This postprint is published in:
Mediterranean journal of mathematics, Volume 17, Issue 4, August 2020, art. 111, 9
DOI: 10.1007/s00009-020-01551-1

LOW PERTURBATIONS FOR A CLASS OF NONUNIFORMLY ELLIPTIC PROBLEMS

ANOUAR BAHROUNI AND DUŠAN D. REPOVŠ

Abstract. We introduce and study a new functional which was motivated by our paper on the Caffarelli-Kohn-Nirenberg inequality with variable exponent (Bahrouni, Rădulescu & Repovš, Nonlinearity 31 (2018), 1518-1534). We also study the eigenvalue problem for equations involving this new functional.

1. Introduction

The Caffarelli-Kohn-Nirenberg inequality plays an important role in studying various problems of mathematical physics, spectral theory, analysis of linear and nonlinear PDEs, harmonic analysis, and stochastic analysis. We refer to Chaudhuri & Ramaswamy [2], Baroni, Colombo & Mingione [4], Colasuonno & Pucci [7], and Colombo & Mingione [8] for relevant applications of the Caffarelli-Kohn-Nirenberg inequality.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary. The following Caffarelli-Kohn-Nirenberg inequality (see Caffarelli, Kohn & Nirenberg [5]) establishes that given $p \in (1, N)$ and real numbers a, b, and q such that

$$-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a+1, \quad q = \frac{Np}{N-p(1+a-b)}\,,$$

there is a positive constant $C_{a,b}$ such that for every $u \in C_c^1(\Omega)$,

$$\left(\int_\Omega |x|^{-bq}|u|^q\,dx\right)^{p/q} \leq C_{a,b}\int_\Omega |x|^{-ap}|\nabla u|^p\,dx\,. \tag{1}$$

This inequality has been extensively studied (see, e.g. Abdellaoui & Peral [1], Chaudhuri & Ramaswamy [2], Bahrouni, Rădulescu & Repovš [3], Catrina & Wang [6], and Mihăilescu, Rădulescu & Stancu [11], and the references therein).

In particular, Bahrouni, Rădulescu & Repovš [3] gives a new version of the Caffarelli-Kohn-Nirenberg inequality with variable exponent. The next theorem is proved under the following assumptions: let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary and suppose that the following hypotheses are satisfied

(A) $a : \overline{\Omega} \to \mathbb{R}$ is a function of class C^1 and there exist $x_0 \in \Omega$, $r > 0$, and $s \in (1, +\infty)$ such that:
  (1) $|a(x)| \neq 0$, for every $x \in \overline{\Omega} \setminus \{x_0\}$;
  (2) $|a(x)| \geq |x - x_0|^s$, for every $x \in B(x_0, r)$;

(P) $p : \overline{\Omega} \to \mathbb{R}$ is a function of class C^1 and $2 < p(x) < N$ for every $x \in \Omega$.

Theorem 1.1. (see Bahrouni, Rădulescu & Repovš [3]) *Suppose that hypotheses (A) and (P) are satisfied. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary. Then there exists a*

Key words and phrases. Caffarelli-Kohn-Nirenberg inequality, eigenvalue problem, critical point theorem, generalized Lebesgue-Sobolev space, Luxemburg norm.

2010 Math. Subj. Classif.: Primary 35J60, Secondary 35J91, 58E30.

positive constant β such that

$$\int_\Omega |a(x)|^{p(x)}|u(x)|^{p(x)}dx \leq \beta \int_\Omega |a(x)|^{p(x)-1}||\nabla a(x)||u(x)|^{p(x)}dx + \beta \left(\int_\Omega |a(x)|^{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega |a(x)|^{p(x)}|\nabla p(x)||u(x)|^{p(x)+1}dx\right) + \beta \int_\Omega |a(x)|^{p(x)-1}|\nabla p(x)||u(x)|^{p(x)-1}dx.$$

for every $u \in C_c^1(\Omega)$.

Motivated by Bahrouni, Rădulescu & Repovš [3], we introduce and study in the present paper a new functional $T : E_1 \to \mathbb{R}$ via the Caffarelli-Kohn-Nirenberg inequality, in the framework of variable exponents. More precisely, we study the eigenvalue problem in which functional T is present. Our main result is Theorem 4.2 and we prove it in Section 5.

2. **Function spaces with variable exponent**

We recall some necessary properties of variable exponent spaces. We refer to Hajek, Santalucia, Vanderwerff & Zizler [10], Musielak [12], Papageorgiou, Rădulescu & Repovš [13], Rădulescu [15], Rădulescu [16], and Rădulescu & Repovš [17], and the references therein.

Consider the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \mid \ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, let

$$p^+ = \sup_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^- = \inf_{x\in\overline{\Omega}} p(x),$$

and define the *variable exponent Lebesgue space* as follows

$$L^{p(x)}(\Omega) = \left\{u \mid \ u \text{ is measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\},$$

with the *Luxemburg norm*

$$|u|_{p(x)} = \inf\left\{\mu > 0 \mid \ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\, dx \leq 1\right\}.$$

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces if and only if $1 < p^- \leq p^+ < \infty$, and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}\,. \tag{2}$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the $p(.)-$modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u|^{p(x)}dx.$$

Proposition 2.1. (see Rădulescu & Repovš [17]) *The following properties hold*
(i) $|u|_{p(x)} < 1(\textit{resp.}, = 1; > 1) \Leftrightarrow \rho(u) < 1(\textit{resp.}, = 1; > 1)$*;*
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$*; and*
(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.

Proposition 2.2. (see Rădulescu & Repovš [17]) *If $u, u_n \in L^{p(x)}(\Omega)$ and $n \in \mathbb{N}$, then the following statements are equivalent*
(1) $\lim_{n\to+\infty} |u_n - u|_{p(x)} = 0$.
(2) $\lim_{n\to+\infty} \rho(u_n - u) = 0$.
(3) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to+\infty} \rho(u_n) = \rho(u)$.

We define the *variable exponent Sobolev space* by
$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid \ |\nabla u| \in L^{p(x)}(\Omega)\}.$$
On $W^{1,p(x)}(\Omega)$ we consider the following norm
$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$
Then $W^{1,p(x)}(\Omega)$ is a reflexive separable Banach space.

3. Functional T

We shall introduce a new functional $T : E_1 \to \mathbb{R}$ motivated by the Caffarelli-Kohn-Nirenberg inequality obtained in Bahrouni, Rădulescu & Repovš [3].

We denote by E_1 the closure of $C_c^1(\Omega)$ under the norm
$$\begin{aligned}\|u\| = \ & ||B(x)|^{\frac{1}{p(x)}}\nabla u(x)|_{p(x)} + |A(x)^{\frac{1}{p(x)}}u(x)|_{p(x)} + \\ & ||D(x)|^{\frac{1}{p(x)+1}}u(x)|_{p(x)+1} + ||C(x)|^{\frac{1}{p(x)-1}}u(x)|_{p(x)-1},\end{aligned}$$
where the potentials A, B, C, and D are defined by
$$\left\{\begin{array}{l} A(x) = |a(x)|^{p(x)-1}|\nabla a(x)| \\ B(x) = |a(x)|^{p(x)} \\ C(x) = |a(x)|^{p(x)-1}|\nabla p(x)| \\ D(x) = B(x)|\nabla p(x)|. \end{array}\right. \tag{3}$$
We now define $T : E_1 \to \mathbb{R}$ as follows
$$\begin{aligned}T(u) = & \int_\Omega \frac{B(x)}{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega \frac{A(x)}{p(x)}|u(x)|^{p(x)}dx \\ & + \int_\Omega \frac{D(x)}{p(x)+1}|u(x)|^{p(x)+1}dx + \int_\Omega \frac{C(x)}{p(x)-1}|u(x)|^{p(x)-1}dx.\end{aligned}$$
The following properties of T will be useful in the sequel.

Lemma 3.1. *Suppose that hypotheses* (A) *and* (P) *are satisfied. Then the functional* T *is well-defined on* E_1. *Moreover,* $T \in C^1(E_1, \mathbb{R})$ *with the derivative given by*
$$\begin{aligned}\langle L(u), v\rangle = \langle T'(u), v\rangle = & \int_\Omega B(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx + \int_\Omega A(x)|u(x)|^{p(x)-2}u(x)v(x)dx \\ & + \int_\Omega D(x)|u(x)|^{p(x)-1}u(x)v(x)dx + \int_\Omega C(x)|u(x)|^{p(x)-3}u(x)v(x)dx,\end{aligned}$$
for every $u, v \in E_1$.

Proof. The proof is standard, see Rădulescu & Repovš [17]. □

Lemma 3.2. *Suppose that hypotheses* (A) *and* (P) *are satisfied. Then the following properties hold*
(i) $L : E_1 \to E_1^*$ *is a continuous, bounded and strictly monotone operator;*
(ii) L *is a mapping of type* (S_+), *i.e. if* $u_n \rightharpoonup u$ *in* E_1 *and*
$$\limsup_{n\to+\infty}\langle L(u_n) - L(u), u_n - u\rangle \leq 0,$$
then $u_n \to u$ *in* E_1.

Proof. (i) Evidently, L is a bounded operator. Recall the following Simon inequalities (see Simon [18]):

$$\begin{cases} |x-y|^p \le c_p \left(|x|^{p-2}x - |y|^{p-2}y \right).(x-y) & \text{for } p \ge 2 \\ |x-y|^p \le C_p \left[\left(|x|^{p-2}x - |y|^{p-2}y \right).(x-y) \right]^{\frac{p}{2}} (|x|^p + |y|^p)^{\frac{2-p}{2}} & \text{for } 1 < p < 2, \end{cases} \tag{4}$$

for every $x, y \in \mathbb{R}^N$, where

$$c_p = (\frac{1}{2})^{-p} \text{ and } C_p = \frac{1}{p-1}.$$

Using inequalities (4) and recalling that $2 < p^-$, we can prove that L is a strictly monotone operator.

(ii) The proof is identical to the proof of Theorem 3.1 in Fan & Zhang [9]. □

4. Main theorem

We recall our Compactness Lemma:

Lemma 4.1. (see Bahrouni, Rădulescu & Repovš [3]) *Suppose that hypotheses* (A) *and* (P) *are satisfied and that* $p^- > 1 + s$. *Then* E_1 *is compactly embeddable in* $L^q(\Omega)$ *for each* $q \in (1, \frac{Np^-}{N+sp^+})$. *Moreover, the same conclusion holds if we replace* $L^q(\Omega)$ *by* $L^{q(x)}(\Omega)$, *provided that* $q^+ < \frac{Np^-}{N+sp^+}$.

We are concerned with the following nonhomogeneous problem

$$\begin{cases} -\text{div}\,(B(x)|\nabla u|^{p(x)-2}\nabla u) + (A(x)|u|^{p(x)-2} + C(x)|u|^{p(x)-3})u = \\ (\lambda|u|^{q(x)-2} - D(x)|u|^{p(x)-1})u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{5}$$

where $\lambda > 0$ is a real number and q is continuous on $\overline{\Omega}$. We assume that q satisfies the following basic inequalities

$$(Q) \qquad 1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}}(p(x)-1) < \max_{x\in\overline{\Omega}} q(x) < \frac{Np^-}{N+sp^+}.$$

We can now state the main result of this paper.

Theorem 4.2. *Suppose that all hypotheses of Lemma 4.1 are satisfied and that inequalities* (Q) *hold. Then there exists* $\lambda_0 > 0$ *such that every* $\lambda \in (0, \lambda_0)$ *is an eigenvalue for problem* (5).

In order to prove Theorem 4.2 (this will be done in the Section 5), we shall need some preliminary results. We begin by defining the functional $I_\lambda : E_1 \to \mathbb{R}$,

$$\begin{aligned} I_\lambda(u) = &\int_\Omega \frac{B(x)}{p(x)}|\nabla u(x)|^{p(x)}dx + \int_\Omega \frac{A(x)}{p(x)}|u(x)|^{p(x)}dx + \int_\Omega \frac{C(x)}{p(x)-1}|u(x)|^{p(x)-1}dx \\ &+ \int_\Omega \frac{D(x)}{p(x)+1}|u(x)|^{p(x)+1}dx - \lambda\int_\Omega \frac{|u(x)|^{q(x)}}{q(x)}dx. \end{aligned}$$

Standard argument shows that $I_\lambda \in C^1(E_1, \mathbb{R})$ and

$$\begin{aligned} \langle I'_\lambda(u), v\rangle = &\int_\Omega B(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx + \int_\Omega A(x)|u(x)|^{p(x)-2}u(x)v(x)dx \\ &+ \int_\Omega D(x)|u(x)|^{p(x)-1}u(x)v(x)dx + \int_\Omega C(x)|u(x)|^{p(x)-3}u(x)v(x)dx \\ &- \lambda\int_\Omega |u(x)|^{q(x)-2}u(x)v(x), \end{aligned}$$

for every $u, v \in E_1$.

Thus the weak solutions of problem (5) coincide with the critical points of I_λ.

Lemma 4.3. *Suppose that all hypotheses of Theorem 4.2 are satisfied. Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda \in (0, \lambda_0)$ *there exist* $\rho, \alpha > 0$ *such that*

$$I_\lambda(u) \geq \alpha \quad \text{for any} \quad u \in E_1 \quad \text{with} \quad \|u\| = \rho.$$

Proof. By Lemma 4.1, there exists $\beta > 0$ such that

$$|u|_{r(x)} \leq \beta \|u\|, \quad \text{for every} \quad u \in E_1 \quad \text{and} \quad r^+ \in (1, \frac{Np^-}{N + sp^+}).$$

We fix $\rho \in (0, \min(1, \frac{1}{\beta}))$. Invoking Proposition 2.1, for every $u \in E_1$ with $\|u\| = \rho$, we can get

$$|u|_{q(x)} < 1.$$

Combining the above relations and Proposition 2.1, for any $u \in E_1$ with $\|u\| = \rho$, we can then deduce that

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{p^+}\left(\int_\Omega B(x)|\nabla u(x)|^{p(x)}dx + \int_\Omega A(x)|u(x)|^{p(x)}dx\right) \\
&+\frac{1}{p^+ + 1}\int_\Omega D(x)|u(x)|^{p(x)+1}dx \\
&+\frac{1}{p^+ - 1}\int_\Omega C(x)|u(x)|^{p(x)-1}dx - \frac{\lambda}{q^-}\int_\Omega |u(x)|^{q(x)}dx \\
&\geq \frac{1}{4^{p^+}(p^+ + 1)}\|u\|^{p^+ + 1} - \lambda\frac{\beta^{q^-}}{q^-}\|u\|^{q^-} \\
&\geq \frac{1}{4^{p^+}(p^+ + 1)}\rho^{p^+ + 1} - \lambda\frac{\beta^{q^-}}{q^-}\rho^{q^-} \\
&= \rho^{q^-}(\tfrac{1}{4^{p^+}(p^+ + 1)}\rho^{p^+ + 1 - q^-} - \lambda\tfrac{\beta^{q^-}}{q^-}).
\end{aligned} \tag{6}$$

Put $\lambda_0 = \frac{\rho^{p^+ + 1 - q^-}}{4^{p^+}(2p^+ + 2)}\frac{q^-}{\beta^{q^-}}$. It now follows from (6) that for any $\lambda \in (0, \lambda_0)$,

$$I_\lambda(u) \geq \alpha \quad \text{with} \quad \|u\| = \rho,$$

and $\alpha = \frac{\rho^{p^+ + 1}}{4^{p^+}(2p^+ + 2)} > 0$. This completes the proof of Lemma 4.3. ☐

Lemma 4.4. *Suppose that all hypotheses of Theorem 4.2 are satisfied. Then there exists* $\varphi \in E_1$ *such that* $\varphi > 0$ *and* $I_\lambda(t\varphi) < 0$*, for small enough* t.

Proof. By virtue of hypotheses (P) and (Q), there exist $\epsilon_0 > 0$ and $\Omega_0 \subset \Omega$ such that

$$q(x) < q^- + \epsilon_0 < p^- - 1, \quad \text{for every} \quad x \in \Omega_0. \tag{7}$$

Let $\varphi \in C_0^\infty(\Omega)$ such that $\overline{\Omega_0} \subset \text{supp}(\varphi)$, $\varphi = 1$ for every $x \in \overline{\Omega_0}$ and $0 \leq \varphi \leq 1$ in Ω. It then follows that for $t \in (0, 1)$,

$$\begin{aligned}
I_\lambda(t\varphi) = &\int_\Omega \frac{t^{p(x)}B(x)}{p(x)}|\nabla\varphi(x)|^{p(x)}dx + \int_\Omega \frac{t^{p(x)}A(x)}{p(x)}|\varphi(x)|^{p(x)}dx + \int_\Omega \frac{t^{p(x)-1}C(x)}{p(x)-1}|\varphi|^{p(x)-1}dx \\
&+ \int_\Omega \frac{t^{p(x)+1}D(x)}{p(x)+1}|\varphi(x)|^{p(x)+1}dx - \lambda\int_\Omega t^{q(x)}\frac{|\varphi(x)|^{q(x)}}{q(x)}dx \\
&\leq \frac{t^{p^- - 1}}{p^- - 1}(\int_\Omega \frac{B(x)}{p(x)}|\nabla\varphi(x)|^{p(x)}dx + \int_\Omega \frac{A(x)}{p(x)}|\varphi(x)|^{p(x)}dx + \int_\Omega \frac{C(x)}{p(x)-1}|\varphi|^{p(x)-1}dx \\
&+ \int_\Omega \frac{D(x)}{p(x)+1}|\varphi(x)|^{p(x)+1}dx) - \lambda t^{q^- + \epsilon_0}\int_\Omega \frac{|\varphi(x)|^{q(x)}}{q(x)}dx.
\end{aligned} \tag{8}$$

Combining (7) and (8), we finally arrive at the desired conclusion.
This completes the proof of Lemma 4.4. ☐

5. **Proof of Theorem 4.2**

In the last section we shall prove the main theorem of this paper.

Let λ_0 be defined as in Lemma 4.3 and choose any $\lambda \in (0, \lambda_0)$.

Again, invoking Lemma 4.3, we can deduce that

$$\inf_{u \in \partial B(0,\rho)} I_\lambda(u) > 0. \tag{9}$$

On the other hand, by Lemma 4.4, there exists $\varphi \in E_1$ such that

$$I_\lambda(t\varphi) < 0 \ \text{ for every small enough t} > 0.$$

Moreover, by Proposition 2.1, when $\|u\| < \rho$, we have

$$I_\lambda(u) \geq \frac{1}{4^{p^+}(p^+ + 1)} \|u\|^{p^+ + 1} - c\|u\|^{q^-},$$

where c is a positive constant. It follows that

$$-\infty < m = \inf_{u \in B(0,\rho)} I_\lambda(u) < 0.$$

Applying Ekeland's variational principle to the functional

$$I_\lambda : B(0,\rho) \to \mathbb{R},$$

we can find a (PS) sequence $(u_n) \in B(0,\rho)$, that is,

$$I_\lambda(u_n) \to m \ \text{ and } \ I_\lambda^{'}(u_n) \to 0.$$

It is clear that (u_n) is bounded in E_1. Thus there exists $u \in E_1$ such that, up to a subsequence,

$$(u_n) \rightharpoonup u \ \text{ in } \ \mathrm{E}_1.$$

Using Theorem 4.1, we see that (u_n) strongly converges to u in $L^{q(x)}(\Omega)$.

So, by the Hölder inequality and Proposition 2.2, we can obtain the following

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{q(x)-2} u_n(u_n - u)dx = \lim_{n\to+\infty} \int_\Omega |u|^{q(x)-2} u(u_n - u)dx = 0.$$

On the other hand, since (u_n) is a (PS) sequence, we can also infer that

$$\lim_{n\to+\infty} \langle I_\lambda^{'}(u_n) - I_\lambda^{'}(u), u_n - u\rangle = 0.$$

Combining the above pieces of information with Lemma 3.2, we can now conclude that

$$u_n \to u \ \text{ in } \ \mathrm{E}_1.$$

Therefore

$$I_\lambda(u) = m < 0 \ \text{ and } \ I_\lambda^{'}(u) = 0.$$

We have thus shown that u is a nontrivial weak solution for problem (5) and that every $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (5).

This completes the proof of Theorem 4.2. □

Acknowledgements

The second author was supported by the Slovenian Research Agency grants P1-0292, J1-7025, J1-8131, N1-0064, N1-0083, and N1-0114. We thank the referee for comments and suggestions.

References

[1] B. Abdellaoui and I. Peral, Some results for quasilinear elliptic equations related to some Caffarelli-Kohn-Nirenberg inequalities, *Commun. Pure Appl. Anal.* **2** (2003), 539-566.
[2] A.N. Chaudhuri, and M. Ramaswamy, An improved Hardy-Sobolev inequality and its application, *Proc. Amer. Math. Soc.* **130** (2002), 89-505.
[3] A. Bahrouni, V. Rădulescu and D.D. Repovš, A weighted anisotropic variant of the Caffarelli-Kohn-Nirenberg inequality and applications, *Nonlinearity.* **31** (2018), no. 4, 1516-1534.
[4] P. Baroni, M. Colombo, and G. Mingione, Non-autonomous functionals, borderline cases and related function classes, *St. Petersburg Math. J.* **27** (2016), 347-379.
[5] L. Caffarelli, R. Kohn, and L. Nirenberg, First order interpolation inequalities with weights, *Compos. Math.* **53** (1984), 259-275.
[6] F. Catrina and Z.Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal function, *Commun. Pure Appl. Math.* **54** (2001), 229-258.
[7] F. Colasuonno and P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, *Nonlinear Anal.* **74** (2011), no. 17, 5962-5974.
[8] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.* **218** (2015), 219-273.
[9] X. Fan and Q. Zhang, Existence of solutions for $p(x)-$Laplacian Dirichlet probem, *Nonlinear Anal.* **52** (2003), 1843-1852.
[10] P. Hajek, V.M. Santalucia, J. Vanderwerff, and V. Zizler, *Biorthogonal Systems in Banach Spaces*, CMC Books in Mathematics, Springer, New York, 2008.
[11] M. Mihăilescu, V. Rădulescu, and D. Stancu, A Caffarelli-Kohn-Nirenberg type inequality with variable exponent and applications to PDE's, *Complex Variables Elliptic Eqns.* **56** (2011), 659-669.
[12] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034, Springer, Berlin, 1983.
[13] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, *Nonlinear Analysis - Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.
[14] P.H. Rabinowitz, Minimax Theorems and Applications to Nonlinear Partial Differential Equations, in: L.Cesari, R. Kannan, H.F. Weinberger (Eds.), *Nonlinear Analysis*, Academic Press, New York, 1978, pp. 161-177.
[15] V.D. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121** (2015), 336-369.
[16] V.D. Rădulescu, Isotropic and anisotropic double-phase problems: old and new, *Opuscula Math.* **39** (2019), 259-279.
[17] V.D. Rădulescu and D.D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton, FL, 2015.
[18] J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, *Journées d'Analyse Non Linéaire (Proc. Conf., Besançon, 1977)*, Lecture Notes in Math., 665, Springer, Berlin, 1978, pp. 205-227 .

(A. Bahrouni) Mathematics Department, University of Monastir, Faculty of Sciences, 5019 Monastir, Tunisia
Email address: bahrounianouar@yahoo.fr

(D.D. Repovš) Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
Email address: dusan.repovs@guest.arnes.si