
- Short selling is permitted.
- We can buy or sell any fraction of the asset.

Under these assumptions, the value of the option will depend only on the price of stock at a given time and on variables that are taken to be known constants [4]. We will denote the option values by \( V(S, t) \), and at this point, we will not specify whether the option is a call or a put. Using Itô’s lemma (2.17) and assuming the stock price follows (2.18), we get the stochastic process followed by \( V \):

\[ dV = \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t. \]  (2.21)

Next, we construct a portfolio where the holder is long one option and short in a number \( \Delta \) of the underlying asset. The value of the portfolio \( \Pi \) is:

\[ \Pi(t) = V - \Delta S. \]  (2.22)

The change in the value of this portfolio in the time interval \( dt \) is:

\[ d\Pi(t) = dV - \Delta dS. \]  (2.23)

By substituting Equation (2.18) and (2.21) in (2.23), we get:

\[ d\Pi(t) = \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt + \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dW_t. \]  (2.24)
The first term in the right-hand side of Equation (2.24) is deterministic while the second term is stochastic since it involves a Wiener process $W_t$. We will make the second term deterministic by choosing $\Delta = \frac{\partial V}{\partial S}$, and Equation (2.24) becomes:

$$d\Pi = \left(\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt$$  \hspace{1cm} (2.25)

To avoid arbitrage opportunities, we should have $d\Pi = r\Pi dt$ where $r$ is the risk-free interest rate. Thus, we have:

$$r\Pi dt = \left(\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) dt$$  \hspace{1cm} (2.26)

Now, we replace $\Pi$ in the above equation by $V - \Delta S$, and $\Delta$ by $\frac{\partial V}{\partial S}$, and then divide both sides by $dt$.

Thus, we have derived the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$  \hspace{1cm} (2.27)

In order to obtain a unique solution for the Equation (2.27) we give the boundary conditions for European options. The boundary condition can be determined by using the Put-Call parity defined in Corollary (2.3).

Thus, we get for:

- Call option: $C(0, t) = 0$, $C(S, t) \to S$ as $(S \to \infty)$
- Put option: $P(0, t) = Ke^{-r(T-t)}$, $P(S, t) \to 0$ as $(S \to \infty)$

**Theorem 2.27 (Black-Scholes formulas).** The solution to the partial differential equation (2.27) is the Black–Scholes formula for the prices of European call and put options:

- Call option: $C(S, t) = SN(d_1) - Ke^{-rT}N(d_2)$
- Put option: $P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$,

where $N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}z^2} dz$ is the cumulative distribution function of the standard normal distribution, and,

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma \sqrt{T-t}}$$  \hspace{1cm} (2.28)
3 Introduction to the Tree Methods

After Black-Scholes revolutionary formula in 1973, the importance of options increased tremendously. There were ongoing research and discussions for pricing formulas. Black-Scholes formula was considered (at that time) as too advanced mathematically and with some economic limitations [17].

In 1979, Cox, Ross, and Rubinstein presented a discrete-time option pricing formula in their paper [16]. Their method was earlier in 1978 suggested by William Sharpe [49] in and it consists of deriving the same results as Black-Scholes formula using only elementary mathematics. The underlying assumption in their model is that the stock price follows a random walk. In their paper, they represent the binomial tree model, also known as the CRR tree, which assumes that in each time step, the stock price moves up or down by a certain probability.

In 1986, Phelim Boyle [7] introduced the trinomial model, which is an extension of the binomial model. The concept is similar, but under the trinomial model stock price can move up, down or remain unchanged.

Thus, trees represent all the possible paths that stock price could take during the life of the option. The advantage of the tree methods is that it is possible to check the stock price at every step in an option’s life and early exercise. This fact makes it a very useful method to price options for which the early exercise is possible and for which there are no analytical pricing formulas, such as American options.

We will begin by introducing the binomial tree.

3.1 Binomial Tree

In the binomial model, the market consists of a stock whose price is $S_t$ and a risk-free bond $B_t$. The binomial model assumes that the price of the stock or the underlying asset follows a binomial distribution, that is, in each period, the price can move up or down. In reality, stock price movements might be more complicated than assumed in the binomial model, but we will show later that the model can provide a good approximation of the real stock prices.

We begin by considering the simplest binomial model, which has only one-period, and then we generalize to a more realistic multi-period model.

3.1.1 One-period Binomial Tree

We consider a binomial tree where the expiration date is just one period away. The risk-free bond will be set to $B_0 = 1$ and is assumed to have the dynamics:

$$B_{t+1} = (1 + r)B_t$$

where $r$ is the riskless rate return.

Its price process is $B_t = (1 + r)^t$ where $t = 0, 1, ..., T$. Occasionally we also call $B$ the money market account.

At time zero, the stock price has the initial value $S_0$ with the possibility to increase to $S_0u$ with probability $p$ or to decrease to $S_0d$ with probability $1 - p$.

The dynamics of the stock can be represented with the following tree:
We refer to $u$ as the \textit{up factor} and $d$ as the \textit{down factor}. The factors $u$ and $d$ should be positive (due to the positivity of the stock price), and we assume that $d < u$ (the case when $d > u$ is achieved just by relabeling the factors; the case when $d = u$ is not random and is an uninteresting model).

The choice of $u$ and $d$ might differ depending on the model, but it must satisfy the no-arbitrage assumption. In this section, we will choose the parameters based on the CRR model [16]. Therefore, the following proposition holds:

\textbf{Proposition 1.} The binomial model of Cox, Ross and Rubinstein is arbitrage-free if and only if:

$$0 < d < 1 + r < u$$

Moreover, in this case the market is complete.

If the above inequalities do not hold (if $d \geq 1 + r$ or $u \leq 1 + r$), we would have arbitrage opportunities. The proof of the proposition can be found in [50].

We have introduced above the main assumption in the binomial pricing model. Before proceeding with the pricing process, we should also consider the market assumptions given in Section 2.2. The distinctive assumption in the binomial model is that the stock can take only two possible values in the next period. This assumption in the Black-Scholes model is replaced by that the stock price follows a Geometric Brownian motion.

Let us now consider a European call option. We denote by $C$ the current value of the call option. The owner of the option has the right (but not the obligation) to buy a share of stock at time one for the strike price $K$. During the life of the option, the stock price $S_0$ can move up to $S_0u$ or can move down to $S_0d$.

Since the one-period binomial model has just two times (time zero and time one), the option expires at time one. Therefore, the payoff of the option at time one can be $C_u = \max\{S_0u - K, 0\}$ if the price goes up or $C_d = \max\{S_0d - K, 0\}$ if the price goes down.

We represent it with the following tree:

$$C_u = \max\{S_0u - K, 0\} \quad \text{with probability } p,$$

$$C_d = \max\{S_0d - K, 0\} \quad \text{with probability } 1 - p.$$
Now, our purpose is to determine the value of the call option at time zero. We will begin by constructing a replicating portfolio. We assume a portfolio consisting of $\Delta$ shares of stock and an amount of riskless bond $B$. At the end of the period, the value of the portfolio will be:

\[ \Delta S_0 + B \]

- $\Delta S_0 u + (1 + r)B$ with probability $p$,
- $\Delta S_0 d + (1 + r)B$ with probability $1 - p$.

In the arbitrage-free market, the replication strategy at time zero must give the same payoff as the call option at time one (regardless of the stock price movement). We will choose $\Delta$ and $B$ such that the value of the portfolio at the end of the period will be equal to the value of the call option. Thus, we must have:

\[ \Delta S_0 u + (1 + r)B = C_u \quad \text{if stock price moves up,} \]
\[ \Delta S_0 d + (1 + r)B = C_d \quad \text{if stock price moves down.} \]

By solving the above equations, we get:

\begin{equation}
\Delta = \frac{C_u - C_d}{S_0(u - d)}, \quad B = \frac{uC_d - dC_u}{(u - d)(1 + r)} \tag{3.2}
\end{equation}

This choice of $\Delta$ and $B$ gives us the hedging portfolio.

Now it must hold that:

\[ C_0 = \Delta S_0 + B \]
\[ = \frac{C_u - C_d}{S_0(u - d)} + \frac{uC_d - dC_u}{(u - d)(1 + r)} \]
\[ = \frac{1}{(1 + r)} \left[ \frac{(1 + r) - d}{u - d} C_u + \frac{u - (1 + r)}{u - d} C_d \right] \tag{3.3} \]

We will simplify the equality (3.3) to the form:

\[ C_0 = \frac{1}{1 + r} [\tilde{p} C_u + \tilde{q} C_d] \tag{3.4} \]

where we have defined,

\[ \tilde{p} = \frac{1 + r - d}{u - d} \quad \text{and} \quad \tilde{q} = 1 - \tilde{p} = \frac{u - 1 - r}{u - d}. \tag{3.5} \]

The probabilities $\tilde{p}$ and $\tilde{q}$ are interpreted as the risk-neutral probabilities. The Equation (3.4) is the risk-neutral pricing formula for the one-period binomial model. The probabilities $\tilde{p}$ and $\tilde{q}$ are strictly positive since we have assumed that
$d < 1 + r < u$, and they sum up to 1. They differ from the actual probabilities $p$ and $q = 1 - p$. If we would keep the probability $p$ during the pricing procedure, then we have to perform a change of measure.

But, we should not be concerned about the probabilities of up and down movement since they are not relevant to our model. In the binomial model, the relevant parameters are $u$ and $d$. In the CRR model, it is common to choose $d = \frac{1}{u}$ but not necessary. We will discuss the choice of parameters later.

Since we have the risk-neutral measure $\tilde{P}$, we can restate the formula (3.4) in terms of an expected value:

$$C_0 = \frac{1}{1 + r} E_{\tilde{P}}[C_1]$$

(3.6)

where $E_{\tilde{P}}$ is the expected value w.r.t. probability measure $\tilde{P}$.

A similar formula also holds for the stock:

$$S_0 = \frac{1}{1 + r} E_{\tilde{P}}[S_1]$$

(3.7)

In the next section, we will generalize the formula (3.7) for $n$ – periods, and we will show that the discounted stock price is a martingale.

We have shown that in the one-period binomial tree model, each option can be replicated and have a unique no-arbitrage price.

### 3.1.2 Multi-period Binomial Tree

We extend our binomial model from the previous section to an $n$-period model. Let $T$ be the maturity time. We suppose that the time intervals have the same length:

$$\Delta t = \frac{T}{n}$$

Again, we consider a financial market consisting of a stock $S$ and a bond $B$.

The dynamics of the bond is $B_n = (1 + r)^n$ where the interest rate $r$ is constant over the time period $[0, T]$. For the stock, we assume that when passing from time $t_{n-1}$ to $t_n$, its value can only go up or go down.

Let $S_n$ be the price of the stock at period $n$:

$$S_n = \begin{cases} 
  uS_{n-1}, & \text{with probability } p, \\
  dS_{n-1} & \text{with probability } 1 - p.
\end{cases}$$

(3.8)

Its possible values at time $t = n$ are:

$$u^nS_0, u^{n-1}dS_0, \ldots, u^{n-j}d^jS_0, \ldots d^nS_0.$$ 

If we take, for example, $n = 4$, a sample path of the stock may be:

$$(S_0, uS_0, udS_0, u^2S_0, u^3dS_0)$$
or

\[(S_0, dS_0, d^2S_0, ud^2S_0, u^2d^2S_0).\]

There are \(\binom{n}{j}\) possible ways to determine the probabilities of obtaining \(j\) up-moves of the stock price from a total of \(n\) up or down moves (\(n\) time steps).

Hence:

\[P(S_n = u^j d^{n-j}S_0) = \binom{n}{j} p^j (1-p)^{n-j} \quad (3.9)\]

where \(j = 0, \ldots, n\), for \(n = 1, \ldots, N\).

Formula (3.9) corresponds to the binomial distribution.

\[\begin{align*}
\begin{array}{c}
\text{Figure 5: n-period binomial tree} \\
\end{array}
\end{align*}\]

The assumptions stated in the one-period model also hold for the multi-period model. On parameters \(u\) and \(d\), we again assume the no-arbitrage condition.

**Theorem 3.1.** In the binomial model, the condition

\[d < 1 + r < u,\]

is equivalent to the existence of an equivalent martingale measure \(\tilde{P}\).

The proof of the theorem is a consequence of the risk-neutral pricing procedure. Now, we consider a European call option. The notation we use is similar to the one-period model. We want to determine the value of the option at time \(t = 0\). For this, we will generalize the risk-neutral valuation established for the one-period case. We construct a portfolio whose payoff replicates the payoff of the option at time \(t = n\). Formula (3.8) states that the stock price \(S_n\) at time \(n\) has two possible
values, $uS_{n-1}$ or $dS_{n-1}$.

Knowing the solution of (3.7), it is straightforward that the stock price at time $n$ is:

$$S_n = \frac{1}{1 + r} E_\tilde{p}[S_{n+1}]$$

(3.10)

If we divide this equation by $(1 + r)^n$, we get:

$$S_n \left(\frac{1}{1 + r}\right)^n = E_\tilde{p} \left[ \frac{S_{n+1}}{(1 + r)^{n+1}} \right]$$

(3.11)

Equation (3.11) states that under the risk-neutral measure, the best estimate based on the information at time $n$ of the value of the discounted stock price at time $n + 1$ is the discounted stock price at time $n$ [50].

Thus, the discounted stock price is a martingale.

The value of the option at time $t = n$ is:

$$C_n = \frac{1}{1 + r} E_\tilde{p}[C_{n+1}]$$

(3.12)

We can proceed recursively to calculate the prices at any period, $n - 1, n - 2, \ldots$, and so on back to 0. So, the value of the option at time $t = 0$ is:

$$C_0 = \left(\frac{1}{1 + r}\right)^n E_\tilde{p}[C_n]$$

(3.13)

By definition, we know that $C_n = (S_n - K)^+$ and therefore we have:

$$C_0 = \left(\frac{1}{1 + r}\right)^n E_\tilde{p}[C_n]$$

$$= \left(\frac{1}{1 + r}\right)^n E_\tilde{p}[(S_n - K)^+]$$

$$= \left(\frac{1}{1 + r}\right)^n \sum_{j=0}^{n} (u^j d^{n-j} S_0 - K)^+ \mathbb{P}(S_n = u^j d^{n-j} S_0)$$

$$= \left(\frac{1}{1 + r}\right)^n \sum_{j=0}^{n} (u^j d^{n-j} S_0 - K)^+ \binom{n}{j} \tilde{p}^j (1 - \tilde{p})^{n-j}$$

(3.14)

Formula (3.14) can be written in different ways. In the following proposition, we write it as a difference of two terms.

**Proposition 2.** The call price of the $n$-period model at time $t = 0$, given by CRR pricing formula equals:

$$C_0 = S_0 \sum_{j=k}^{n} \binom{n}{j} \tilde{p}^j (1 - \tilde{p})^{n-j} - \frac{K}{(1 + r)^n} \sum_{j=k}^{n} \binom{n}{j} \tilde{p}^j (1 - \tilde{p})^{n-j}$$

(3.15)

where

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \hat{p} = \frac{\tilde{p} u}{1 + r}$$

and $k$ is the smallest integer $j$ such that $S_0 u^j d^{n-j} > K$. 

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Furthermore, we can write the formula (6.5) as:

$$C = S_0\Phi(n, k, \tilde{p}) - K\frac{1}{(1+r)^n}\Phi(n, k, \tilde{p})$$  \hspace{1cm} (3.16)$$

where

$$\Phi(n, k, p) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

The formula (6.6) above is called the *exact pricing formula* by Cox, Ross and Rubinstein and it is similar to the Black-Scholes formula. We will establish their relationship in Chapter 4. The corresponding put price of the European option can be obtained using the put-call parity.

Hence, for the put option we have:

$$P = K\frac{1}{(1+r)^n}\Phi(n, k, \tilde{p}) - S_0\Phi(n, k, \tilde{p})$$  \hspace{1cm} (3.17)$$

### 3.2 The Calibration of the Binomial Tree

The risk-neutral binomial tree is specified by the parameters $p$, $u$, $d$, and $r$. The model can be calibrated by determining these parameters from observing the real market. We have already mentioned that the arbitrage price of the derivative does not depend on the parameter $p$ and therefore, it is not needed to be calibrated. The risk-free interest rate is inherited from the model. So we focus on calibrating the parameters $u$ and $d$.

The time period $[0, T]$ is divided into $n$ discrete equal intervals denoted by $\Delta t = \frac{T}{n}$. We write the continuously-compounded interest rate factor as $e^{r\Delta t}$.

The parameters $u$ and $d$ must be calibrated to fit the mean and the variance of the underlying stock.

Usually, when estimating the volatility of a stock, it is assumed that the stock prices are lognormally distributed.

By matching the discrete-time mean return and variance under the risk-neutral measure to the continuous-time mean return and variance, we get:

i) Matching mean:

$$E_{\tilde{p}}[S_{t+\Delta t}] = [\tilde{p}S_t u + (1-\tilde{p})S_t d] = S_t e^{r\Delta t}$$  \hspace{1cm} (3.18)$$

which implies that:

$$\tilde{p} = \frac{e^{r\Delta t} - d}{u - d}$$  \hspace{1cm} (3.19)$$

Note that this formula is similar to the one derived in previous section, but now the risk-free return is written in terms of a continuously-compounded interest rate.

ii) Matching variance:

$$Var_{\tilde{p}}\left[ ln\left( \frac{S_{t+\Delta t}}{S_t} \right) \right] = \sigma^2 \Delta t.$$  \hspace{1cm} (3.20)$$
The variance in (3.20) is consistent with the Black-Scholes formula. Thus, the stock volatility $\sigma$ provides us with a simple way of calibrating $u$ and $d$. The approximate values of $u$ and $d$ are:

$$
\begin{align*}
    u &= e^{\sigma\sqrt{\Delta t}} \\
    d &= \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}.
\end{align*}
$$

The approximate values of $u$ and $d$ are:

$$
\begin{align*}
    u &= e^{\sigma\sqrt{\Delta t}} \\
    d &= \frac{1}{u} = e^{-\sigma\sqrt{\Delta t}}.
\end{align*}
$$

The above parameter choice does not completely satisfy the model since $\tilde{p} = \frac{e^{r\Delta t} - d}{u - d}$ is not necessarily in $[0, 1]$ unless $\Delta t$ is small enough. Therefore multiple parameter choices exist.

We will examine some of them in Chapter 4.

### 3.3 Pricing American Options with Trees

In this section, we examine the pricing of American options with the CRR tree. An American option can be exercised at any time before or at the expiration date $T$. The possibility of early exercise at any time actually complicates the valuation of American options. Thus, there is no closed-form solution for their valuation. However, there are several numerical methods that give a good approximation for the value of American options. Besides the valuation, there is an additional problem of choosing the optimal exercise time.

The binomial model provides a possibility to chose the exercise time optimally. In order to value an American option, we need to work backwards through the binomial tree to compare the exercise value and decide whether the exercise time is optimal. The optimal exercise time is denoted by $\tau \in [0, T]$ and it should belong to the class of stopping times.

**Definition 3.2 (Stopping time).** A random variable $\tau$ with values in $\{0, ..., T\}$ on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}_t)$ is called a stopping time if

$$
\{\tau \leq t\} \in \mathcal{F}_n \quad \text{for all} \quad t \in 0, ..., T.
$$

This means that the holder’s decision to exercise the option is only based on the information available up to time $t$.

Let us assume a discrete market where the no-arbitrage assumption holds. First, we consider an American call option with a price process $C_t$ where $t \leq T$.

**Proposition 3.** The price of an American call option in the CRR arbitrage-free model coincides with the arbitrage-free price of a European call option with the same expiry date $T$ and strike price $K$.

To prove the above proposition, it is sufficient to show that the American call option should never be exercised before maturity since otherwise, the writer of the option could make a risk-less profit.

Further argumentation can be found in [40] (see page 57).
Thus, we will focus on the American put option. In their case, an early exercise might be optimal. Recall that an American put option gives its holder the right to sell the underlying at any time before or at maturity. We denote its terminal value by:

\[ P^A_T = (K - S_T)^+ \]

**Proposition 4.** The arbitrage-free price \( P^A_t \) of an American put options equals:

\[ P^A_t = \max_{t \leq \tau \leq T} E^\mathbb{P}_{\tau} \left( \frac{1}{1 + r} (K - S_{\tau})^+ | \mathcal{F}_t \right) \quad \forall t \leq T. \quad (3.22) \]

We can show that the price \( P^A_t \) can be obtained by working recursively, for \( t < T - 1 \) so that:

\[ P^A_t = \max \left\{ (K - S_t, \frac{1}{1 + r} E^\mathbb{P}_{t+1} (P^A_{t+1} | \mathcal{F}_t) \right\} \quad (3.23) \]

This can be argued by otherwise constructing a portfolio at time \( T - 1 \) that gives a risk-free profit at time \( T \):

\[ P^A_{T-1} = \max \left\{ K - S_{T-1}, \frac{1}{1 + r} E^\mathbb{P}_{T} (K - S_{T})^+ | \mathcal{F}_T \right\} \]

Similarly, we can construct next the portfolio for the period \([T-2, T-1]\). We can proceed with the recursive procedure for as many steps as we need.

So, the CRR model provides a simple way to value American options since the valuation problem is reduced to a one-period case. We have the arbitrage-free price for American put option for \( t = 0, ..., T - 1 \), given by:

\[ P^A_t = \max \left\{ K - S_t, \frac{1}{1 + r} \left( \hat{p} P^A_{t+1} + (1 - \hat{p}) P^A_{t+1} \right) \right\} \quad (3.24) \]

where \( P^A_{t+1} \) and \( P^A_{t+1} \) denote the value of American put in the next step where the stock price can move up or down, respectively.

### 3.4 Trinomial Tree

The trinomial model is an extension of the binomial model that includes a third possible state of the price. Under the trinomial model, the prices can move up, down or remain unchanged. So the number of possible prices at time \( t \) will escalate faster than in the binomial model as \( t \) increases.

In the trinomial model, the market consists of a non-risky asset \( B_t \) and of a risky asset \( S_t \) with the price dynamics:

\[
S_t = \begin{cases} 
S_{t-1}u, & \text{with probability } p_u, \\
S_{t-1}, & \text{with probability } p_m = 1 - p_u - p_d, \\
S_{t-1}d & \text{with probability } p_d.
\end{cases} \quad (3.25)
\]
where $p_u, p_d > 0$ and $p_u + p_d < 1$.

Although the trinomial tree is constructed similarly to the binomial tree, the same hedging and replication argument do not hold here.

In the trinomial model, it is not possible to set up a unique replication portfolio. The risk-neutral probability exists, but it is not unique. So, there can only exist a set or a range of arbitrage-free prices [33].

Thus, the trinomial model is an example of an incomplete market. There are different ways to deal with the incomplete market in the trinomial model. One way is to add another risky asset $S$ in the portfolio, such that the portfolio would consist of two risky-assets $S^1, S^2$ and one riskless asset $B$ (see p.63, [43]).

Another way is to try to find a portfolio that replicates the derivative as close as possible. This can be done by creating super-replicating portfolios whose terminal values dominate that of the corresponding option payoff and then choose the cheapest strategy as the best from the set of all super-replicating strategies.

One could also create a sub-replicating portfolio and choose the most expensive strategy (see Chapter 13, [6]).

However, the most often used way in practice is to use a risk-neutral approach and fix the parametrization. So, for the option price at time step zero it must hold that:

$$C = e^{-r\Delta t} [p_u C_u + p_m C_m + p_d C_d].$$  \hspace{1cm} (3.26)

In the trinomial model we have to deal with six unknown parameters $p_u, p_d, p_m, u, d$ and $m$. They can be found by forming a system of equations from the sum of probabilities, expected return of the stock and the variance of the logarithm of the
The condition (3.28) or the no-arbitrage condition can be written in the following explicit form:

\[ 1 - p_u - p_d + p_u u + p_d d = e^{r\Delta t} \]  

(3.30)

The purpose of the first trinomial tree presented by Boyle (1986) was to enhance accuracy and speed over the binomial trees [7]. The original parameters of CRR for \( u \) and \( d \) cannot be used even if \( m = 1 \) since some of the probabilities would not lie in the interval between 0 and 1.

Boyle suggested to use a dispersion parameter \( \lambda > 1 \) to increase \( u \) and lower \( d \), since the original parameter of CRR for \( u \) cannot be used [8].

His parameter choice was:

\[ u = e^{\lambda \sigma \sqrt{\Delta t}} \]

\[ d = e^{-\lambda \sigma \sqrt{\Delta t}} = \frac{1}{u} \]

\[ m = 1 \]

(3.31)

However, this parameterization can give negative probabilities for small values of \( \lambda \). Boyle found that there is a range of parameter values that produces acceptable values for all probabilities and that the best results were obtained when \( \lambda \) was set so that the probabilities were roughly equal.

In 1991, Kamrad and Ritchken [34] presented a model with few simplifications to Boyle’s model that would correct the possible problem of negative probabilities.

Another parameterization that is often used is to set \( p_u = p_d = \frac{1}{6} \) and \( p_m = \frac{2}{3} \) (Derman et al., 1996) such that:

\[ u = e^{\sigma \sqrt{3\Delta t}}, \quad d = \frac{1}{u}, \quad m = 1, \]

(3.32)

\[ p_u = \sqrt{\frac{\Delta t}{12\sigma^2}} \left( r - \frac{1}{2}\sigma^2 \right) + \frac{1}{6}, \]

(3.33)

\[ p_m = \frac{2}{3}, \]

(3.34)

\[ p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}} \left( r - \frac{1}{2}\sigma^2 \right) + \frac{1}{6} \]

(3.35)

Thus we can construct several kinds of trinomial trees with a different choice of parameters. We will introduce the most convenient ones in Chapter 4.
The pricing procedure of European options with the trinomial tree is similar to the binomial tree. Again, backward induction is used and can be derived from the risk-neutral principle.

For a European call at node \((n, j)\) where \(n\) represents the time position and \(j\) the space position on the tree, we have [58]:

\[
C_{n,j} = e^{-r\Delta t} \left[ p_u C_{n+1,j+1} + p_m C_{n+1,j} + p_d C_{n+1,j-1}\right] \tag{3.36}
\]

The backward algorithm can be used for both European call and put options. Using it, we can calculate the value of the option at interior nodes of the tree. We can also calculate the option value at time \(n\) using the Equation (6.16).

In literature, the trinomial tree is considered equivalent to the explicit finite difference method [25]. The explicit finite difference method was first shown by Brennan and Schwartz in (1978, [9]) The idea behind finite difference methods is to simplify the partial differential equation by replacing the partial differentials with finite differences. Brennan and Schwartz applied the explicit finite difference method to solve the Black-Scholes partial differential equation. Recall the Black-Scholes partial differential equation:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf = 0.
\]

We assume that the life of the option is \(T\), and we divide it into \(N\) equal intervals of length \(\Delta t = \frac{T}{N}\). Hence, we have a total of \(N + 1\) time intervals. For the stock price, we consider \(M + 1\) equally spaced increments \(\Delta S\) of the price and denote by \(S_{\text{max}}\) the highest price value that can be reached. From time points and stock price points, we construct a grid that consists of \((N + 1)(M + 1)\) points. A point \((i, j)\) in the grid denotes time \(i\Delta t\) and stock price \(j\Delta S\), and \(f_{i,j}\) denotes the value of the option at \((i, j)\) point. In the explicit framework, it is assumed that the values \(\frac{\partial f}{\partial S}\) and \(\frac{\partial^2 f}{\partial S^2}\) at point \((i, j)\) are the same as at point \((i+1, j)\).

By solving the derivatives \(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial S}, \frac{\partial^2 f}{\partial S^2}\) and collecting terms \(f_{i,j}\) we obtain:

\[
f_{i,j}(1 + r\Delta t) = f_{i+1,j-1} \left( -\frac{1}{2} rj\Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \right) + f_{i+1,j} \left( 1 - \sigma^2 j^2 \Delta t \right) + f_{i+1,j+1} \left( \frac{1}{2} rj\Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \right) \tag{3.37}
\]

Similarly, like in the tree method, we move from the end of the grid to the beginning, and at each node, we calculate the option value [51]. We can rewrite the above equation in terms of parameters:

\[
a = \frac{1}{1 + r\Delta t} \left( -\frac{1}{2} rj\Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \right)
\]

\[
b = \frac{1}{1 + r\Delta t} \left( 1 - \sigma^2 j^2 \Delta t \right)
\]

\[
c = \frac{1}{1 + r\Delta t} \left( \frac{1}{2} rj\Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \right)
\]
and obtain:

\[ f_{i,j} = a f_{i+1,j-1} + b f_{i+1,j} + c f_{i+1,j+1} \]

We can already see the similarity to the trinomial tree. The sum of probabilities is equal to 1. Moreover \( \frac{1}{1 + r \Delta t} \) is an approximation of \( e^{-r \Delta t} \). We can write the parameters in terms of up, down and unchanged probabilities:

\[
\begin{align*}
p_u &= \frac{1}{2} r j \Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t \\
p_m &= 1 - \sigma^2 j^2 \Delta t \\
p_d &= -\frac{1}{2} r j \Delta t + \frac{1}{2} \sigma^2 j^2 \Delta t.
\end{align*}
\]

Rubinstein in his paper [46], noticed that Kamrad-Ritchken parametrization (4.42) is almost the same as the parametrization that Brennan and Schwartz use and gives a theorem in their relationship. The explicit finite difference method can be slower than other schemes [56]. In general, finite difference methods suffer from oscillations originating by non-smooth boundary conditions, and this is also mutual in tree methods. However, there are many ways the finite difference method can be improved to be faster and more accurate. The most used one is the smoothing technique.

In Chapter 4, we will discuss further the parameter choice and the convergence of the model.
4 Convergence of the Tree Models to the Black-Scholes Model

In the previous chapter, we introduced the very first trees that were established in their original papers. Many other trees have been introduced by now, and they were designed to mostly match higher moments and approximate the Black-Scholes model. In this chapter, we examine the convergence of the binomial tree model for European options. We have mentioned that the binomial method is easy to implement and can be used for pricing different option types. However, in many cases, the convergence of the method is irregular and slow. Therefore, later there were introduced more advanced tree models with an improved order of convergence.

We begin by studying the convergence of the CRR tree and showing its irregularities.

4.1 Convergence of the Binomial Tree Models to the Black-Scholes model

In Section 3.1, we have given the exact pricing formula by Cox, Ross and Rubinstein. In their seminal paper, they also consider the limiting case of the model (see page 246-254 in [16]).

When modelling, it might seem natural to consider a period of time as one day, but in the real market, trading happens almost continuously. So, the stock price could take multiple values by the end of the day, although the price can make a very small percentage change over periods. As trading takes time more frequently, \( \Delta t = \frac{T}{n} \) gets closer to zero, or equivalently \( n \) tends to infinity. When \( n \) tends to infinity, the binomial model approximates the Black-Scholes model. We had already done the first step of fitting the binomial model to the continuous-time model when we parametrized it in Section 3.2.

Let us rewrite the CRR option price formula as given in (6.6):

\[
C_{CRR} = S_0 \Phi(n, k, \tilde{p}) - Ke^{-rT} \Phi(n, k, \hat{p})
\]  

(4.1)

and also the corresponding formula given by Black and Scholes in Theorem 6.3:

\[
C_{BS} = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2).
\]  

(4.2)

We can already see a similarity between the two formulas above. However, the difference between them remains in the fact that the CRR formula is dependent on the integer \( n \), whereas the Black-Scholes formula is independent of \( n \). But, when \( n \) tends to infinity, the CRR model converges to the Black-Scholes [19].

Therefore we have,

\[
\lim_{n \to \infty} C_{CRR} = C_{BS}.
\]  

(4.3)

To show this convergence, we will use the De Moivre-Laplace Theorem, which is a special case of the Central Limit Theorem.
Theorem 4.1 (De Moivre-Laplace Theorem). Let \( Y_n \) be a binomial random variable. For every \( n \geq 1 \) and for every \( x \in \mathbb{R} \) we have:

\[
\lim_{n \to \infty} P \left( \frac{Y_n - np}{\sqrt{np(1-p)}} \leq x \right) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz \tag{4.4}
\]

where the binomial random variable \( Y_n \sim B(n, p) \) with parameters \( n \in \mathbb{N} \) and \( p \in [0, 1] \) is defined via \( P(Y = j) = \binom{n}{j} p^j (1-p)^{n-j} \).

Using the above theorem, we expect that the binomial distribution converges to the standard normal distribution as \( n \to \infty \). Now, we will give the convergence theorem for the parameter choice as given in (3.21).

Theorem 4.2. Let \( u = e^{\sigma \sqrt{\Delta t}} \) and \( d = e^{-\sigma \sqrt{\Delta t}} \) where \( \sigma > 0 \) and let the \( C_{CRR} \) be the option price as in (4.1) Then,

\[
\lim_{\Delta t \to 0} C_{CRR} = S\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{4.5}
\]

where the limit \( \Delta t \to 0 \) is equivalent to \( n \to \infty \) such that \( T = n\Delta t \) is constant and \( d_1 \) and \( d_2 \) are defined as in (6.3).

Proof. Recall from (6.6) that the binomial distribution is defined as:

\[
\Phi(n, k, p) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}.
\]

Since,

\[
S_0 u^k d^{n-k} > K, \text{ it follows that}
\]

\[
k \ln u + (n - k) \ln d + \ln S_0 > K
\]

\[
k \ln u - k \ln d > \ln K - \ln S_0 - n \ln d
\]

\[
k > - \frac{\ln \left( \frac{S_0}{K} \right) + n \ln d}{\ln \left( \frac{u}{d} \right)}
\]

\[
k = - \frac{\ln \left( \frac{S_0}{K} \right) + n \ln d}{\ln \left( \frac{u}{d} \right)} + \xi \tag{4.6}
\]

where \( \xi \) is added to make \( k \) an integer.

So \( \Phi(n, k, p) \) is the probability that the binomial process \( X_p \) with parameter \( p \) has values greater than or equal to \( k \):

\[
\Phi(n, k, p) = P(X_p \geq k)
\]

We have:

\[
C_{CRR} = SP (X_\hat{p} \geq k) - Ke^{-rT} P (X_\hat{p} \geq k)
\]
and we must show that $P(X \geq k) \rightarrow \Phi(d_1)$ and $P(X \geq k) \rightarrow \Phi(d_2)$ as $\Delta t \rightarrow 0$. We only prove the latter relation since the proof for the other one is similar.

First, we will write the Taylor expansion for $\tilde{p}$ w.r.t. $\Delta t$:

$$\tilde{p} = e^{r\Delta t - e^{-\sigma \sqrt{\Delta t}}} = \frac{(1 + r\Delta t) - \left(1 - \sigma \sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2}\right) + O(\sqrt{\Delta t})}{(1 + \sigma \sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2}) - \left(1 - \sigma \sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2} + O(\Delta t)\right)} = \frac{\sigma + \left(r - \frac{\sigma^2}{2}\right) \sqrt{\Delta t}}{2\sigma + O(\Delta t)} \text{ as } \Delta t \rightarrow 0. \tag{4.7}$$

It follows that,

$$\lim_{\Delta t \rightarrow 0} \tilde{p} = \frac{1}{2}, \text{ and } \lim_{\Delta t \rightarrow 0} \frac{2\tilde{p} - 1}{\sqrt{\Delta t}} = \frac{r - \frac{\sigma^2}{2}}{\sigma}. \tag{4.8}$$

Also from $E\left[\ln\left(\frac{S_k}{S_0}\right)\right]$ and $Var\left[\ln\left(\frac{S_k}{S_0}\right)\right]$ we get:

$$\lim_{\Delta t \rightarrow 0} n\left(\tilde{p} \ln \frac{u}{d} + \ln d\right) = \lim_{\Delta t \rightarrow 0} \frac{T}{\Delta t} \left(\tilde{p} 2\sigma \sqrt{\Delta t} - \sigma \sqrt{\Delta t}\right) \tag{4.9}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{T}{\Delta t} (2\tilde{p} - 1) \sigma = \left(r - \frac{\sigma^2}{2}\right) T \tag{4.10}$$

and,

$$\lim_{\Delta t \rightarrow 0} n \tilde{p} \left(1 - \tilde{p}\right) \left(\ln \frac{u}{d}\right)^2 = \lim_{\Delta t \rightarrow 0} \frac{T}{\Delta t} \tilde{p} \left(1 - \tilde{p}\right) \left(2\sigma \sqrt{\Delta t}\right)^2 \tag{4.11}$$

$$= \lim_{\Delta t \rightarrow 0} 4\tilde{p} \left(1 - \tilde{p}\right) \sigma^2 T = \sigma^2 T. \tag{4.12}$$

To apply the De Moivre-Laplace theorem, we need to normalize $X_{\tilde{p}}$:

$$P(X_{\tilde{p}} \geq k) = 1 - P(X_{\tilde{p}} < k) = 1 - P\left(\frac{X_{\tilde{p}} - \tilde{p}}{\sqrt{\tilde{p}(1 - \tilde{p})}} < \frac{k - \tilde{p}}{n\tilde{p}(1 - \tilde{p})}\right). \tag{4.13}$$

Now we apply to the above formula the results we’ve found in Equations (4.6), (4.9), (4.11) and $\ln \frac{S_k}{S_0} = 2\sigma \sqrt{\Delta t} \rightarrow 0$ as $\Delta t \rightarrow 0$:

$$\frac{k - \tilde{p}}{\sqrt{\tilde{p}(1 - \tilde{p})}} = \frac{-\ln \left(\frac{S}{R}\right) - n \ln d - n\tilde{p} \ln \left(\frac{S}{R}\right) - \xi \ln \left(\frac{u}{d}\right)}{\ln \left(\frac{S}{R}\right) \sqrt{n\tilde{p}(1 - \tilde{p})}} = \frac{-\ln \left(\frac{S}{R}\right) - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} \text{ as } \Delta t \rightarrow 0. \tag{4.14}$$
By letting the limit of $\Delta t \to 0$ or $n \to \infty$ in (4.13), we obtain:

$$\lim_{\Delta t \to 0} P(X_p \geq k) = 1 - \Phi \left( -\ln \left( \frac{S}{K} \right) - \left( r - \frac{\sigma^2}{2} \right) \frac{\ln T}{\sigma \sqrt{T}} \right).$$

(4.15)

And, because $1 - \Phi (-x) = \Phi (x)$:

$$\lim_{\Delta t \to 0} P(X_p \geq k) = \Phi \left( \ln \left( \frac{S}{K} \right) - \left( r - \frac{\sigma^2}{2} \right) \frac{\ln T}{\sigma \sqrt{T}} \right) = \Phi (d_2)$$

(4.16)

that concludes the proof. □

After Cox et al., the proof of the convergence was also derived by other authors (see Rendleman and Bartter [45], Hsia [24] etc.). However, it took many years to derive the first results on the rate of its convergence.

In (2000, [23]), Heston and Zhou studied the convergence rate for the CRR model with parameters $u = e^{\sigma \sqrt{\Delta t}}$ and $d = e^{-\sigma \sqrt{\Delta t}}$ and showed that the error or difference between the binomial model and Black-Scholes model is $O \left( \frac{1}{\sqrt{n}} \right)$ for a general class of options.

They gave the following proposition.

**Proposition 5.** Let $C_{CRR}$ and $C_{BS}$ be the binomial and continuous-time prices of a European call option respectively. Then:

$$C_{CRR} = C_{BS} + O \left( \frac{1}{\sqrt{n}} \right)$$

(4.17)

The proof of the above proposition can be found in their paper ([23], page 58). This proposition is formulated for call options but also holds for put options, and it states that the convergence rate is at least $\frac{1}{\sqrt{n}}$. They claim that $\frac{1}{\sqrt{n}}$ is the best possible uniform convergence rate for European option, and that the binomial model cannot converge faster than $\frac{1}{\sqrt{n}}$ at the nodes near expiration. However, they also claim that it is possible for the binomial model to converge at the $\frac{1}{n}$ rate at current nodes.

Heston and Zhou propose two approaches on how to achieve the maximum possible convergence rate of $\frac{1}{n}$ for non-smooth payoff functions, but they don’t provide an exact formula for the coefficients in the error expansion. The rate of convergence depends on the smoothness of the payoff function [23].

Also, Leisen and Reimer in [38], seem to provide evidence that the binomial solution can convergence faster than $\frac{1}{\sqrt{n}}$ and achieves rate of $\frac{1}{n}$. Still, they did not give an explicit formula for this, and it is not clear that this holds for general non-smooth payoff functions.

The first two results on the exact formula for the error term are given by Francine and Marc Diener (2004, [19]) and Walsh (2003, [55]). Walsh considered a more general class of options and a different parameter choice.
Francine and Marc Diener worked with the CRR model, and they computed the first term of the asymptotic expansion of the price of a European call option. They showed that the convergence rate for CRR trees is $O\left(\frac{1}{n}\right)$.

We state their result in the following theorem.

**Theorem 4.3.** In the $n$-period CRR binomial model for a European call option with strike price $K$, $S_0 = 1$ and maturity $T = 1$, the binomial price $C_{CRR}$ at $t = 0$ is:

$$C_{CRR} = C_{BS} + \frac{e^{-\frac{d_2^2}{2}}}{24\sigma \sqrt{2\pi}} A - 12\sigma^2 \left(\frac{\Delta_n^2 - 1}{n}\right) + O\left(\frac{1}{n\sqrt{n}}\right)$$

where $\Delta_n = 1 - 2\left\{\frac{\ln\left(\frac{u}{d}\right) + n\ln d}{\ln(\frac{u}{d})}\right\}$,

$$A = -\sigma^2 (6 + d_1^2 + d_2^2) + 4 (d_1^2 - d_2^2) r - 12r^2,$$

$\{x\}$ is the fractional part of $x$ and $C_{BS}$ is the Black-Scholes price where with coefficients $d_1, d_2$ as in Theorem 6.3.

The factor $\Delta_n$ shows how the strike value is positioned between the terminal nodes. So, the coefficient of $\frac{1}{n}$ in the error depends on the quantity of $\Delta_n$ which oscillates between $-1$ and $1$ as $n$ tends to $\infty$. This is the reason why we have oscillation of the CRR binomial price around the Black-Scholes model. Controlling $\Delta_n$ might help in getting a smooth convergence [14].

![Figure 7: European call option computed with CRR tree: $S_0 = 45$, $K = 40$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, $C_{BS} = 9.8695$.](image)
4.2 Alternative Binomial Models

Over the years, there was ongoing research in order to improve the binomial model. There are by now, more than 11 alternative versions of the original binomial model presented by Cox, Ross and Rubinstein. It is interesting to see how many different approaches and input parameters can be technically correct [13]. Actually, it can be confusing to treat all of them coherently and with a unified notation since they might have distinct assumptions, opposite notation, and were published over several years. However, we will present a few of the models and try to examine their convergence. Despite the models’ differences, mostly on parameter choice, they all give the same result in the limit. More on alternative binomial trees can be found in Mark Joshi’s book ([33], Chapter 28) and papers [31], [30], [13].

4.2.1 The Jarrow-Rudd model

Jarrow and Rudd (1983, [28]) constructed a binomial tree which is also known as the equal-probability model. They used a very similar approach to Rendleman and Bartter (1979, [45]) and are often combined together as a model in the literature. The model is based on matching the first two moments of the discrete and continuous-time log-return processes. By solving the equations from matching the first two moments similar as in Section 3.2, they obtain:

\[ p = \frac{1}{2} \]

\[ u = e^{(r^2 - \frac{\sigma^2}{2}) \Delta t + \sigma \sqrt{\Delta t}} \]

\[ d = e^{(r^2 - \frac{\sigma^2}{2}) \Delta t - \sigma \sqrt{\Delta t}} \] (4.19)

These formulas do not guarantee the absence of arbitrage because they are established arbitrarily without conditioning on the no-arbitrage equation. Years later, Jarrow and Turnbull (2000, [29]) derived the same model and fixed the no-arbitrage condition.

Now, in the no-arbitrage condition \( \hat{p} = \frac{e^{r \Delta t} - d}{u - d} \), we substitute the values in (6.11) and we obtain:

\[ \hat{p} = \frac{e^{\frac{r^2}{2} \Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}} \] (4.20)

Moreover, if we calculate the limit of the above equation we get:

\[ \lim_{\Delta t \to 0} \hat{p} = \lim_{\Delta t \to 0} \frac{e^{\frac{r^2}{2} \Delta t} - e^{-\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}} = \frac{1}{2} \] (4.21)

**Remark 4.4.** Jarrow and Rudd used a different approach from Black and Scholes. However, their choice of parameters ensures that the limiting model is the Black-Scholes model.
Figure 8: European call option computed with Jarrow-Rudd tree: $S_0 = 45$, $K = 40$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, $C_{BS} = 9.8695$.

4.2.2 The Tian model

Tian proposed a model (1993, [53]) that matches the first three moments of the binomial model to the first three moments of a lognormal distribution. The model assumes that in a risk-neutral world, the stock price follows a stochastic process which is given by following the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW$$

where risk-free rate $r$ and volatility $\sigma$ are both constant.

He did a logarithmic transformation of the above process. Then by applying Itô’s formula to the above SDE, we get:

$$d\ln S(t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW$$

(4.22)

It follows that the stock price is lognormally distributed. The return $\ln \left\{ \frac{S(t)}{S_0} \right\}$ is normally distributed with mean $\left( r - \frac{\sigma^2}{2} \right) t$ and variance $\sigma^2 t$. For the $m$th non-central moment of the stock price $S(t)$, the following formula holds:

$$E[S(t)^m | S_0] = S_0 \exp \left\{ \left( mr + (m-1)\frac{\sigma^2}{2} \right) t \right\}$$

(4.23)

This formula holds for the continuous-time, and to approximate it in the discrete-time binomial model we use $\Delta t = \frac{T}{n}$. 

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So, the binomial parameters \( u, d, p \) are chosen such that the discrete-time model converges to the log-normal distribution of the stock price in continuous-time. We have:

\[
\begin{align*}
p + (1 - p) &= 1 \\
p u + (1 - p) d &= M = e^{r \Delta t} \\
p u^2 + (1 - p) d^2 &= M^2 V = (e^{r \Delta t})^2 e^{\sigma^2 \Delta t} \\
p u^3 + (1 - p) d^3 &= M^3 V^3 = (e^{r \Delta t})^3 e^{\sigma^3 \Delta t}.
\end{align*}
\]  

(4.24)

Tian denoted \( M = e^{r \Delta t} \) and \( V = e^{\sigma^2 \Delta t} \). He suggested to add the fourth equation in (4.24), which is the third moment of the discrete-time process matching that of a continuous-time process. Then by solving the system of equations (4.24), he got:

\[
\begin{align*}
p &= \frac{M - d}{u - d}, q = 1 - p = \frac{u - M}{u - d} \\
u &= \frac{M V}{2} \left( V + 1 + \sqrt{V^2 + 2V + 3} \right) \\
d &= \frac{M V}{2} \left( V + 1 - \sqrt{V^2 + 2V + 3} \right).
\end{align*}
\]  

(4.25)

Figure 9: European call option computed with Tian binomial tree: \( S_0 = 45, K = 40, \sigma = 0.25, r = 0.1, T = 1, C_{BS} = 9.8695 \).

Remark 4.5. Tian model differs from the CRR model in two aspects. The choice \( ud = 1 \) in the CRR was for simplicity, whereas Tian chooses the correct third
moments where $ud = (MV)^2$. In the CRR model the variance is correct in the limit case and when $\Delta t \to 0$, whereas in the Tian model, both mean and variance are correct for any given $\Delta t$.

As the number of steps increases, the Tian model converges to the Black-Scholes model.

**Proposition 6.** The binomial model approaches presented by CRR (1979), Jarrow-Rudd (1983) and Tian (1993), all converge with order one.

The proof is shown in [38].

### 4.2.3 The Leisen-Reimer model

Leisen and Reimer, in their paper [38], examined the convergence of the already existing binomial models to the Black-Scholes model. They proved that the CRR model, Jarrow-Rudd model and Tian model, all converge with order $O\left(\frac{1}{n}\right)$. Besides, they found an upper bound of the approximation error in the CRR model, which holds for a generic binomial tree and can also be used for Jarrow-Rudd model and Tian model. However, the convergence of these models to the Black-Scholes in the limit as $\Delta t \to 0$ is not smooth. Thus, Leisen and Reimer defined a new binomial model such that it converges smoothly to the Black-Scholes model, and they succeed this with a second ordered convergence.

Let us rewrite the CRR option pricing formula as Leisen and Reimer did:

$$C = S_0 \Phi(n, k, \hat{p}) - K r_n^{-n} \Phi(n, k, p) \tag{4.26}$$

where:

$$p = \frac{r_n - d}{u - d}, \quad \hat{p} = \frac{u d}{d}, \quad \text{and} \quad k = \frac{\ln \left(\frac{K}{S_0}\right) - n \ln d}{\ln u - \ln d} \tag{4.27}$$

Here $k$ is calculated in a way similar to Equation (4.6), but Leisen and Reimer denoted by $k$ the number of upward movements of the stock price that exceed strike price in an $n$-step binomial tree.

Usually, in all approaches of the binomial model, the probability $p$ is approximated with the standard normal function $N(z)$, where all input arguments are determined by some adjustment function $z = h(k, n, p)$. But, Leisen and Reimer came up with the idea of an inverse transformation of the adjustment function where $h(k, n, p)$ specifies the distribution parameter $h^{-1}(z) = p$ to approximate $P = N(z)$ with $P \approx 1 - \Phi(k, n, p)$ [38]. In their paper, they developed three formulas using different approximation but the most famous one is the Peizer-Pratt method:

$$h^{-1}(z) = \frac{1}{2} \pm \left[ \frac{1}{4} - \frac{1}{4} \exp \left\{ - \left( \frac{z}{(n + \frac{1}{3})} \right)^2 \left( n + \frac{1}{6} \right) \right\} \right]^{1/2} \tag{4.28}$$

The above formula holds for an odd number of steps. By solving the above equations, Leisen and Reimer derived the following parameters which guarantee the convergence of the binomial model:

$$\hat{p} = h^{-1}(d_1), \quad p = h^{-1}(d_2)$$
Leisen and Reimer did not give a proof of the second order convergence. The proof was given many years later by Joshi (2010, [31]) for odd number of steps, and then Xiao (2010, [57]) extended the proof for even number of steps [37].

\[ u = r_n \frac{\hat{p}}{p} \quad \text{and} \quad d = \frac{r_n - pu}{1 - p} \]  

\[ (4.29) \]

\[ S_0 = 45, \ K = 40, \ \sigma = 0.25, \ r = 0.1, \ T = 1, \ C_{BS} = 9.8695. \]

\[ \text{Figure 10: European call option computed with Leisen-Reimer tree: } S_0 = 45, \ K = 40, \ \sigma = 0.25, \ r = 0.1, \ T = 1, \ C_{BS} = 9.8695. \]

\[ 4.2.4 \quad \text{The Trigeorgis model} \]

The Trigeorgis model (1991, [54]) is designed based on a log-transformation of the Black-Scholes model.

He considers the following process:

\[ \frac{dS(t)}{S(t)} = \alpha dt + \sigma dW \]  

\[ (4.30) \]

where \( \alpha \) is the expected value, \( \sigma \) is the standard deviation and \( W \) is a Wiener process.

Let’s denote \( X(t) = \ln S(t) \) which follows a Brownian motion. Under the risk-neutral probability, we have \( \alpha = r \) and the process:

\[ dX = \ln \left( \frac{S(t + dt)}{S(t)} \right) = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dW \]  

\[ (4.31) \]

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The process \( \ln \left( \frac{S(t+\Delta t)}{S(t)} \right) \) or equivalently \( dX \) is normally distributed and we write the expected values and variance as:

\[
E[dX] = \left( r - \frac{\sigma^2}{2} \right) \Delta t
\]

\[
Var[dX] = \sigma^2 \Delta t
\]

(4.32)

We approximate the continuous-time process by the discrete-time process by letting \( \Delta t = \frac{T}{n} \). In each interval, the process \( X \) moves up by an amount \( \Delta X = H \) with risk-neutral probability \( p \), or moves down by the same amount \( \Delta X = -H \) with probability \( 1 - p \) [54].

The expected value and the variance of the discrete-time process are:

\[
E[\Delta X] = pH + (1-p)H = 2pH - H
\]

\[
Var[\Delta X] = H^2 - (2pH - H)^2 = H^2 - (E[\Delta X])^2
\]

(4.33)

To have consistency between the discrete-time process and continuous-time process, their expected values and variances must be equal:

\[
2pH - H = 2p\Delta X - \Delta X = \Delta X(2p - 1) = \left( r - \frac{\sigma^2}{2} \right) \Delta t
\]

\[
4pH^2 - 4p^2H^2 = 4p(\Delta X)^2(1-p) = \sigma^2 \Delta t
\]

(4.34)

Solving the above equations we obtain:

\[
\Delta X = \sqrt{\sigma^2 \Delta t + \left( r - \frac{\sigma^2}{2} \right)^2 (\Delta t)^2}
\]

\[
p = \frac{1}{2} \left[ 1 + \left( r - \frac{\sigma^2}{2} \right) \frac{\Delta t}{\Delta X} \right]
\]

(4.35)

Here \( u = e^X \) and \( d = \frac{1}{u} = e^{-X} \).

**Remark 4.6.** In the limit the Trigeorgis’s risk-neutral probability converges to \( \frac{1}{2} \), \( \lim_{\Delta t \to 0} p = \frac{1}{2} \). Thus, the model is arbitrage-free in the limit [13].

### 4.2.5 The Chang-Palmer model

Chang and Palmer (2007, [14]) studied the convergence rate for the \( n \)-period binomial model where the parameters \( u \) and \( d \) are more generic than those used in the CRR model. Actually, they slightly generalize the convergence theorem given by Deiner and Deiner (2004, [19]). They also developed a new binomial model known as the center binomial model. First, we give the generalized class of binomial models as introduced by Chang and Palmer.

**Definition 4.7.** Let \( \Delta t = \frac{T}{n} \) and \( \lambda_n \) be an arbitrary bounded function of \( n \). We consider the \( n \)-period binomial model where:

\[
u = e^{\sigma \sqrt{\Delta t} + \lambda_n \sigma^2 \Delta t}
\]

\[
d = e^{-\sigma \sqrt{\Delta t} + \lambda_n \sigma^2 \Delta t}
\]

(4.36)

with initial stock price \( S_0 \).
Remark 4.8. We observe that the choice \( \lambda_n = 0 \) gives the Cox, Ross and Rubinstein model, and the choice \( \lambda_n = \frac{r}{\sigma^2} - \frac{1}{2} \) gives the Jarrow and Rudd model.

Now, we give the main theorem in [14].

**Theorem 4.9** (Main theorem - Chang and Palmer). Let \( u \) and \( d \) be defined by Equation (6.15). The price of a European call option satisfies:

\[
C(n) = C_{BS} + \frac{S_0 e^{-d^2/2}}{24\sigma\sqrt{2\pi T}} A_n - \frac{12\sigma^2 T (\Delta_n^2 - 1)}{n} + O\left(\frac{1}{n}\right) \tag{4.37}
\]

where

\[
\Delta_n = 1 - 2 \left\{ \frac{\log \left( \frac{S_0}{K} \right) + n \log d}{\log \left( \frac{u}{d} \right)} \right\}
\]

\[
A_n = -\sigma^2 T (6 + d^2_1 + d^2_2) + 4T (d^2_1 - d^2_2) (r - \lambda_n \sigma^2) - 12T^2 (r - \lambda_n \sigma^2)^2 \tag{4.38}
\]

We don’t give the proof of theorem since it is very technical, although it is considered slightly easier than Deiner and Deiner’s theorem proof. Chang and Palmer in their theorem show the result of the remaining term to be \( O\left(\frac{1}{n}\right) \). However, many obtained that the remaining term is actually \( O\left(\frac{1}{n\sqrt{n}}\right) \).

Figure 11: European call option computed with Chang-Palmer tree: \( S_0 = 45, K = 40, \sigma = 0.25, r = 0.1, T = 1, C_{BS} = 9.8695 \).

### 4.3 Convergence of the Trinomial Tree

The convergence of the trinomial model has been studied less than the convergence of the binomial model. Thus, there are not many analytical results on the rate of
the convergence of the trinomial model. However, from numerical results and simulations, it is clear that the trinomial model price converges to the Black-Scholes model price for a large number of \( n \). According to these results, the trinomial model converges faster to the Black-Scholes model than the binomial model. This comes from the fact that the trinomial tree can be seen as a binomial tree regarded at every second-time node only [46]. A binomial tree after two-time steps has exactly three distinct nodes, which is equal to the number of nodes after one step in the trinomial model [18]. This means that the trinomial tree model requires half as many steps as the binomial tree model. More on the relationship between binomial and trinomial tree models can be found in Rubinstein’s paper [46]. Therefore another possible way to construct a trinomial tree is to consider two steps of a binomial tree as a single step of a trinomial tree. This model could then be applied to all standard binomial trees with constant volatility, e.g. CRR, Jarrow-Rudd, Trigeorgis, Tian, etc.

For example, a two-step representation of the CRR binomial model is:

\[
\begin{align*}
  u &= e^{\sigma \sqrt{2 \Delta t}} \\
  d &= e^{-\sigma \sqrt{2 \Delta t}} \\
  m &= 1
\end{align*}
\]

and transition probabilities,

\[
\begin{align*}
  p_u &= \left( \frac{e^{r \Delta t} - e^{-\sigma \sqrt{2 \Delta t}}}{e^{\sigma \Delta t} - e^{-\sigma \sqrt{2 \Delta t}}} \right)^2 \\
  p_d &= \left( \frac{e^{-\sigma \sqrt{2 \Delta t}} - e^{r \Delta t}}{e^{\sigma \Delta t} - e^{-\sigma \sqrt{2 \Delta t}}} \right)^2 \\
  p_m &= 1 - p_u - p_d
\end{align*}
\]

In Section 3.4, we have introduced Boyle’s trinomial tree with parameters:

\[
\begin{align*}
  u &= e^{\lambda \sigma \sqrt{\Delta t}} \\
  d &= e^{-\lambda \sigma \sqrt{\Delta t}} = \frac{1}{u} \\
  m &= 1
\end{align*}
\]

and transition probabilities,

\[
\begin{align*}
  p_u &= \frac{u(V + M^2 - M) - (M - 1)}{(u^2 - 1)(u - 1)} \\
  p_d &= \frac{u^2(V + M^2 - M) - u^3(M - 1)}{(u^2 - 1)(u - 1)} \\
  p_m &= 1 - p_u - p_d
\end{align*}
\]

where \( V = e^{\sigma^2 \Delta t} \) and \( M = e^{r \Delta t} \).

Boyle found that for a range of parameter values, the accuracy of the trinomial...
model with 5-time intervals was comparable to that of the CRR method with 20-
time intervals ([8], p.6). A numerical study in the paper [44], shows that that order
of convergence for a trinomial model constructed from CRR binomial model is 1.5,
whereas the order of convergence for Boyle’s trinomial model is 1.85.
The authors conclude that in general, the order of convergence for trinomial models
is higher than one [44].

Another famous trinomial model was introduced by Kamrad and Ritchken (1991,
[34]), which was intended to improve Boyle’s model.
They gave the following parametrization for the model:

\[ u = e^{\lambda \sigma \sqrt{\Delta t}} \]
\[ d = e^{-\lambda \sigma \sqrt{\Delta t}} = \frac{1}{u} \]
\[ m = 1 \]

with transition probabilities,

\[ p_u = \frac{1}{2\lambda^2} + \frac{\left( r - \frac{1}{2} \sigma^2 \sqrt{\Delta t} \right)}{2\lambda \sigma} \]
\[ p_d = \frac{1}{2\lambda^2} - \frac{\left( r - \frac{1}{2} \sigma^2 \sqrt{\Delta t} \right)}{2\lambda \sigma} \]
\[ p_m = 1 - \frac{1}{\lambda^2} \]  

(4.42)
This model corrects Boyle’s problem of negative probabilities and for any values of \( \lambda, \lambda \geq 1 \) we get a feasible set of probabilities. Kamrad and Ritchken compare their model to the binomial model by numerical examples. It results that the trinomial model is more accurate than the binomial. The trinomial model converges faster, and the error is smaller in comparison with the binomial model [34]. However, one should consider that the trinomial model is computationally more expensive.

We introduce two more parametrizations of the trinomial tree given by Tian (1993, [53]). The first one is known as an equal probability tree with up and down parameters equal to \( \frac{1}{3} \), and it consists of matching the first two moments:

\[
\begin{align*}
    m &= \frac{M(3 - V)}{2} \\
    K &= \frac{M(V + 3)}{4} \\
    u &= K + \sqrt{K^2 - m^2} \\
    d &= K - \sqrt{K^2 - m^2}
\end{align*}
\]

where \( V = e^{\sigma^2 \Delta t} \) and \( M = e^{r \Delta t} \).

The second parametrization given by Tian ([53], p.568) drops the equal probability
constraint and matches the first four moments [12]:

\[ u = K + \sqrt{K^2 - m^2} \]
\[ m = MV^2 \]
\[ d = K - \sqrt{K^2 - m^2} \] (4.44)

and,

\[ p_u = \frac{md - M(m + d) + M^2V}{(u - d)(u - m)} \]
\[ p_d = \frac{um - M(m + u) + M^2V}{(u - d)(m - d)} \] (4.45)

where \( K = \frac{M}{2} (V^4 + V^3) \), \( V = e^{\sigma^2 \Delta t} \) and \( M = e^{r \Delta t} \).

Figure 14: European call option computed with Tian trinomial tree: \( S_0 = 45 \), \( K = 40 \), \( \sigma = 0.25 \), \( r = 0.1 \), \( T = 1 \), \( C_{BS} = 9.8695 \).

From numerical results obtained by Chan et al. [12], the Tian’s trinomial tree is less effective than the Kamrad-Ritchken’s tree and Boyle’s tree.

We can also conclude this by comparing the figures of each of the trees above, where Tian’s trinomial tree obviously suffers from more oscillations.

Numerical results comparing binomial and trinomial trees mentioned above to the Black-Scholes price are shown in Table 1.
Figure 15: A comparison of the convergence of some binomial and trinomial models when \( S_0 = 45, K = 40, \sigma = 0.25, r = 0.1, T = 1, C_{BS} = 9.8695 \).
The Kamrad-Ritchken trinomial tree seems to have the closest value to the Black-Scholes model.

<table>
<thead>
<tr>
<th>Model</th>
<th>CRR</th>
<th>JR</th>
<th>LR</th>
<th>CP</th>
<th>Tian B</th>
<th>Tian T</th>
<th>Boyle</th>
<th>KR</th>
</tr>
</thead>
</table>

Table 1: A comparison of prices obtained from different models computed after \( n \) steps when \( S_0 = 45, K = 40, \sigma = 0.25, r = 0.1, T = 1, C_{BS} = 9.8695 \)
4.4 Accelerating Convergence

We established the convergence of binomial and trinomial tree methods in the previous sections. Although we have seen different parametrization of the trees, the rate of convergence remains slow in usual models. Moreover, convergence is oscillatory. Therefore to deal with the slow convergence of the trees, there have been introduced some techniques that accelerate the convergence.

Since there are not many results on the acceleration techniques for option pricing, we will rely on the paper by Chan et al [12]. The most commonly used techniques to accelerate the convergence are smoothing, Richardson extrapolation, control variate, and truncation. These techniques can be used independently or combined, giving 16 combinations of accelerating techniques that can be used to price options using trees [12].

4.4.1 Smoothing

The smoothing technique was proposed by Broadie and Detemple (1996, [10]). Their idea is simple. Let’s say we use a CRR method to calculate the option value. Just before maturity, we replace this value with the Black-Scholes value. To get fewer oscillations, we apply it in all the nodes close to maturity. Their paper concludes that after applying this technique, the error is reduced and the convergence is smoother.

The results of the smoothing technique can be seen in Figure 16.

![Figure 16: Convergence of the American put option with and without smoothing technique; computed with Tian binomial: $S_0 = 100$, $K = 90$, $r = 0.05$, $\sigma = 0.30$ and $T = 0.5$. The true price is taken as 3.345 [47].](image)

Figure 16: Convergence of the American put option with and without smoothing technique; computed with Tian binomial: $S_0 = 100$, $K = 90$, $r = 0.05$, $\sigma = 0.30$ and $T = 0.5$. The true price is taken as 3.345 [47].
4.4.2 Richardson extrapolation

Richardson extrapolation technique in option pricing was also presented by Broadie and Detemple [10]. However, for this subsection, we will use the explanation given by Joshi [32]. Richardson extrapolation is considered a very useful technique for increasing the speed of the convergence. It works by eliminating the error term [15]. Let \( C_n \) be the option price generated by a tree after \( n \) steps and let \( C_{n}^{RE} \) be the price with Richardson extrapolation applied.

We have

\[
C_n = C_{True} + \frac{\epsilon}{n} + O\left(\frac{1}{n}\right) \tag{4.46}
\]

where \( \epsilon \) is a constant and \( C_{True} \) is the correct price. (The above equation is not generally true for American put options in binomial trees. However, the technique can still be used [15].) The Richardson extrapolated value is constructed as follows [15]:

\[
C_{n}^{RE} = C_{True} + O\left(\frac{1}{n}\right) \tag{4.47}
\]

which was obtained by a weighted sum of the \( n \) step tree price and a price generated by a tree with \( \lfloor \frac{n}{2} \rfloor \) steps:

\[
C_{n}^{RE} = \omega C_n + (1 - \omega)C_{\lfloor \frac{n}{2} \rfloor} \tag{4.48}
\]

Now, we want to cancel the error term \( \epsilon \) and we get for \( \omega \):

\[
0 = \omega \frac{\epsilon}{n} + (1 - \omega)\frac{\epsilon}{\lfloor \frac{n}{2} \rfloor} \tag{4.49}
\]

Solving the above gives:

\[
\omega = \begin{cases} 
2 & \text{if } n \text{ is even} \\
\frac{2n}{n+1} & \text{if } n \text{ is odd.}
\end{cases} \tag{4.50}
\]

For \( n \) even, \( \omega = 2 \) we have the extrapolated value:

\[
C_n^{RE} = 2C_n - C_{\lfloor \frac{n}{2} \rfloor} \tag{4.51}
\]

Richardson extrapolation works well with smoothing technique because the smoothing reduces the oscillations, which improves the performance of Richardson extrapolation [15].
Figure 17: Convergence of the American put option with and without Richardson extrapolation technique; computed with CRR binomial tree: $S_0 = 45$, $K = 50$, $\sigma = 0.3$, $r = 0.05$, $T = 1$. True price is taken as 2.322.
Source: Wolfram Demonstrations Project.
4.4.3 Truncation

The truncation technique is based on the idea that trees should be pruned in order to not waste time computing nodes that are not of our interest [15]. There are a few methods on how to apply this technique. We will follow the method suggested by Andricopoulos et al. (2004, [1]).

This method is also known as the standard deviation method. It works by selecting to truncate nodes in the tree based on standard deviations in the log space from the present value of the strike price, or standard deviations from the future value of the current stock price, or both [15].

Thus, we have three ways to set the boundaries of the truncation.

Let $\xi$ denote the standard deviation.

The boundaries from truncation based on the strike price $K e^{-r(T-t)}$ are [15]:

upper bound: $S_{max} = Ke^{-r(T-t)+\xi \sigma \sqrt{(T-t)}}$

lower bound: $S_{min} = Ke^{-r(T-t)-\xi \sigma \sqrt{(T-t)}}$  \hspace{1cm} (4.52)

The boundaries from the truncation based on the stock price $S_0 = e^{rt}$ are [15]:

upper bound: $S_{max} = S_0 e^{rt+\xi \sigma \sqrt{t}}$

lower bound: $S_{min} = S_0 e^{rt-\xi \sigma \sqrt{t}}$  \hspace{1cm} (4.53)

The boundaries from truncation based both in strike price and stock price are [15]:

upper bound: $S_{max} = \min \left( S_0 e^{rt+\xi \sigma \sqrt{t}}, Ke^{-r(T-t)+\xi \sigma \sqrt{(T-t)}} \right)$

lower bound: $S_{min} = \max \left( S_0 e^{rt-\xi \sigma \sqrt{t}}, Ke^{-r(T-t)-\xi \sigma \sqrt{(T-t)}} \right)$  \hspace{1cm} (4.54)

The reasoning behind this idea is that, if the number of standard deviations is large enough, the node’s value does not make a much difference to the value of the option.

For further discussion on the truncation technique and its results see Chen and Joshi [15].

4.4.4 Control variates

The control variate technique was established by Hull and White (1988, [26]). It helps in reducing variance, and it is mostly used for American options.

Given a binomial tree, we price both American and Europeans options.

Based on this technique, the size of the error in the tree price for American option is related to the size of the error when the same tree is used to price a European option [47].

Let $P_A$ be the price for an American put option generated by a tree, $P_E$ be the price generated by the same tree for the European put option and $P_{BS}$ be the Black-Scholes price for European option [30].

The price generated by the control variate technique (error controlled price) $P_{CV}$ is:

$$P_{CV} = P_A + (P_{BS} + P_E)$$
However, in a study by Joshi (2007, [30]) for American options, it was shown that the control variate technique is inferior to the Richardson extrapolation and smoothing technique. Therefore we will not give results for this method here. In a study by Joshi [30] the best performing binomial tree (for American put) when applying smoothing, Richardson extrapolation and truncation techniques was the Tian third moment matching tree. In the previous section, we mentioned that the best performing trinomial trees (without acceleration techniques) are Kamrad-Ritchken tree and Boyle’s tree (when $\lambda = 1.3$). According to [12], when applying acceleration techniques (for American put) the best performing trinomial trees are Tian fourth moment-matching tree and Tian Equal-Probability tree (with Richardson extrapolation, smoothing, truncation). Overall, the best combination of acceleration techniques was concluded [12] to be smoothing, Richardson extrapolation and truncation. But if only one acceleration technique has to be used then the truncation and control variate techniques perform better individually than smoothing or Richardson extrapolation [12]. For numerical results, see Chan et al. [12].
5 Conclusion

First, we introduced the main concepts of option theory. We gave the necessary assumptions in order to proceed with pricing models. We showed how the stock price is modelled and gave the definition of the Brownian motion. Afterwards, we presented the revolutionary formula of Black and Scholes that would later serve as a benchmark for the tree model.

In Chapter 3, we established the binomial tree model. The binomial model is considered as an easier and more intuitive framework than the Black-Scholes model. The no-arbitrage arguments and martingales helped us derive the Cox-Ross-Rubinstein model, the first tree introduced for pricing options. Another advantage of the tree method is that it can price American options, and we showed the pricing formula. A surprising fact was that the up and down movement probabilities were not explicitly defined but instead followed by a no-arbitrage argument. After the Cox-Ross-Rubinstein publication, other academics have explored the concept of the tree method. Thus, we presented other constructions of trees and discussed the diversity of tree parametrization. Briefly, we have touched upon the trinomial trees but without giving the no-arbitrage argument. Our main goal was to prove the convergence of the binomial model to the Black-Scholes model as the number of steps $n$ increases.

With the help of a version of Central Limit Theorem, we proved this convergence. Next, we gave some results about the order of the convergence. It was established that the binomial tree convergences with order $1/n$. Also, we gave some characteristics of the convergence and the behaviour of some trees. We concluded that all parametrizations convergence pretty much with the same rate. There are many attempts to accelerate the convergence of trees, and we presented some proposed acceleration techniques. However, due to many possible combinations and different specifications of the option, it is difficult to conclude what model and method is more efficient. More studies on the convergence rate and the approximation error might be needed. We conclude that trees have the potential to be investigated further.
Razširjeni povzetek v slovenščini

6.1 Uvod


Vnaprej določena cena v pogodbi se imenuje izvršilna cena in je označena s $K$; datum v pogodbi se imenuje zapadlost in je označen s $T$. Uveljavitev pogodbe s strani kupca se imenuje izvršitev opcije; v nasprotnem primeru se opcija opusti. Obstajajo tri glavne skupine opcij na podlagi pogodbenih specifikacij. Vendar bomo omenili samo evropske in ameriške opcije. Evropske opcije so unovčljive samo na datum zapadlosti. Lahko jih ovrednotimo z Black-Scholesovim modelom, ki ponuja rešitev z zaprto formulo. Ameriške opcije se lahko izvšijo kadar koli pred datumom zapadlosti. Ameriške opcije dajejo imetniku več pravic kot pa evropske in so lahko s tem bolj dragocene. Izračun cen za ameriške opcije je bolj zapleten in ne obstaja rešitev z zaprto formulo.

Naj bo osnovno sredsto delnica. Potem evropska nakupna opcija imetniku daje pravico do nakupa osnovne delnice za izvršilno ceno $K$ v času $T$. S $S_T$ označimo ceno delnice ob času $T$. Pri času $t = 0$, poznamo samo izvršilno ceno $K$, vendar ne poznamo cene delnice $S_T$. Z vidika imetnika opcije je izplačilo $C$ ob času datuma izteka $T$ evropske nakupne opcije podano z naslednjo enačbo:

$$C(S_T) = (S_T - K^+) = max\{S_T - K, 0\}.$$ Evropska prodajna opcija daje imetniku pravico prodaje osnovne delnice za izvršilno ceno $K$ ob času zapadlosti $T$. Sedaj je izplačilo $P$ za imetnika prodajne opcije ob času datuma izteka $T$ podano z naslednjo enačbo:

$$P(S_T) = (K - S_T^+) = max\{K - S_T, 0\}.$$
6.2 Modeli oblikovanja cen

Za določanje cen opcij moramo uporabiti nekaj poenostavljenih predpostavk. Ta najbolj pomembna je predpostavka o *neobstoju arbitraže*. To predpostavko razširimo v smislu obstoja *ekvivalentne martingalske mere*.

**Definicija 6.2.** Ekvivalentna martingalska mera je verjetnostna mera \( Q \) na \((\Omega, \mathcal{F})\) tako da:

1. \( Q \) je ekvivalentna \( P \), to je \( Q(A) \neq 0 \) natanko tedaj, ko je \( P(A) \neq 0 \);
2. diskontirana cena \( \tilde{S}_t = \frac{S_t}{(1+R)^t} \), \( R \) je netvegana obrestna mera, je \( Q \)-martingal.

Po martingalskih lastnostih imamo

\[
\tilde{S}_k = E^Q \left[ \tilde{S}_n \mid \mathcal{F}_k \right], \quad 0 \leq k \leq n \leq N, \tag{6.1}
\]

tako sledi,

\[
E^Q \left[ \tilde{S}_n \right] = E^Q \left[ E^Q \left[ \tilde{S}_n \mid \mathcal{F}_0 \right] \right] = \tilde{S}_0, \quad n \leq N. \tag{6.2}
\]

Enačba (6.2) nam poda *do tveganja nevtralno vrednost* in \( Q \) je *do tveganja nevtralna verjetnost*.

**6.2.1 Black-Scholesov model**

Kot naslednje uvedemo Black-Scholesovo formulo za določitev cen evropskih opcij. Prva predpostavka Black-Scholesovega modela je, da cena delnice sledi geometričnemu Brownovemu gibanju (za druge predpostavke glej [4]).

**Trditev 6.3 (Black-Scholesova formula).** Za cene evropskih nakupnih in prodajnih opcij imamo:

- **Nakupna opcija:** \( C(S, t) = SN(d_1) - Ke^{-rT}N(d_2) \)
- **Prodajna opcija:** \( P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \),

kjer je \( N(d) = \frac{1}{\sqrt{2\pi}} \int_{-d}^{d} e^{-\frac{1}{2}z^2} dz \) *kumulativna porazdelitvena funkcija standardne normalne porazdelitve in*

\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}}, \tag{6.3}
\]

\[
d_2 = \frac{\ln \left( \frac{S}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) (T-t)}{\sigma \sqrt{T-t}}.
\]

Black-Scholesova formula poda premijo za evropske opcije v zveznem času.
6.2.2 Drevesne metode

Leta 1979 so Cox, Ross in Rubinstein predstavili enačbo za določanje cene opcije z diskretnim časom [16]. Osnovna predpostavka njihovega modela je, da cena delnic sledi slučajnemu sprehodu. V svojem delu so predstavili binomski drevesni model, tudi poznan kot CRR drevo, ki predpostavlja, da se v vsakem časovnem koraku cena delnice z določeno verjetnostjo giblje gor ali dol. Časovna perioda \([0, T]\) je razdeljena na \(n\) diskretnih enakih intervalih označenih z \(\Delta t = T/n\). Obrestni faktor pri zveznem obrestovanju zapišemo kot \(e^{r\Delta t}\). V binomskem modelu je trg sestavljen iz delnice, katere cena je \(S_t\) in iz netvegane obveznice \(B_t\). Ob času 0 ima delnica začetno vrednost \(S_0\) z možnostjo povečanja na \(S_0u\) z verjetnostjo \(p\) ali zmanjšanja na \(S_0d\) z verjetnostjo \(1-p\). Ustrezajoči faktor naraščanja in faktor padanja, ki mora izpolnjevati nearbitražno predpostavko.

Trditev 6.4. V binomskem modelu je pogoj

\[ d < 1 + r < u, \]  

enakovreden obstoju ekvivalentne martingalske verjetnosti \(\hat{P}\).

Izplačilo opcije ob času ena je lahko

\[ C_u = \max\{S_0u - K, 0\} \]  

če cena naraste ali

\[ C_d = \max\{S_0d - K, 0\} \]  

če cena pade. Ob času \(n\) ima cena delnice \(S_n\) dve možni vrednosti, \(uS_{n-1}\) ali \(dS_{n-1}\).

Trditev 6.5. Premija za nakupno opcijo v n-obdobjem modelu ob času \(t = 0\), podana s CRR modelom je enaka:

\[ C_0 = S_0 \sum_{j=k}^{n} \binom{n}{j} \hat{p}^j (1 - \hat{p})^{n-j} - \frac{K}{1+r} \sum_{j=k}^{n} \binom{n}{j} \hat{p}^j (1 - \hat{p})^{n-j} \]  

kjer

\[ \hat{p} = \frac{1 + r - d}{u - d}, \quad \hat{p} = \frac{\hat{p}u}{1 + r} \]

in je \(k\) najmanjše tako, da je \(S_0 w^j d^{n-j} > K\). Nadaljnje lahko enačbo (6.5) zapišemo kot:

\[ C = S_0 \Phi(n, k, \hat{p}) - K \frac{1}{(1+r)^n} \Phi(n, k, \hat{p}) \]  

kjer

\[ \Phi(n, k, p) = \sum_{j=k}^{n} \binom{n}{j} p^j (1 - p)^{n-j}. \]

Že lahko opazimo podobnost med Black-Scholesovo formulijo (6.3) in CRR enačbo (6.6) za določanje cen opcij. Kadar \(n\) teži v neskončnost, CRR model konvergira.
proti Black-Scholesovemu modelu \[19\].
Zato imamo,
\[
\lim_{n \to \infty} C_{CRR} = C_{BS}. \tag{6.7}
\]
Za prikaz te konvergence bomo uporabili De Moivre-Laplaceov izrek ki je poseben primer centralnega limitiranega izreka.

**Trditev 6.6.** Naj bo \( u = e^{\sigma \sqrt{\Delta t}} \) in \( d = e^{-\sigma \sqrt{\Delta t}} \) kjer \( \sigma > 0 \) in naj bo \( C_{CRR} \) cena opcije. Potem,
\[
\lim_{\Delta t \to 0} C_{CRR} = S \Phi(d_1) - Ke^{-rT} \Phi(d_2) \tag{6.8}
\]
pri čemer je limita, ko gre \( \Delta t \to 0 \) enaka kot limita ko gre \( n \to \infty \) tako, da je \( T = n \Delta t \) konstanten ter sta \( d_1 \) in \( d_2 \) definirana kot v (6.3).

Čeprav je bil dokaz o tej konvergenci izpeljan v prvotnem članku Coxa et al. \[16\], je trajalo mnogo let za izpeljavo prvih rezultatov stopnje te konvergence. V (2000, \[23\]) sta Heston in Zhou preučevala stopnjo konvergence za CRR model s parametri \( u = e^{\sigma \sqrt{\Delta t}} \) in \( d = e^{-\sigma \sqrt{\Delta t}} \) in pokazala da je napaka ali razlika med binomskim modelom in Black-Scholesovim modelom \( O\left(\frac{1}{\sqrt{n}}\right) \) za splošno vrsto opcij. Podala sta naslednji rezultat.

**Trditev 6.7.** Naj bosta \( C_{CRR} \) binomska in \( C_{BS} \) Black-Sholesova cena evropske nakupne opcije. Potem je
\[
C_{CRR} = C_{BS} + O\left(\frac{1}{\sqrt{n}}\right) \tag{6.9}
\]
V delu trdita, da je \( \frac{1}{\sqrt{n}} \) najboljša možna stopnja konvergence za evropsko opcijo in da binomski model ne more konvergirati hitreje kot \( \frac{1}{\sqrt{n}} \) pri drevesnem vozlišču blizu času datuma izteka. Trdita pa tudi, da je možno, da binomski model konvergira s stopnjo \( \frac{1}{n} \) pri trenutnem drevesnem vozlišču.

Heston in Zhou sta predlagala dva pristopa, kako doseči največjo možno stopnjo konvergence \( \frac{1}{n} \) za natančne izplačilne funkcije, vendar nista podala zveze za koeficiente pri oceni napak. Prva dva rezultata o natančnem izrazi za izračun reda napake so podalai Francine in Marc Diener (2004, \[19\]) ter Walsh (2003, \[55\]). Francine in Marc Diener sta delala s CRR modelom in izračunala prvi red asimptotičnega razvoja cene evropske nakupne opcije. Pokazala sta, da je stopnja konvergence za CRR drevesa \( O\left(\frac{1}{n}\right) \). Njun rezultat je naveden v naslednjem izreku.

**Trditev 6.8.** V \( n \)-periodnem CRR binomskem modelu za evropsko nakupno opcijo z izvršilno ceno \( K \), \( S_0 = 1 \) in časom dospetja \( T = 1 \), je binomska cena \( C_{CRR} \) pri \( t = 0 \):
\[
C_{CRR} = C_{BS} + \frac{e^{-d_2^2}}{24 \sigma \sqrt{2\pi}} \frac{A - 12\sigma^2 (\Delta^2_n - 1)}{n} + O\left(\frac{1}{n\sqrt{n}}\right) \tag{6.10}
\]
kjer \( \Delta_n = 1 - 2 \left\{ \ln\left(\frac{S_0}{K}\right)+n\ln d - \ln\left(\frac{e}{2}\right) \right\} \),
\[
A = -\sigma^2 (6 + d_1^2 + d_2^2) + 4 (d_1^2 - d_2^2) r - 12r^2,
\]
55
\{x\} je celi del od \(x\) in \(C_{BS}\) je Black-Scholesova cena, kjer so koeficienti \(d_1, d_2\) enaki kot v trditvi 6.3. Faktor \(\Delta_n\) prikazuje, kje se nahaja izvršilna cena postavljena med končnimi drevesnimi vozlišči.

Ta stopnja konvergence velja za počasno. Poleg tega je konvergenca nihajna. V

\begin{align*}
\frac{\partial S}{\partial t} + rS \frac{\partial S}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 S}{\partial S^2} - rV &= 0, \\
S(0, S) &= S_0, \\
S(T, S) &= K \Phi(d_2), \\
V(0, S) &= 0.
\end{align*}


**Jarrow-Ruddov model.** Njihov model je znan kot model enake verjetnosti in temelji na ujemanju prvih dveh momentov diskretnega in časovno zveznega lognormalnega procesa. Z reševanjem enačb dobljenih iz ujemajočih se momentov dobimo:

\[ p = \frac{1}{2}, \]

\[ u = e^{(r - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t}}, \]

\[ d = e^{(r - \frac{\sigma^2}{2})\Delta t - \sigma \sqrt{\Delta t}} \quad (6.11) \]

![Figure 18: Evropska prodajna opcija izračunana s CRR drevesom: \(S_0 = 45, K = 40, \sigma = 0.25, r = 0.1, T = 1, C_{BS} = 9.8695\).](image-url)

pretakajoči graf, ki prikazuje izvedene CRR in BS cene po številkah korakov.
Taka izbira parametrov zagotavlja, da je drevesni model enak Black-Scholesovemu modelu v limiti.

**Tianov model.** Tian je predlagal model (1993, [53]) ki prve tri momente binomskega modela veže na prve tri momente lognormalne porazdelitve. Binomski parametri $u$, $d$, $p$ so izbrani tako, da se limitni model v diskretnem času približuje lognormalni porazdelitvi cene delnice v zveznem času. Tian je označil $M = e^{r \Delta t}$ in $V = e^{\sigma^2 \Delta t}$. Z reševanjem sistema enačb dobljenim iz ujemajočih se momentov je dobil naslednje:

$$\begin{align*}
p &= \frac{M - d}{u - d}, \\
qu &= 1 - p = \frac{u - M}{u - d}, \\
u &= \frac{MV}{2} \left( V + 1 + \sqrt{V^2 + 2V + 3} \right), \\
d &= \frac{MV}{2} \left( V + 1 - \sqrt{V^2 + 2V + 3} \right).
\end{align*}$$

(6.12)

Ko se število korakov povečuje, Tianov model konvergira proti Black-Scholesovemu modelu.

**Leisen-Reimerov model.** Leisen in Reimer sta v svojem članku [38], preučila konvergenco že obstoječih binomskih modelov proti Black-Scholesovemu modelu. Dokazala sta da CRR model, Jarrow-Ruddov model in Tianov model konvergira z redom $O(\frac{1}{n})$. Leisen in Reimer sta opredelila nov binomski model, tako da gladko konvergira proti Black-Scholesovemu modelu. To jima je uspelo s konvergenco drugega reda, za katero pa nista podala dokaza. V vseh pristopih binomskega modela je verjetnost $p$ aproksimirana s standardno normalno funkcijo $N(z)$, kjer vse vhodne argumente določa neka prilagoditvena funkcija $z = h(k, n, p)$. Toda Leisen in Reimer sta prišla do ideje o inverzni transformaciji prilagoditvene funkcije, kjer $h(k, n, p)$ določa porazdelitveni parameter $h^{-1}(z) = p$ kot približek za $P = N(z)$ z $P \approx 1 - \Phi(k, n, p) [38]$. V članku sta razvila tri enačbe z različnimi približki pri čemer je najbolj znana Peizer-Prattova metoda:

$$h^{-1}(z) = \frac{1}{2} \pm \left[ \frac{1}{4} - \frac{1}{4} \exp \left\{ - \left( \frac{z}{n + \frac{1}{3}} \right)^2 \left( n + \frac{1}{6} \right) \right\} \right]^{\frac{1}{2}}$$

(6.13)

Zgornja enačba velja za neparno število korakov. Z reševanjem zgornje enačbe sta Leisen in Reimer izpeljala naslednje parametre, ki zagotavljajo kvadratično konvergenco binomskega modela:

$$\tilde{p} = h^{-1}(d_1), \quad p = h^{-1}(d_2)$$

$$u = \frac{r_n \tilde{p}}{p} \quad \text{in} \quad d = \frac{r_n - pu}{1 - p}$$

(6.14)

**Chang-Praimerjeva metoda.** Chang in Palmer (2007, [14]) sta preučevala stopnjo konvergence za $n$-periodni binomski model, kjer sta parametra $u$ in $d$ bolj posplošena kot v CRR modelu. Naj bosta $\Delta t = \frac{T}{n}$ in $\lambda_n$ poljubna omejena funkcija od $n$. Upoštevala sta $n$-periodni binomski model kjer:

$$\begin{align*}
u &= e^{\sigma \sqrt{\Delta t} + \lambda_n \sigma^2 \Delta t} \\
d &= e^{-\sigma \sqrt{\Delta t} + \lambda_n \sigma^2 \Delta t}
\end{align*}$$

(6.15)
z začetno cenom delnice $S_0$.

Razen binomskih dreves so v tem delu obravnavana tudi trinomska drevesa. Trinomski model je razširitev binomskega modela, ki vključuje tretje možno stanje cene. Leta 1986 ga je predstavil Phelim Boyle [7]. Pri trinomskem modelu se lahko cene gibljejo navzgor, navzdol ali ostanejo nespremenjene. Tako bo število možnih cen v času $t$ naraščalo hitreje kot v binomnem modelu med povečevanjem $t$. V trinomskem drevesu za ceno opcije v časovnem koraku nič mora veljati, da:

$$C = e^{-r\Delta t} [p_u C_u + p_m C_m + p_d C_d].$$

(6.16)

Namen prvega trinomskega drevesa predstavljenega s strani Boyla (1986) je bil povečati natančnost in hitrost nad binomskimi drevesi [7].

**Boyleov model.** Prvotna parametra CRR modela $u$ in $d$ ne moreta biti uporabljena tudi če je $m = 1$, ker nekatere verjetnosti ne bi ležale v intervalu med 0 in 1. Boyle je predlagal uporabiti razpršitveni parameter $\lambda > 1$ za povečavo $u$ in zmanjšanja $d$, kajti prvotni parameter v CRR za $u$ ne more biti uporabljen [8].

Njegova izbira parametrov je bila:

$$u = e^{\lambda \sigma \sqrt{\Delta t}},$$

$$d = e^{-\lambda \sigma \sqrt{\Delta t}} = \frac{1}{u},$$

$$m = 1$$

(6.17)

Vendar lahko ta parametrizacija daje negativne verjetnosti za majhne vrednosti $\lambda$. Tako obstajajo tudi različne parametrizacije trinomskih dreves. Najbolj priljubljeni sta Kamrad-Ritchkenovo drevo in Tianovo drevo.

**Kamrad-Ritchkenov model.** Kamrad in Ritchken (1991, [34]), sta nameravala izboljšati Boyleov model. Njun model popravi Boyleov problem negativnih verjetnosti in za vsako vrednost $\lambda$, $\lambda \geq 1$ dobimo smiseln nabor verjetnosti. Za model sta podala naslednjo parametrizacijo:

$$u = e^{\lambda \sigma \sqrt{\Delta t}},$$

$$d = e^{-\lambda \sigma \sqrt{\Delta t}} = \frac{1}{u},$$

$$m = 1$$

(6.18)

**Tianov trinomski model** Tian je predstavil dve različni parametrizaciji (1993, [53]) za trinomsko drevo. Prva je znaša kot enako verjetnostno drevo z zgornjimi in spodnjimi parametri enakimi $\frac{1}{3}$, in je dobljena ob uporabi ujemanja prvih dveh momentov:

$$m = \frac{M(3-V)}{2},$$

$$K = \frac{M(V+3)}{4},$$

$$u = K + \sqrt{K^2 - m^2},$$

$$d = K - \sqrt{K^2 - m^2}.$$
Druga parametrizacija podana s strani Tiana ([53], p.568) odstrani omejitve enake verjetnosti in je dobljena z ujemanjem prvih štirih momentov [12]:

\[ u = K + \sqrt{K^2 - m^2} \]
\[ m = MV^2 \]
\[ d = K - \sqrt{K^2 - m^2} \]

kjer so \( V = e^{\sigma \Delta t} \), \( M = e^{r \Delta t} \) in \( K = \frac{M}{2} (V^4 + V^3) \).

Analitičnih rezultatov o hitrosti konvergence trinomskega modela ni veliko. Iz numeričnih rezultatov in simulacij je sicer razvidno, da cena trinomskega modela konvergira proti ceni Black-Scholesovega modela za velike vrednosti \( n \). Glede na te rezultate trinomski model hitreje konvergira proti Black-Scholesovemu modelu kot binomski model. To izhaja iz dejstva, da lahko trinomsko drevo obravnavamo kot binomsko drevo, ki se obravnava samo na vsakem drugem drevesnem vozlišču [46]. Vendar je treba upoštevati, da je trinomski model računsko obravnavo dražji. Slika 19, prikazuje primerjavo konvergence dreves modelov omenjenih v tem delu. Številčni rezultati konvergence so prikazani v tabeli 2.

![Diagram](Image)

**Figure 19:** Primerjava konvergenc nekaterih binomskih in trinomskih modelov kadar \( S_0 = 45 \), \( K = 40 \), \( \sigma = 0.25 \), \( r = 0.1 \), \( T = 1 \), \( C_{BS} = 9.8695 \).

Kamrad-Ritchkenov trinomski drevesni model je še najbližji vrednosti Black-Scholesovega modela.
<table>
<thead>
<tr>
<th>Model</th>
<th>CRR</th>
<th>JR</th>
<th>LR</th>
<th>CP</th>
<th>Tian B</th>
<th>Tian T</th>
<th>Boyle</th>
<th>KR</th>
</tr>
</thead>
</table>

Table 2: Primerjava cen pridobljenih z različnimi modeli izračunanimi po n korakih kadar $S_0 = 45$, $K = 40$, $\sigma = 0.25$, $r = 0.1$, $T = 1$, $C_{BS} = 9.8695$

### 6.3 Zaključek

Na splošno ostaja stopnja konvergence drevesnih metod počasna. Za spopadanje s počasno konvergenco dreves so bile uvedene nekatere tehnike, ki pospešujejo konvergenco. Najpogosteje uporabljene tehnike za pospešitev konvergence so glajenje, Richardsonova ekstrapolacija, kontrolna spremenljivka in okrnitev.

Te tehnike je mogoče uporabiti samostojno ali kombinirano, tako da dobimo 16 kombinacij pospeševalnih tehnik, ki jih lahko uporabimo za določanje cen opcij z uporabo dreves [12]. Vendar je zaradi številnih možnih kombinacij in različnih specifikacij opcij težko sklepati, kateri model in metoda sta najbolj učinkovita. Potrebne bodo dodatne študije o stopnji konvergence in aproksimaciji napake.
References


