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ROBIN DOUBLE-PHASE PROBLEMS WITH SINGULAR AND SUPERLINEAR TERMS

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Abstract. We consider a nonlinear Robin problem driven by the sum of p-Laplacian and q-Laplacian (i.e. the (p, q)-equation). In the reaction there are competing effects of a singular term and a parametric perturbation $\lambda f(z, x)$, which is Carathéodory and $(p-1)$-superlinear at $x \in \mathbb{R}$, without satisfying the Ambrosetti-Rabinowitz condition. Using variational tools, together with truncation and comparison techniques, we prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Robin problem

$$(P_\lambda) \qquad \left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma} + \lambda f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega,\ u > 0,\ \lambda > 0,\ 0 < \gamma < 1,\ 1 < q < p. \end{array} \right\}$$

For every $r \in (1, \infty)$, we denote by Δ_r the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \text{ for all } u \in W^{1,r}(\Omega).$$

The differential operator of (P_λ) is the sum of p-Laplacian and q-Laplacian. Such an operator is not homogeneous and it appears in the mathematical models of various physical processes. We mention the works of Cherfils & Ilyasov [2] (reaction-diffusion systems) and Zhikov [22] (elasticity theory). The potential function $\xi \in L^\infty(\Omega)$ satisfies $\xi(z) \geqslant 0$ for almost all $z \in \Omega$. In the reaction (the right-hand side of (P_λ)), we have the combined effects of two nonlinearities of different nature. One nonlinearity is the singular term $u^{-\gamma}$ and the other nonlinearity is the parametric term $\lambda f(z, x)$, where $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, the mapping $x \mapsto f(z, x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $+\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). In the boundary condition, $\dfrac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative corresponding to the (p, q)-Laplace differential operator. Then according to the nonlinear Green's identity (see Gasinski & Papageorgiou [3, p. 210]), we have

$$\frac{\partial u}{\partial n_{pq}} = (|Du|^{p-2}Du + |Du|^{q-2}Du, n) \text{ for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ (with $0 < \alpha < 1$) satisfies $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

In the past, nonlinear singular problems were studied only in the context of Dirichlet equations driven by the p-Laplacian (a homogeneous differential operator). We mention the works of Giacomoni, Schindler & Takač [6], Papageorgiou, Rădulescu & Repovš [11, 12], Papageorgiou & Smyrlis [17], Papageorgiou & Winkert [18], and Perera & Zhang [20]. Nonlinear elliptic problems with unbalanced growth have been studied recently by Papageorgiou, Rădulescu and Repovš [13, 14, 16]. Double-phase transonic flow problems with variable growth have been considered by Bahrouni,

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Rădulescu and Repovš [1]. A comprehensive study of semilinear singular problems can be found in the book of Ghergu & Rădulescu [5].

Using variational methods based on the critical point theory together with suitable truncation and comparison techniques, we prove a bifurcation type result, describing in a precise way the dependence of the set of positive solutions of (P_λ) on the parameter. So, we produce a critical parameter value $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least two positive solutions, for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution and for $\lambda > \lambda^*$ there are no positive solutions for problem (P_λ).

2. Mathematical background and hypotheses

Let X be a Banach space. By X^* we denote the topological dual of X. Given $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds

> "Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that
> $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1 + ||u_n||)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
> admits a strongly convergent subsequence."

This is a compactness type condition on the functional φ, which leads to the minimax theory of the critical values of $\varphi(\cdot)$.

The two main spaces in the analysis of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. By $||\cdot||$ we denote the norm on the Sobolev space $W^{1,p}(\Omega)$. We have

$$||u|| = \left[||u||_p^p + ||Du||_p^p\right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

We will also consider another order cone (closed convex cone) in $C^1(\overline{\Omega})$, namely the cone

$$\hat{C}_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega},\ \frac{\partial u}{\partial n}|_{\partial\Omega\cap u^{-1}(0)} \leqslant 0\right\}.$$

This cone has a nonempty interior

$$\operatorname{int} \hat{C}_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega\cap u^{-1}(0)} < 0\right\}.$$

To take care of the Robin boundary condition, we will also use the "boundary" Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant \infty)$. More precisely, on $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$ we can define in the usual way the Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant \infty)$. We know that there exists a continuous, linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map" such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. We have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)\ \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \text{ and } \ker \gamma_0 = W_0^{1,p}(\Omega).$$

The trace map γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in \left[1, \frac{(N-1)p}{N-p}\right)$ if $N > p$ and into $L^q(\partial\Omega)$ for all $q \geqslant 1$ if $p \geqslant N$. In the sequel, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

For every $r \in (1, +\infty)$, let $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ be defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,r}(\Omega).$$

The following proposition summarizes the main properties of this map (see Gasinski & Papageorgiou [3]).

Proposition 2.1. *The map $A_r(\cdot)$ is bounded (that is, maps bounded sets to bounded sets) continuous, monotone (hence maximal monotone, too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,r}(\Omega)$ and $\limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle$, then $u_n \to u$ in $W^{1,r}(\Omega)$.*

Evidently, the $(S)_+$-property is useful in verifying the C-condition.

Now we introduce the conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

$H(\xi)$: $\xi \in L^\infty(\Omega)$ and $\xi(z) \geqslant 0$ for almost all $z \in \Omega$.

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

H_0: $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 2.1. *When $\beta \equiv 0$ we have the usual Neumann problem.*

The next two propositions can be found in Papageorgiou & Rădulescu [10].

Proposition 2.2. *If $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for almost all $z \in \Omega$ and $\xi \not\equiv 0$, then $c_0||u||^p \leqslant ||Du||_p^p + \int_\Omega \xi(z)|u|^p dz$ for some $c_0 > 0$ and all $u \in W^{1,p}(\Omega)$.*

Proposition 2.3. *If $\beta \in L^\infty(\partial\Omega)$, $\beta(z) \geqslant 0$ for σ-almost all $z \in \partial\Omega$ and $\beta \not\equiv 0$, then $c_1||u||^p \leqslant ||Du||_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma$ for some $c_1 > 0$ and all $u \in W^{1,p}(\Omega)$.*

In what follows, let $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ be defined by

$$\gamma_p(u) = ||Du||_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

If hypotheses $H(\xi), H(\beta), H_0$ hold, then from Propositions 2.2 and 2.3 we can infer that

$$c_2||u||^p \leqslant \gamma_p(u) \text{ for some } c_2 > 0 \text{ and all } u \in W^{1,p}(\Omega). \tag{1}$$

As we have already mentioned in the introduction, our approach involves also truncation and comparison techniques. So, the next strong comparison principle, a slight variant of Proposition 4 of Papageorgiou & Smyrlis [17], will be useful.

Proposition 2.4. *If $\hat{\xi} \in L^\infty(\Omega)$ with $\hat{\xi}(z) \geqslant 0$ for almost all $z \in \Omega, h_1, h_2 \in L^\infty(\Omega)$,*

$$0 < c_3 \leqslant h_2(z) - h_1(z) \text{ for almost all } z \in \Omega,$$

and the functions $u_1, u_2 \in C^1(\overline{\Omega})\backslash\{0\}, u_1 \leqslant u_2, u_1^{-\gamma}, u_2^{-\gamma} \in L^\infty(\Omega)$ satisfy

$$-\Delta_p u_1 - \Delta_q u_1 + \hat{\xi}(z)u_1^{p-1} - u_1^{-\gamma} = h_1 \text{ for almost all } z \in \Omega,$$
$$-\Delta_p u_2 - \Delta_q u_2 + \hat{\xi}(z)u_2^{p-1} - u_2^{-\gamma} = h_2 \text{ for almost all } z \in \Omega,$$

then $u_2 - u_1 \in \operatorname{int}\hat{C}_+$.

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_0(z,x)| \leqslant a_0(z)[1 + |x|^{r-1}] \text{ for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$ and $1 < r \leqslant p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leqslant p \end{cases}$ (the critical Sobolev exponent corresponding to p).

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega F_0(z,u)dz \text{ for all } u \in W^{1,p}(\Omega) \text{ (recall that } q < p).$$

The next proposition can be found in Papageorgiou & Rădulescu [9] and essentially is an outgrowth of the nonlinear regularity theory of Lieberman [7].

Proposition 2.5. *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \text{ for all } ||h||_{C^1(\overline{\Omega})} \leqslant \rho_0,$$

then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u + h) \text{ for all } ||h|| \leqslant \rho_1.$$

The next fact about ordered Banach spaces is useful in producing upper bounds for functions and can be found in Gasinski & Papageorgiou [4, Problem 4.180, p. 680].

Proposition 2.6. *If X is an ordered Banach space with positive (order) cone K,*

$$\operatorname{int} K \neq \emptyset \text{ and } e \in \operatorname{int} K$$

then for every $u \in X$ we can find $\lambda_u > 0$ such that $\lambda_u e - u \in K$.

Under hypotheses $H(\xi), H(\beta), H_0$, the differential operator $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with the Robin boundary condition, has a principal eigenvalue $\hat{\lambda}_1(p) > 0$ which is isolated, simple and admits the following variational characterization:

$$\hat{\lambda}_1(p) = \inf\left\{\frac{\gamma_p(u)}{||u||_p^p} : u \in W^{1,p}(\Omega), u \neq 0\right\}. \tag{2}$$

The infimum is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. By $\hat{u}_1(p)$ we denote the positive, L^p-normalized (that is, $||\hat{u}_1(p)||_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(p) > 0$. The nonlinear Hopf theorem (see, for example, Gasinski & Papageorgiou [3, p. 738]) implies that $\hat{u}_1(p) \in D_+$.

Let us fix some basic notation which we will use throughout this work. So, if $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$ and the for $u \in W^{1,p}(\Omega)$ we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that

$$u^{\pm} \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

If $\varphi \in C^1(W^{1,p}(\Omega), \mathbb{R})$, then by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in W^{1,p}(\Omega) : \varphi'(u) = 0\}.$$

Also, if $u, y \in W^{1,p}(\Omega)$, with $u \leqslant y$, then we define

$$[u, y] = \{h \in W^{1,p}(\Omega) : u(z) \leqslant h(z) \leqslant y(z) \text{ for almost all } z \in \Omega\},$$
$$[u) = \{h \in W^{1,p}(\Omega) : u(z) \leqslant h(z) \text{ for almost all } z \in \Omega\},$$
$$\operatorname{int}_{C^1(\overline{\Omega})}[u, y] = \text{the interior in the } C^1(\overline{\Omega})\text{-norm of } [u, y] \cap C^1(\overline{\Omega}).$$

Now we introduce our hypotheses on the perturbation $f(z, x)$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $f(z, x) \leqslant a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$ and all $x \geqslant 0$ with $a \in L^\infty(\Omega), p < r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x\to+\infty} \frac{F(z, x)}{x^p} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) there exists $\tau \in ((r - p)\max\left\{\frac{N}{p}, 1\right\}, p^*)$ such that

$$0 < \hat{\beta}_0 \leqslant \liminf_{x\to+\infty} \frac{f(z, x)x - pF(z, x)}{x^\tau} \text{ uniformly for almost all } z \in \Omega;$$

(iv) for every $\vartheta > 0$, there exists $m_\vartheta > 0$ such that

$$m_\vartheta \leqslant f(z, x) \text{ for almost all } z \in \Omega \text{ and all } x \geqslant \vartheta;$$

(v) for every $\rho > 0$ and $\lambda > 0$, there exists $\hat{\xi}_\rho^\lambda > 0$ such that for almost all $z \in \Omega$, the function $x \mapsto f(z, x) + \hat{\xi}_\rho^\lambda x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 2.2. *Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis, without any loss of generality we may assume that*

$$f(z, x) = 0 \text{ for almost all } z \in \Omega \text{ and all } x \leqslant 0. \tag{3}$$

From hypotheses $H(f), (ii), (iii)$ it follows that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for almost all } z\in\Omega.$$

Hence, for almost all $z \in \Omega$ the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. However, this superlinearity of $f(z,\cdot)$ is not expressed using the well-known AR-condition. We recall that the AR-condition (unilateral version due to (3)) says that there exist $q > p$ and $M > 0$ such that

$$0 < qF(z,x) \leqslant f(z,x)x \text{ for almost all } z\in\Omega \text{ and all } x \geqslant M, \tag{3a}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M). \tag{3b}$$

Integrating (3a) and using (3b), we obtain the weaker condition

$$\begin{aligned} & c_4 x^q \leqslant F(z,x) \text{ for almost all } z\in\Omega \text{ all } x\geqslant M, \text{ and some } c_4 > 0, \\ \Rightarrow\quad & c_4 x^{q-1} \leqslant f(z,x) \text{ for almost all } z\in\Omega \text{ and all } x \geqslant M. \end{aligned}$$

So, the AR-condition dictates an at least $(q-1)$-polynomial growth for $f(z,\cdot)$. Here we replace the AR-condition with hypothesis $H(f)(iii)$ which is less restrictive and permits superlinear nonlinearities with "slower" growth near $+\infty$. For example the function

$$f(x) = x^{p-1}\ln(1+x) \text{ for all } x \geqslant 0.$$

(for the sake of simplicity we have dropped the z-dependence) satisfies hypotheses $H(f)$, but fails to satisfy the AR-condition.

We introduce the following sets:

$$\begin{aligned} &\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\}, \\ &S_\lambda = \text{the set of positive solutions of } (P_\lambda). \end{aligned}$$

Also we set

$$\lambda^* = \sup \mathcal{L}.$$

3. Some auxiliary Robin problems

Let $\eta > 0$. First we examine the following auxiliary Robin problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \eta \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega,\ u > 0. \end{array} \right\} \tag{5}$$

Proposition 3.1. *If hypotheses $H(\xi), H(\beta), H_0$ hold, then for every $\eta > 0$ problem (5) has a unique solution $\tilde{u}_\eta \in D_+$, the mapping $\eta \mapsto \tilde{u}_\eta$ is strictly increasing (that is, $\eta < \eta' \Rightarrow \tilde{u}_{\eta'} - \tilde{u}_\eta \in \operatorname{int}\hat{C}_+$) and*

$$\tilde{u}_\eta \to 0 \textit{ in } C^1(\overline{\Omega}) \textit{ as } \eta \to 0^+.$$

Proof. Consider the map $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle V(u), h\rangle = \langle A_p(u), h\rangle + \langle A_q(u), h\rangle + \int_\Omega \xi(z)|u|^{p-2}uh dz + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh d\sigma \quad \text{for all } u,h \in W^{1,p}(\Omega). \tag{6}$$

Evidently, $V(\cdot)$ is continuous, strictly monotone (hence maximal monotone, too) and coercive (see (1)). Therefore $V(\cdot)$ is surjective (see Gasinski & Papageorgiou [3, Corollary 3.2.31, p. 319]). So, we can find $\tilde{u}_\eta \in W^{1,p}(\Omega), \tilde{u}_\eta \neq 0$ such that

$$V(\tilde{u}_\eta) = \eta.$$

The strict monotonicity of $V(\cdot)$ implies that $\tilde{u}_\eta$ is unique. We have

$$\langle V(\tilde{u}_\eta), h\rangle = \eta \int_\Omega h dz \text{ for all } h \in W^{1,p}(\Omega). \tag{7}$$

In (7) we choose $h = -\tilde{u}_\eta^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & c_2 ||\tilde{u}_\eta^-||^p \leqslant 0 \text{ (see (1))}, \\ \Rightarrow \quad & \tilde{u}_\eta \geqslant 0, \ \tilde{u}_\eta \neq 0. \end{aligned}$$

From (7) we have

$$\left\{ \begin{array}{l} -\Delta_p \tilde{u}_\eta(z) - \Delta_q \tilde{u}_\eta(z) + \xi(z)\tilde{u}_\eta(z)^{p-1} = \eta \text{ for almost all } z \in \Omega, \\ \dfrac{\partial \tilde{u}_\eta}{\partial n_{pq}} + \beta(z)\tilde{u}_\eta^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{8}$$

From (8) and Proposition 7 of Papageorgiou & Rădulescu [9] we deduce that

$$\tilde{u}_\eta \in L^\infty(\Omega).$$

Then the nonlinear regularity theory of Lieberman [7] implies that

$$\tilde{u}_\eta \in C_+ \backslash \{0\}.$$

From (8) we have

$$\begin{aligned} & \Delta_p \tilde{u}_\eta(z) + \Delta_q \tilde{u}_\eta(z) \leqslant ||\xi||_\infty \tilde{u}_\eta(z)^{p-1} \text{ for almost all } z \in \Omega, \\ \Rightarrow \quad & \tilde{u}_\eta \in D_+ \text{ (see Pucci \& Serrin [21, pp. 111, 120])}. \end{aligned}$$

Suppose that $0 < \eta_1 < \eta_2$ and let $\tilde{u}_{\eta_1}, \tilde{u}_{\eta_2} \in D_+$ be the corresponding solutions of problem (5). We have

$$\begin{aligned} & -\Delta_p \tilde{u}_{\eta_1} - \Delta_q \tilde{u}_{\eta_1} + \xi(z)\tilde{u}_{\eta_1}^{p-1} = \eta_1 < \eta_2 = -\Delta_p \tilde{u}_{\eta_2} - \Delta_q \tilde{u}_{\eta_2} + \xi(z)\tilde{u}_{\eta_2} \\ & \text{for almost all } z \in \Omega, \\ \Rightarrow \quad & \tilde{u}_{\eta_2} - \tilde{u}_{\eta_1} \in \operatorname{int} \hat{C}_+ \text{ (see Proposition 2.4)}, \\ \Rightarrow \quad & \eta \mapsto \tilde{u}_\eta \text{ is strictly increasing from } (0, +\infty) \text{ into } C^1(\overline{\Omega}). \end{aligned}$$

Finally, let $\eta_n \to 0^+$ and let $\tilde{u}_n = \tilde{u}_{\eta_n} \in D_+$ be the corresponding solutions of (5). As before, via Proposition 7 of Papageorgiou & Rădulescu [9], we can find $c_5 > 0$ such that

$$||\tilde{u}_n||_\infty \leqslant c_5 \text{ for all } n \in \mathbb{N}.$$

Then from Lieberman [7] we infer that there exist $\alpha \in (0,1)$ and $c_6 > 0$ such that

$$\tilde{u}_n \in C^{1,\alpha}(\overline{\Omega}), \ ||\tilde{u}_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_6 \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, the monotonicity of the sequence $\{\tilde{u}_n\}_{n\geqslant 1} \subseteq D_+$ and that for $\eta = 0, u \equiv 0$ is the only solution of (5) we obtain

$$\tilde{u}_n \to 0 \text{ in } C^1(\overline{\Omega}).$$

The proof is now complete. ☐

Using Proposition 3.1, we see that we can find $\eta_0 > 0$ such that

$$\eta \leqslant \tilde{u}_\eta(z)^{-\gamma} \text{ for all } z \in \overline{\Omega}, \quad 0 < \eta \leqslant \eta_0. \tag{9}$$

We consider the following purely singular problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \ u > 0, \ 0 < \gamma < 1. \end{array} \right\} \tag{10}$$

In the first place, by a solution of (10) we understand a weak solution, that is, a function $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & u^{-\gamma}h \in L^1(\Omega) \text{ and } \langle A_p(u), h\rangle + \langle A_q(u), h\rangle + \int_\Omega \xi(z)u^{p-1}hdz + \int_{\partial\Omega} \beta(z)u^{p-1}hd\sigma \\ & = \int_\Omega u^{-\gamma}hdz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned}$$

In fact, using the nonlinear regularity theory, we will be able to establish more regularity for the solution of (10), which in fact, is a strong solution (that is, the equation can be interpreted pointwise almost everywhere on Ω).

Proposition 3.2. *If hypotheses $H(\xi), H(\beta), H_0$ hold, then problem* (10) *admits a unique solution $v \in D_+$.*

Proof. Let $\eta \in (0, \eta_0]$ (see (9)) and recall that $\tilde{u}_\eta \in D_+$. So $m_\eta = \min_{\overline{\Omega}} \tilde{u}_\eta > 0$ and

$$\begin{aligned} & \eta \leqslant \tilde{u}_\eta^{-\gamma} \leqslant m_\eta^{-\gamma} \text{ (see (9))}, \\ \Rightarrow \quad & \tilde{u}_\eta^{-\gamma} \in L^\infty(\Omega). \end{aligned} \tag{11}$$

We consider the following truncation of the reaction in problem (10):

$$k(z,x) = \begin{cases} \tilde{u}_\eta(z)^{-\gamma} & \text{if } x \leqslant \tilde{u}_\eta(z) \\ x^{-\gamma} & \text{if } \tilde{u}_\eta(z) < x. \end{cases} \tag{12}$$

This is a Carathéodory function. We set $K(z,x) = \int_0^x k(z,s)ds$ and consider the C^1-functional $\Psi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega K(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (12) and (11), we see that $\Psi(\cdot)$ is coercive. Also the Sobolev embedding theorem and the compactness of the trace map, imply that $\Psi(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $v \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \Psi(v) = \inf\{\Psi(u) : u \in W^{1,p}(\Omega)\}, \\ \Rightarrow \quad & \Psi'(v) = 0, \\ \Rightarrow \quad & \langle A_p(v), h\rangle + \langle A_q(v), h\rangle + \int_\Omega \xi(z)|v|^{p-2}vhdz + \int_{\partial\Omega} \beta(z)|v|^{p-2}vhd\sigma = \\ & \int_\Omega k(z,v)hdz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{13}$$

In (13) we choose $(\tilde{u}_\eta - v)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(v), (\tilde{u}_\eta - v)^+\rangle + \langle A_q(v), (\tilde{u}_\eta - v)^+\rangle + \int_\Omega \xi(z)|v|^{p-2}v(\tilde{u}_\eta - v)^+dz + \\ & \int_{\partial\Omega} \beta(z)|v|^{p-2}v(\tilde{u}_\eta - v)^+d\sigma = \int_\Omega \tilde{u}_\eta^{-\gamma}(\tilde{u}_\eta - v)^+dz \text{ (see (12))} \\ \geqslant \quad & \int_\Omega \eta(\tilde{u}_\eta - v)^+dz \text{ (see (9) and recall that } 0 < \eta \leqslant \eta_0) \\ = \quad & \langle A_p(\tilde{u}_\eta), (\tilde{u}_\eta - v)^+\rangle + \langle A_q(\tilde{u}_\eta), (\tilde{u}_\eta - v)^+\rangle + \int_\Omega \xi(z)\tilde{u}_\eta^{p-1}(\tilde{u}_\eta - v)^+dz + \\ & \int_{\partial\Omega} \beta(z)\tilde{u}_\eta^{p-1}(\tilde{u}_\eta - v)^+d\sigma \text{ (see Proposition 3.1)}, \\ \Rightarrow \quad & \tilde{u}_\eta \leqslant v. \end{aligned} \tag{14}$$

Then from (12), (13), (14) we obtain

$$\left\{ \begin{array}{l} -\Delta_p v(z) - \Delta_q v(z) + \xi(z)v(z)^{p-1} = v(z)^{-\gamma} \text{ for almost all } z \in \Omega, \\ \dfrac{\partial v}{\partial n_{pq}} + \beta(z)v^{p-1} = 0 \text{ on } \partial\Omega \end{array} \right\} \tag{15}$$

(see Papageorgiou & Rădulescu [8]).

From (14) we have $v^{-\gamma} \leqslant \tilde{u}_\eta^{-\gamma} \in L^\infty(\Omega)$ (see (11)). So, from (15) and [9] we have $v \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [7] implies that $v \in C_+$. Hence it follows from (14) that

$$v \in D_+.$$

Next, we show that this positive solution is unique. To this end, let $\hat{v} \in W^{1,p}(\Omega)$ be another positive solution of (10). Again we have $\hat{v} \in D_+$. Then

$$\begin{aligned}
& \langle A_p(v), (\hat{v}-v)^+\rangle + \langle A_q(v), (\hat{v}-v)^+\rangle + \int_\Omega \xi(z) v^{p-1}(\hat{v}-v)^+ dz + \\
& \int_{\partial\Omega} \beta(z) v^{p-1}(\hat{v}-v)^+ d\sigma \\
= \; & \int_\Omega v^{-\gamma}(\hat{v}-v)^+ dz \\
\geqslant \; & \int_\Omega \hat{v}^{-\gamma}(\hat{v}-v)^+ dz \\
= \; & \langle A_p(\hat{v}), (\hat{v}-v)^+\rangle + \langle A_q(\hat{v}), (\hat{v}-v)^+\rangle + \int_\Omega \xi(z) \hat{v}^{p-1}(\hat{v}-v)^+ dz + \\
& \int_{\partial\Omega} \beta(z) \hat{v}^{p-1}(\hat{v}-v)^+ d\sigma \\
\Rightarrow \; & \hat{v} \leqslant v.
\end{aligned}$$

Interchanging the roles of v and $\hat{v}$ in the above argument, we obtain

$$\begin{aligned}
& v \leqslant \hat{v}, \\
\Rightarrow \; & v = \hat{v}.
\end{aligned}$$

This proves the uniqueness of the positive solution of the purely singular problem (10). ☐

Next, we consider the following nonlinear Robin problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) + \xi(z) u(z)^{p-1} = v(z)^{-\gamma} + 1 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z) u^{p-1} = 0 \text{ on } \partial\Omega, \; u > 0. \end{array} \right\} \tag{16}$$

Proposition 3.3. *If hypotheses $H(\xi), H(\beta), H_0$ hold, then problem* (16) *admits a unique solution $\overline{u} \in D_+$ and $v \leqslant \overline{u}$.*

Proof. We know that $v^{-\gamma} \in L^\infty(\Omega)$ (see (11) and (14)). Then the existence and uniqueness of the solution $\overline{u} \in W^{1,p}(\Omega)\backslash\{0\}, \overline{u} \geqslant 0$ of (16) follow from the surjectivity and strict monotonicity of the map $V(\cdot)$ (see the proof of Proposition 3.1). The nonlinear regularity theory and the nonlinear Hopf's theorem imply that $\overline{u} \in D_+$.

Moreover, we have

$$\begin{aligned}
& \langle A_p(\overline{u}), (v-\overline{u})^+\rangle + \langle A_q(\overline{u}), (v-\overline{u})^+\rangle + \int_\Omega \xi(z) \overline{u}^{p-1}(v-\overline{u})^+ dz + \\
& \int_{\partial\Omega} \beta(z) \overline{u}^{p-1}(v-\overline{u})^+ d\sigma \\
= \; & \int_\Omega [v^{-\gamma}+1](v-\overline{u})^+ dz \text{ (see (16))} \\
\geqslant \; & \int_\Omega v^{-\gamma}(v-\overline{u})^+ dz \\
= \; & \langle A_p(v), (v-\overline{u})^+\rangle + \langle A_q(v, (v-\overline{v})^+)\rangle + \int_\Omega \xi(z) v^{p-1}(v-\overline{v})^+ dz + \\
& \int_{\partial\Omega} \beta(z) v^{p-1}(v-\overline{v})^+ d\sigma \\
\Rightarrow \; & v \leqslant \overline{u}.
\end{aligned}$$

The proof is now complete. ☐

4. Positive solutions

In this section we prove the bifurcation-type theorem described in the Introduction.

Proposition 4.1. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, then $\mathcal{L} \neq \emptyset$ and $S_\lambda \subseteq D_+$.*

Proof. Let $v \in D_+$ be the unique positive solution of the auxiliary problem (10) (see Proposition 3.2) and $\overline{u} \in D_+$ the unique solution of (16) (see Proposition 3.3). We know that $v \leqslant \overline{u}$ (see Proposition 3.3). Since $\overline{u} \in D_+$, hypothesis $H(f)(i)$ implies that

$$0 \leqslant f(z, \overline{u}(z)) \leqslant c_7 \text{ for some } c_7 > 0 \text{ and almost all } z \in \Omega.$$

So, we can find $\lambda_0 > 0$ small such that

$$0 \leqslant \lambda f(z, \overline{u}(z)) \leqslant 1 \text{ for almost all } z \in \Omega \text{ and all } 0 < \lambda \leqslant \lambda_0. \tag{17}$$

We consider the following truncation of the reaction in problem (P_λ)

$$\vartheta_\lambda(z,x) = \begin{cases} v(z)^{-\gamma} + \lambda f(z, v(z)) & \text{if } x < v(z) \\ x^{-\gamma} + \lambda f(z,x) & \text{if } v(z) \leqslant x \leqslant \overline{u}(z) \\ \overline{u}(z)^{-\gamma} + \lambda f(z, \overline{u}(z)) & \text{if } \overline{u}(z) < x. \end{cases} \tag{18}$$

This is a Carathéodory function. We set $\theta_\lambda(z,x) = \int_0^x \vartheta_\lambda(z,s)ds$ and consider the functional $\mu_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ $(\lambda \in (0, \lambda_0])$ defined by

$$\mu_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega \theta_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Since $0 \leqslant \overline{u}^{-\gamma} \leqslant v^{-\gamma} \in L^\infty(\Omega)$, we see that $\mu_\lambda \in C^1(W^{1,p}(\Omega))$. Also, it is clear from (18) and (1), that $\mu_\lambda(\cdot)$ is coercive. In addition, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \mu_\lambda(u_\lambda) = \inf\left\{\mu_\lambda(u) : u \in W^{1,p}(\Omega)\right\}, \\ & \Rightarrow \mu_\lambda'(u_\lambda) = 0, \\ \Rightarrow \quad & \langle A_p(u_\lambda), h\rangle + \langle A_q(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\ = \quad & \int_\Omega \vartheta_\lambda(z, u_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{19}$$

In (19) first we choose $h = (u_\lambda - \overline{u})^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(u_\lambda), (u_\lambda - \overline{u})^+\rangle + \langle A_q(u_\lambda), (u_\lambda - \overline{u})^+\rangle + \int_\Omega \xi(z) u_\lambda^{p+}(u_\lambda - \overline{u})^+ dz + \\ & \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\lambda - \overline{u}) d\sigma \\ & = \int_\Omega [\overline{u}^{-\gamma} + \lambda f(z, \overline{u})](u_\lambda - \overline{u})^+ dz \text{ (see (18)))} \\ & \leqslant \int_\Omega [\overline{u}^{-\gamma} + 1](u_\lambda - \overline{u})^+ dz \text{ (see (17))} \\ & \leqslant \int_\Omega [v^{-\gamma} + 1](u_\lambda - \overline{u})^+ dz \text{ (since } v \leqslant \overline{u}) \\ & = \langle A_p(\overline{u}), (u_\lambda - \overline{u})^+\rangle + \langle A_q(\overline{u}), (u_\lambda - \overline{u})^+\rangle + \int_\Omega \xi(z)\overline{u}^{p-1}(u_\lambda - \overline{u})^+ dz \\ & + \int_{\partial\Omega} \beta(z)\overline{u}^{p-1}(u_\lambda - \overline{u})^+ d\sigma \text{ (see Proposition 3.3)}, \\ & \Rightarrow u_\lambda \leqslant \overline{u}. \end{aligned}$$

Next, in (19) we choose $h = (v - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A_p(u_\lambda), (v-u_\lambda)^+\rangle + \langle A_q(u_\lambda), (v-u_\lambda)^+\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda(v-u_\lambda)^+dz + \\
&\int_{\partial\Omega}\beta(z)|u_\lambda|^{p-2}u_\lambda(v-u_\lambda)^+d\sigma \\
&= \int_\Omega [v^{-\gamma} + \lambda f(z,v)](v-u_\lambda)^+dz (\text{see } (18)) \\
&\geqslant \int_\Omega v^{-\gamma}(v-u_\lambda)^+dz (\text{since } f \geqslant 0) \\
&= \ \langle A_p(v), (v-u_\lambda)^+\rangle + \langle A_q(v), (v-u_\lambda)^+\rangle + \int_\lambda \xi(z)v^{p-1}(v-u_\lambda)^+dz \\
&+ \int_{\partial\Omega}\beta(z)v^{p-1}(v-u_\lambda)^+d\sigma \text{ (see Proposition 3.2)}, \\
&\Rightarrow v \leqslant u_\lambda.
\end{aligned}$$

So, we have proved that

$$u_\lambda \in [v, \overline{u}]. \tag{20}$$

From (18), (19), (20) it follows that

$$\left\{ \begin{array}{l} -\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \xi(z)u_\lambda(z)^{p-1} = u_\lambda(z)^{-\gamma} + \lambda f(z, u_\lambda(z)) \\ \text{for almost all } z \in \Omega, \\ \dfrac{\partial u_\lambda}{\partial n_{pq}} + \beta(z)u_\lambda^{p-1} = 0 \text{ on } \partial\Omega, \ (\text{see [8]}). \end{array} \right\} \tag{21}$$

From (21) and Proposition 3.1 of Papageorgiou & Rădulescu [9], we have that $u_\lambda \in L^\infty(\Omega)$. So, the nonlinear regularity theory of Lieberman [7] implies that $u_\lambda \in D_+$ (see (20)). Therefore we have proved that

$$(0, \lambda_0] \leqslant \mathcal{L} \neq \emptyset \text{ and } S_\lambda \subseteq D_+.$$

The proof is now complete. □

Next, we establish a lower bound for the elements of S_λ.

Proposition 4.2. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, $\lambda \in \mathcal{L}$ and $u \in S_\lambda$, then $v \leqslant u$.*

Proof. From Proposition 4.1 we know that $u \in D_+$. Then Proposition 3.1 implies that for $\eta > 0$ small we have $\tilde{u}_\eta \leqslant u$. So, we can define the following Carathéodory function

$$e(z,x) = \left\{ \begin{array}{ll} \tilde{u}_\eta(z)^{-\gamma} & \text{if } x < \tilde{u}_\eta(z) \\ x^{-\gamma} & \text{if } \tilde{u}_\eta(z) \leqslant x \leqslant u(z) \\ u(z)^{-\gamma} & \text{if } u(z) < x. \end{array} \right. \tag{22}$$

We set $E(z,x) = \int_0^x e(z,s)ds$ and consider the functional $d : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$d(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega E(z,u)dz \text{for all } u \in W^{1,p}(\Omega).$$

As before, we have $d \in C^1(W^{1,p}(\Omega))$. Also, $d(\cdot)$ is coercive (see (22)) and weakly lower semicontinuous. Hence we can find $\hat{v} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& d(\hat{u}) = \inf\{d(u) : u \in W^{1,p}(\Omega)\}, \\
\Rightarrow\ & d'(\hat{v}) = 0, \\
\Rightarrow\ & \langle A_p(\hat{v}), h\rangle + \langle A_q(\hat{v}), h\rangle + \int_\Omega \xi(z)|\hat{v}|^{p-2}\hat{v}hdz + \int_{\partial\Omega}\beta(z)|\hat{v}|^{p-2}\hat{v}hd\sigma = \\
& \int_\Omega e(z,\hat{v})hdz \text{ for all } h \in W_{1,p}(\Omega).
\end{aligned} \tag{23}$$

In (23) first we choose $h = (\hat{v} - u)^+ \in W^{1,p}(\Omega)$. Exploiting the fact that $u \in S_\lambda$ and recalling that $f \geqslant 0$, we obtain $\hat{v} \leqslant u$. Next in (23) we test with $h = (\tilde{u}_\eta - v)^+ \in W^{1,p}(\Omega)$. Using (22), (9) and Proposition 3.1, we obtain $\tilde{u}_\eta \leqslant \hat{v}$. Therefore

$$\hat{v} \in [\tilde{u}_\eta, u]. \tag{24}$$

From (22), (23), (24) and Proposition 3.2, we conclude that

$$\begin{aligned} & \hat{v} = v, \\ \Rightarrow \quad & v \leqslant u \text{ for all } u \in S_\lambda. \end{aligned}$$

The proof is now complete. □

Now we can deduce a structural property of $\mathcal{L}$.

Proposition 4.3. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, $\lambda \in \mathcal{L}$, $0 < \mu < \lambda$ and $u_\lambda \in S_\lambda \subseteq D_+$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq D_+$ such that $u_\lambda - u_\mu \in \operatorname{int} \hat{C}_+$.*

Proof. From Proposition 4.2 we know that $v \leqslant u_\lambda$. Then we can define the following Carathéodory function

$$\hat{k}_\mu(z,x) = \begin{cases} v(z)^{-\gamma} + \mu f(z, v(z)) & \text{if } x < v(z) \\ x^{-\gamma} + \mu f(z,x) & \text{if } v(z) \leqslant x \leqslant u_\lambda(z) \\ u_\lambda(z)^{-\gamma} + \mu f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{25}$$

We set $\hat{K}_\mu(z,x) = \int_0^x \hat{k}_\mu(z,s)ds$ and consider the C^1-functional $\hat{\Psi}_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\Psi}_\mu(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega \hat{K}_\mu(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently, $\hat{\Psi}_\mu(\cdot)$ is coercive (see (25)) and sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\Psi}_\mu(u_\mu) = \inf\left\{\hat{\Psi}_\mu(u) : u \in W^{1,p}(\Omega)\right\}, \\ \Rightarrow \quad & \hat{\Psi}'_\mu(u_\mu) = 0, \\ \Rightarrow \quad & \langle A_p(u_\mu), h\rangle + \langle A_q(u_\mu), h\rangle + \int_\Omega \xi(z)|u_\mu|^{p-2}u_\mu h dz + \int_{\partial\Omega} \beta(z)|u_\mu|^{p-2}u_\mu h d\sigma \\ = \quad & \int_\Omega \hat{k}_\mu(z, u\mu)h dz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{26}$$

In (26) first we choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. Using (25), the fact that $\mu < \lambda$ and that $f \geqslant 0$ and recalling that $u_\lambda \in S_\lambda$, we conclude that $u_\mu \leqslant u_\lambda$. Next, in (26) we choose $h = (v - u_\mu)^+ \in W^{1,p}(\Omega)$. From (25), the fact that $f \geqslant 0$ and Proposition 3.2, we infer that $v \leqslant u_\mu$. Therefore we have proved that

$$u_\mu \in [v, u_\lambda]. \tag{27}$$

From (25), (26), (27) it follows that

$$u_\mu \in S_\mu \subseteq D_+ \text{(see Proposition 4.1)}.$$

Let $\rho = ||u_\lambda||_\infty$ and let $\hat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis $H(f)(v)$. We have

$$\begin{aligned}
& -\Delta_p u_\lambda(z) - \Delta_q u_\mu(z) + \left[\xi(z) + \hat{\xi}_\rho^\lambda\right] u_\mu(z)^{p-1} - u_\mu(z)^{-\gamma} \\
= \ & \mu f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\mu(z)^{p-1} \\
= \ & \lambda f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\mu(z)^{p-1} - (\lambda - \mu) f(z, u_\mu(z)) \\
< \ & \lambda f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} \text{ (recall that } \lambda > \mu) \\
\leqslant \ & \lambda f(z, u_\lambda(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} \text{ (see (27) and hypothesis } H(f)(v)) \\
= \ & -\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \left[\xi(z) + \hat{\xi}_\rho^\lambda\right] u_\lambda(z)^{p-1} - u_\lambda(z)^{-\lambda} \text{ for almost all } z \in \Omega \qquad (28) \\
& \text{(recall that } u_\lambda \in S_\lambda).
\end{aligned}$$

We know that

$$0 \leqslant u_\mu^{-\gamma},\ u_\lambda^{-\gamma} \leqslant v^{-\gamma} \in L^\infty(\Omega).$$

Also, from hypothesis $H(f)(iv)$ and since $u_\mu \in D_+$, we have

$$0 < c_8 \leqslant (\lambda - \mu) f(z, u_\mu(z)) \text{ for almost all } z \in \Omega.$$

Invoking Proposition 2.4, from (28) we conclude that

$$u_\lambda - u_\mu \in \operatorname{int} \hat{C}_+.$$

The proof is now complete. □

Proposition 4.4. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, then $\lambda^* < +\infty$.*

Proof. On account of hypotheses $H(f)(i) \to (iv)$, we can find $\lambda_0 > 0$ big such that

$$x^{-\gamma} + \lambda_0 f(z, x) \geqslant x^{p-1} \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \qquad (29)$$

Let $\lambda > \lambda_0$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq D_+$ (see Proposition 4.1). Then $m_\lambda = \min_{\overline{\Omega}} u_\lambda > 0$. For $\delta \in (0,1)$ we set $m_\lambda^\delta = m_\lambda + \delta$ and for $\rho = ||u_\lambda||_\infty$ let $\hat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis $H(f)(v)$. We have

$$\begin{aligned}
& -\Delta_p m_\lambda^\delta - \Delta_q m_\lambda^\delta + [\xi(z) + \hat{\xi}_\rho](m_\lambda^\delta)^{p-1} - (m_\lambda^\delta)^{-\gamma} \\
= \ & [\xi(z) + \hat{\xi}_\rho^\lambda] m_\lambda^{p-1} - m_\lambda^{-\gamma} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{as } \delta \to 0^+ \\
< \ & \xi(z) m_\lambda^{p-1} + (1 + \hat{\xi}_\rho^\lambda) m_\lambda^{p-1} - m_\lambda^{-\gamma} + \chi(\delta) \\
\leqslant \ & \lambda_0 f(z, m_\lambda) + [\xi(z) + \hat{\xi}_\rho^\lambda] m_\lambda^{p-1} + \chi(\delta) \text{ (see (29))} \\
\leqslant \ & \lambda_0 f(z, u_\lambda) + [\xi(z) + \hat{\xi}_\rho^\lambda] u_\lambda^{p-1} + \chi(\delta) \text{ (see hypothesis } H(f)(v)) \\
= \ & \lambda f(z, u_\lambda) + [\xi(z) + \hat{\xi}_\rho^\lambda] u_\lambda^{p-1} - (\lambda - \lambda_0) f(z, u_\lambda) + \chi(\delta) \\
= \ & \lambda f(z, u_\lambda) + [\xi(z) + \hat{\xi}_\rho^\lambda] u_\lambda^{p-1} \text{ for } \delta \in (0,1) \text{ small} \\
& \text{(recall that } u_\lambda \in D_+ \text{and see } H(f)(iv)) \\
= \ & -\Delta_p u_\lambda - \Delta_q u_\lambda + [\xi(z) + \hat{\xi}_\rho^\lambda] u_\lambda^{p-1} - u_\lambda^{-\gamma}. \qquad (30)
\end{aligned}$$

Since $(\lambda - \lambda_0) f(z, u_\lambda) - \chi(\delta) \geqslant c_9 > 0$ for almost all $z \in \Omega$ and for $\delta \in (0,1)$ small (just recall that $u_\lambda \in D_+$ and use hypothesis H(f)(iv), invoking Proposition 2.4, from (30) we infer that

$$u_\lambda - m_\lambda^\delta \in \operatorname{int} \hat{C}_+ \text{ for all } \delta \in (0,1) \text{ small enough.}$$

However, this contradicts the definition of m_λ. It follows that $\lambda \notin \mathcal{L}$ and so $\lambda^* \leqslant \lambda_0 < +\infty$. □

Therefore we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*].$$

Proposition 4.5. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has at least two positive solutions*

$$u_0,\ \hat{u} \in D_+,\ u_0 \neq \hat{u}.$$

Proof. Let $0 < \mu < \lambda < \eta < \lambda^*$. According to Proposition 4.3, we can find $u_\eta \in S_\eta \subseteq D_+$, $u_0 \in S_\lambda \subseteq D_+$ and $u_\mu \in S_\mu \subseteq D_+$ such that

$$\begin{aligned} & u_\eta - u_0 \in \operatorname{int} \hat{C}_+ \text{ and } u_0 - u_\mu \in \operatorname{int} \hat{C}_+, \\ & \Rightarrow u_0 \in \operatorname{int}_{C^1(\hat{\Omega})}[u_\mu, u_\eta]. \end{aligned} \tag{31}$$

We introduce the following Carathéodory function

$$\tilde{\tau}_\lambda(z,x) = \begin{cases} u_\mu(z)^{-\gamma} + \lambda f(z, u_\mu(z)) & \text{if } x < u_\mu(z) \\ x^{-\gamma} + \lambda f(z,x) & \text{if } u_\mu(z) \leqslant x \leqslant u_\eta(z) \\ u_\eta(z)^{-\gamma} + \lambda f(z, u_\eta(z)) & \text{if } u_\eta(z) < x. \end{cases} \tag{32}$$

Set $\tilde{T}_\lambda(z,x) = \int_0^x \tilde{\tau}_\lambda(z,s)ds$ and consider the C^1-functional $\tilde{\Psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\Psi}_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\lambda \tilde{T}_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (32) and the nonlinear regularity theory, we can easily check that

$$K_{\tilde{\Psi}_\lambda} \subseteq [u_\mu, u_\eta] \cap D_+. \tag{33}$$

Also, consider the Carathéodory function

$$\tau^*_\lambda(z,x) = \begin{cases} u_\mu(z)^{-\gamma} + \lambda f(z, u_\mu(z)) & \text{if } x \leqslant u_\mu(z) \\ x^{-\gamma} + \lambda f(z,x) & \text{if } u_\mu(z) < x. \end{cases} \tag{34}$$

We set $T^*_\lambda(z,x) = \int_0^x \tau^*_\lambda(z,s)ds$ and consider the C^1-functional $\Psi^*_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi^*_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega T^*_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

For this functional using (34), we show that

$$K_{\Psi^*_\lambda} \subseteq [u_\mu) \cap D_+. \tag{35}$$

From (32) and (34) we see that

$$\tilde{\Psi}_\lambda\Big|_{[u_\mu,u_\eta]} = \Psi^*_\lambda\Big|_{[u_\mu,u_\eta]} \text{ and } \tilde{\Psi}'_\lambda\Big|_{[u_\mu,u_\eta]} = (\Psi^*_\lambda)'\Big|_{[u_\mu,u_\lambda]}. \tag{36}$$

From (33), (35), (36), it follows that without any loss of generality, we may assume that

$$K_{\Psi^*_\lambda} \cap [u_\mu, u_\eta] = \{u_0\}. \tag{37}$$

Otherwise it is clear from (34) and (35) that we already have a second positive smooth solution for problem (P_λ) and so we are done.

Note that $\tilde{\Psi}_\lambda(\cdot)$ is coercive (see (32)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \tilde{\Psi}_\lambda(\hat{u}_0) = \inf\left\{\tilde{\Psi}_\lambda(u) : u \in W^{1,p}(\Omega)\right\}, \\ & \Rightarrow \hat{u}_0 \in K_{\tilde{\Psi}_\lambda}, \\ & \Rightarrow \hat{u}_0 \in K_{\Psi^*_\lambda} \cap [u_\mu, u_\eta] \text{ (see (33),(36))}, \\ & \Rightarrow \hat{u}_0 = u_0 \in D_+ \text{ (see (37))}, \\ & \Rightarrow u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \Psi^*_\lambda \text{ (see (31))}, \\ & \Rightarrow u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \Psi^*_\lambda \text{ (see Proposition 2.5)}. \end{aligned} \tag{38}$$

We assume that $K_{\Psi^*_\lambda}$ is finite. Otherwise on account of (34) and (35) we see that we already have an infinity of positive smooth solutions for problem (P_λ) and so we are done. Then (38) implies that we can find $\rho \in (0,1)$ small. such that

(39) $$\Psi_\lambda^*(u_0) < \inf\{\Psi_\lambda^*(u) : ||u-u_0|| = \rho\} = m_\lambda^*$$
(see Papageorgiou, Rădulescu & Repovš [15, Theorem 5.7.6, p. 367]).

On account of hypothesis $H(f)(ii)$ we have

(40) $$\Psi_\lambda^*(t\hat{u}_1(p)) \to -\infty \text{ as } t \to +\infty.$$

Claim 1. $\Psi_\lambda^*(\cdot)$ *satisfies the C - condition.*

Let $\{u_n\}_{n\geqslant 1} \subseteq \mathrm{W}^{1,p}(\Omega)$ be a sequence such that

(41) $$|\Psi_\lambda^*(u_n)| \leqslant c_{10} \text{ for some } c_{10} > 0 \text{ and all } n \in \mathbb{N},$$

(42) $$(1+||u_n||)(\Psi_\lambda^*)'(u_n) \to 0 \text{ in } \mathrm{W}^{1,p}(\Omega)^*.$$

From (42) we have

(43) $$\begin{aligned}&|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h\,dz + \int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_n h d\sigma \\ &- \int_\Omega \tau_\lambda^*(z,u_n)h\,dz| \leqslant \frac{\epsilon_n||h||}{1+||u_n||} \text{ for all } h \in W^{1,p}, \text{ with } \epsilon_n \to 0^+.\end{aligned}$$

Choosing $h = -u_n^- \in W^{1,p}(\Omega)$, we obtain

(44) $$\begin{aligned}&\gamma_p(u_n^-) + ||Du_n^-||_q^q \leqslant c_{11}||u_n^-|| \text{ for some } c_{11} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (34))} \\ \Rightarrow\ &\{u_n^-\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (1) and recall that } 1 < p).\end{aligned}$$

Next in (43) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

(45) $$\begin{aligned}&-\gamma_p(u_n^+) - ||Du_n^+||_q^q + \int_\Omega \tau_\lambda^*(z,u_n)u_n^+ dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N}, \\ &\Rightarrow -\gamma_p(u_n^+) - ||Du_n^+||_q^q + \int_{\{u_n\leqslant u_\mu\}}[u_\mu^{-\gamma} + \lambda f(z,u_\mu)]u_n^+ dz \\ &+ \int_{\{u_\mu<u_n\}}[u_n^{-\gamma} + \lambda f(z,u_n)]u_n^+ dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (34))}\end{aligned}$$

On the other hand from (41) and (44), we have

$$\begin{aligned}&\gamma_p(u_n^+) + \frac{p}{q}||Du_n^+||_q^q - \int_{\{u_n\leqslant u_\mu\}} p[u_\mu^{-\gamma} + \lambda f(z,u_p)]u_n^+\,dz \\ &- \int_{\{u_\mu<u_n\}}\left[\frac{p}{1-\gamma}(u_n^{1-\gamma} - u_\mu^{1-\gamma}) + p(\lambda F(z,u_n) - \lambda F(z,u_\mu))\right]dz \leqslant \epsilon_n\end{aligned}$$
for all $n \in \mathbb{N}$ (see (34)).

(46) $$\begin{aligned}&\Rightarrow \gamma_p(u_n^+) + \frac{p}{q}||Du_n^+||_p^p - \int_{\{u_n\leqslant u_\mu\}} p[u_\mu^{-\gamma} + \lambda f(z,u_\mu)]u_n^+ dz \\ &- \int_{\{u_p<u_n\}}\left[\frac{p}{1-\gamma}u_n^{1-\gamma} + \lambda pF(z,u_n)\right]dz \leqslant c_{12} \text{ for some } c_{12} > 0 \text{ and all } n \in \mathbb{N}.\end{aligned}$$

We add (45) and (46). Since $p > q$, we obtain

(47) $$\begin{aligned}&\lambda\int_{\{u_\mu<u_n\}}[f(z,u_n)u_n^+ - pF(z,u_n)]dz \leqslant (p-1)\int_{\{u_n\leqslant u_\mu\}}[u_\mu^{-\gamma} + \lambda f(z,u_\mu)]u_n^+ dz \\ &+ \left(\frac{p}{1-\gamma} - 1\right)\int_{\{u_\mu<u_n\}} u_n^{1-\gamma} dz \\ \Rightarrow\ &\lambda\int_\Omega [f(z,u_n^+)u_n^+ - pF(z,u_n^+)]dz \leqslant c_{13}\left[||u_n^+||_1 + 1\right]\end{aligned}$$
for some $c_{13} > 0$, all $n \in \mathbb{N}$.

On account of hypotheses $H(f)(i),(iii)$ we can find $\hat{\beta}_1 \in (0, \hat{\beta}_0)$ and $c_{14} > 0$ such that

$$\hat{\beta}_1 x^\tau - c_{14} \leqslant f(z,x) - pF(z,x) \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{48}$$

Using (48) in (47), we obtain

$$||u_n^+||_\tau^\tau \leqslant c_{15}\left[||u_n^+||_\tau + 1\right] \text{ for some } c_{15} > 0 \text{ and all } n \in \mathbb{N},$$

$$\Rightarrow \{u_n^+\}_{n\geqslant 1} \leqslant L^\tau(\Omega) \text{ is bounded.} \tag{49}$$

First assume $N \neq p$. From hypothesis $H(f)(iii)$ it is clear that we may assume without any loss of generality that $\tau < r < p^*$. Let $t \in (0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p*}.$$

Then from the interpolation inequality (see Papageorgiou & Winkert [19, Proposition 2.3.17, p. 116]), we have

$$\begin{aligned} & ||u_n^+||_r \leqslant ||u_n^+||_\tau^{1-t} ||u_n^+||_{p^*}^t, \\ \Rightarrow \quad & ||u_n^+||_r^r \leqslant c_{16} ||u_n^+||^{tr} \text{for some } c_{16} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (49)).} \end{aligned} \tag{50}$$

From hypothesis $H(f)(i)$ we have

$$f(z,x)x \leqslant c_{17}[1 + x^r] \text{ for all } z \in \Omega, \text{ all } x \geqslant 0 \text{ and some } c_{17} > 0. \tag{51}$$

From (43) with $h = u_n^+ \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned} & \gamma_p(u_n^+) + ||Du_n^+||_q^q - \int_\Omega \tau_\lambda^*(z,u_n)u_n^+ dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \gamma_p(u_n^+) + ||Du_n^+||_q^q \leqslant \int_\Omega [(u_n^+)^{1-\gamma} + f(z,u_n^+)u_n^+] dz + c_{18} \\ & \text{for some } c_{18} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (34))} \\ \leqslant \quad & c_{19}\left[1 + ||u_n^+||_r^r\right] \text{ for some } c_{19} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (51))} \\ \leqslant \quad & c_{20}[1 + ||u_n^+||^{tr}] \text{ for some } c_{20} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (50)).} \end{aligned} \tag{52}$$

The hypothesis on τ (see $H(f)(iii)$) implies that $tr < p$. So, from (52) we infer that

$$\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded,}$$

$$\Rightarrow \{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (44)).} \tag{53}$$

If $N = p$, then $p^* = +\infty$ and from the Sobolev embedding theorem, we know that $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leqslant s < \infty$. Then in order for the previous argument to work, we replace $p^* = +\infty$ by $s > r > \tau$ and let $t \in (0,1)$ as before such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{s},$$

$$\Rightarrow tr = \frac{s(r-\tau)}{s-\tau}.$$

Note that $\frac{s(r-\tau)}{s-\tau} \to r - \tau$ as $s \to +\infty$. But $r - \tau < p$ (see hypothesis H(f)(iii)). We choose $s > r$ big so that $tr < p$. Then again we have (53).

Because of (53) and by passing to a subsequence if neccesary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \tag{54}$$

In (43) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (54). Then

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0,$$

$$\Rightarrow \limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle] \leqslant 0$$
(since $A_q(\cdot)$ is monotone)
$$\Rightarrow \limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leqslant 0,$$
$$\Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.1)}.$$

Therefore $\Psi_\lambda^*(\cdot)$ satisfies the C-condition. This proves the claim.

Then (39), (40) and Claim permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\Psi_\lambda^*} \leqslant [u_\mu) \cap D_+ (\text{see } (35)) \ , m_\lambda^* \leqslant \Psi_\lambda^*(\hat{u}) \text{ (see (39))} \ .$$

Therefore $\hat{u} \in D_+$ is a second positive solution of problem (P_λ) $(\lambda \in (0, \lambda^*))$ distinct from $u_0 \in D_+$. □

Next, we examine what can be said in the critical parameter λ^*.

Proposition 4.6. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq (0, \lambda^*)$ be such that $\lambda_n < \lambda^*$. We can find $u_n \in S_{\lambda_n} \subseteq D_+$ for all $n \in \mathbb{N}$.

We consider the following Carathéodory function

$$\mu_n(z,x) = \begin{cases} v(z)^{-\gamma} + \lambda_n f(z, v(z)) & \text{if } x \leqslant v(z) \\ x^{-\gamma} + \lambda_n f(z,x) & \text{if } v(z) < x. \end{cases} \tag{55}$$

We set $M_n(z,x) = \int_0^x \mu_n(z,x)ds$ and consider the C^1-functional $j_n : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$j_n(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega M_n(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Also, we consider the following truncation of $\mu_n(z,\cdot)$

$$\hat{\mu}_n(z,x) = \begin{cases} \mu_n(z,x) & \text{if } x \leqslant u_{n+1}(z) \\ \mu_n(z, u_{n+1}(z)) & \text{if } u_{n+1}(z) < x \end{cases} \tag{56}$$

(recall that $v \leqslant u_{n+1}$ for all $n \in \mathbb{N}$, see Proposition 4.2). This is a Carathéodory function. We set $\hat{M}_n(z,x) = \int_0^x \hat{\mu}_n(z,s)ds$ and consider the C^1-functional $\hat{J}_n : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{J}_n(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}||Du||_q^q - \int_\Omega \hat{M}_n(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (55), (56) and (1) , it is clear that $\hat{J}_n(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_n \in W^{1,p}(\Omega)$ such that

$$\hat{J}_n(\hat{u}_n) = \inf\left\{\hat{J}_n(u) : u \in W^{1,p}(\Omega)\right\}. \tag{57}$$

Then we have

$$\begin{aligned} \hat{J}_n(\hat{u}_n) &\leqslant \hat{J}_n(v) \\ &\leqslant \frac{1}{p}\gamma_p(v) + \frac{1}{q}||Dv||_q^q - \frac{1}{1-\gamma}\int_\Omega v^{1-\gamma}dz \\ &\quad \text{(see (55), (56) and recall that } f \geqslant 0) \\ &\leqslant \langle A_p(v), v\rangle + \langle A_q(v), v\rangle - \int_\Omega v^{1-\gamma}dz = 0 \qquad (58) \\ &\quad \text{(see Proposition 3.2).} \end{aligned}$$

From (57) we have

$$\hat{u}_n \in K_{\hat{J}_n} \subseteq [v, u_{n+1}] \cap D_+ \text{ for all } n \in \mathbb{N} \text{ (see (56))}. \tag{59}$$

Similarly, using (55) we obtain

$$K_{j_n} \subseteq [v) \cap D_+. \tag{60}$$

Note that

$$J_n|_{[v,u_{n+1}]} = \hat{J}_n|_{[v,u_{n+1}]} \text{ and } J'_n|_{[v,u_{n+1}]} = \hat{J}'_n|_{[v,u_{n+1}]} \text{ (see (55), (56)).}$$

Then from (58), (59), (60), we have

$$J_n(\hat{u}_n) \leqslant 0 \text{ for all } n \in \mathbb{N} \tag{61}$$

$$\begin{aligned} &\langle A_p(\hat{u}_n), h\rangle + \langle A_q(\hat{u}_n), h\rangle + \int_\Omega \xi(z)\hat{u}_n^{p-1}hdz + \int_{\partial\Omega} \beta(z)\hat{u}_n^{p-1}hd\sigma = \int_\Omega \mu_n(z,\hat{u}_n)hdz \\ &\text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{62}$$

Using (61), (62) and reasoning as in the Claim in the proof of Proposition 4.5, we show that

$$\{\hat{u}_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\hat{u}_n \xrightarrow{w} \hat{u}_* \text{ in } W^{1,p}(\Omega) \text{ and } \hat{u}_n \to \hat{u}_* \text{ in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \tag{63}$$

In (62) we choose $h = \hat{u}_n - \hat{u}_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (63). Then as before (see the proof of Proposition 4.5), we obtain

$$\hat{u}_n \to \hat{u}_* \text{ in } W^{1,p}(\Omega). \tag{64}$$

In (62) we pass to the limit as $n \to \infty$ and use (64). Then

$$\langle A_p(\hat{u}_*), h\rangle + \langle A_q(\hat{u}_*), h\rangle + \int_\Omega \xi(z)\hat{u}_*^{p-1}hdz + \int_{\partial\Omega} \beta(z)\hat{u}_*^{p-1}hdz$$

$$= \int_\Omega [\hat{u}_*^{-\gamma} + \lambda^* f(z,\hat{u}_*)]hdz \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (55), (60)),}$$

$$\Rightarrow \hat{u}_* \in S_{\lambda^*} \subseteq D_+ \text{ and so } \lambda^* \in \mathcal{L}.$$

The proof is now complete. ☐

From this proposition it follows that

$$\mathcal{L} = (0, \lambda*].$$

The next bifurcation-type theorem summarizes our findings and provides a complete description of the dependence of the set of positive solutions of problem (P_λ) on the parameter $\lambda > 0$.

Theorem 4.7. *If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions*

$$u_0,\, \hat{u} \in D_+,\ u_0 \neq \hat{u};$$

(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $\hat{u}_* \in D_+$;*
(c) *for all $\lambda > \lambda^*$ problem (P_λ) does not have any positive solutions.*

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