TORSION TABLE FOR THE LIE ALGEBRA $\mathfrak{nil}_n$

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Abstract. We study the Lie ring $\mathfrak{nil}_n$ of all strictly upper-triangular $n\times n$ matrices with entries in $\mathbb{Z}$. Its complete homology for $n\leq 8$ is computed.

We prove that every p^m-torsion appears in $H_*(\mathfrak{nil}_n;\mathbb{Z})$ for $p^m\leq n-2$. For $m=1$, Dwyer proved that the bound is sharp, i.e. there is no p-torsion in $H_*(\mathfrak{nil}_n;\mathbb{Z})$ when prime $p>n-2$. In general, for $m>1$ the bound is not sharp, as we show that there is 8-torsion in $H_*(\mathfrak{nil}_8;\mathbb{Z})$.

As a sideproduct, we derive the known result, that the ranks of the free part of $H_*(\mathfrak{nil}_n;\mathbb{Z})$ are the Mahonian numbers (=number of permutations of $[n]$ with k inversions), using a different approach than Kostant. Furthermore, we determine the algebra structure (cup products) of $H^*(\mathfrak{nil}_n;\mathbb{Q})$.

1. Introduction

Let $\mathfrak{nil}_n$ be the Lie algebra of integral $n\times n$ strictly upper-triangular matrices. The complete homology $H_k(\mathfrak{nil}_n;\mathbb{Z})$ is known only for $n\leq 6$ [4]: Here $0^a 2^b 3^c 4^d \dots$

$k\backslash n$	2	3	4	5	6
0	0^1	0^1	0^1	0^1	0^1
1	0^1	0^2	0^3	0^4	0^5
2		0^2	$0^5 2^1$	$0^9 2^2$	$0^{14} 2^3$
3		0^1	$0^6 2^1$	$0^{15} 2^8 3^2$	$0^{29} 2^{20} 3^4$
4			0^5	$0^{20} 2^{10} 3^3$	$0^{49} 2^{47} 3^{13} 4^3$
5			0^3	$0^{22} 2^{10} 3^3$	$0^{71} 2^{79} 3^{26} 4^9$
6			0^1	$0^{20} 2^8 3^2$	$0^{90} 2^{118} 3^{35} 4^{12}$
7				$0^{15} 2^2$	$0^{101} 2^{138} 3^{36} 4^{12}$
8				0^9	$0^{101} 2^{118} 3^{35} 4^{12}$
9				0^4	$0^{90} 2^{79} 3^{26} 4^9$
10				0^1	$0^{71} 2^{47} 3^{13} 4^3$
11					$0^{49} 2^{20} 3^4$
12					$0^{29} 2^3$
13					0^{14}
14					0^5
15					0^1

Table 1. Elementary divisor decomposition of $H_*(\mathfrak{nil}_n;\mathbb{Z})$

denotes $\mathbb{Z}^a\oplus\mathbb{Z}_2^b\oplus\mathbb{Z}_3^c\oplus\mathbb{Z}_4^d\oplus\dots$ The main reason why computations for larger n are exceedingly difficult is that the chain complex $C_*=\Lambda^*\mathfrak{nil}_n$ is immense. It has $2^{\binom{n}{2}}$

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generators, which is more than 2 million for $n=7$. In this paper, we divide C_* into numerous direct summands $[\![w]\!]$ (corresponding to sequences $w\in\{1,\ldots,n\}^n$ with $w_1+\ldots+w_n=\binom{n+1}{2}$) and show how many of them are isomorphic (up to dimension shift), many are contractible and many are obtained from smaller ones as the cone of a chain map $[\![w']\!]\xrightarrow{\cdot t}[\![w']\!]$. The direct summands corresponding to permutations of $(1,\ldots,n)$ are generated by just one element, hence those only contribute to the free part. We show that any other direct summand contributes only to torsion, so we obtain $H^*(\mathfrak{nil}_n;\mathbb{Q})$.

We derive a stronger property of $H_*(\mathfrak{nil}_n;\mathbb{Z})$ then that of Dwyer in [1, p.524, remarks below Proposition 1.3]. We also prove the first half of Conjecture 1.16 by Jöllenbeck in [4, p.141] and disprove its second half.

The complex. Let e_{ij} be the matrix with all entries 0 except 1 in the position (i,j). The chain complex $C_*=\Lambda^*\mathfrak{nil}_n$, due to Chevalley (1948), is generated by wedges $e_{a_1b_1}\wedge\ldots\wedge e_{a_kb_k}$, where $1\le a_i<b_i\le n$ for all i. From now on, for the sake of brevity, we shall omit the $\wedge$ symbols. The boundary is defined by

$$\partial(e_{a_1b_1}\ldots e_{a_kb_k})=\textstyle\sum_{i<j}(-1)^{i+j}[e_{a_ib_i},e_{a_jb_j}]e_{a_1b_1}\ldots\widehat{e_{a_ib_i}}\ldots\widehat{e_{a_jb_j}}\ldots e_{a_kb_k},$$

where $[e_{ab},e_{cd}]$ equals e_{ad} if $b=c$, equals $-e_{cb}$ if $a=d$, and equals 0 otherwise.

AMT. For some computations later on, we shall use algebraic Morse theory, so we include a short review of it. To a chain complex of free modules (C_*,∂_*) we associate a weighted digraph Γ_{C_*} (vertices are basis elements of C_*, weights of edges are nonzero entries of matrices ∂_*). Then we carefully select a matching $\mathcal{M}$ in this digraph, so that its edges have invertible weights and if we reverse the direction of every $e\in\mathcal{M}$ in Γ_{C_*}, the obtained digraph $\Gamma_{C_*}^{\mathcal{M}}$ contains no directed cycles and no infinite paths in two adjacent degrees. Under these conditions (i.e. if $\mathcal{M}$ is a *Morse matching*), the AMT theorem ([8], [4], [6]) provides a homotopy equivalent complex $(\mathring{C}_*,\mathring{\partial}_*)$, spanned by the unmatched vertices in $\Gamma_{C_*}^{\mathcal{M}}$, and with the boundary $\mathring{\partial}_*$ of $v\in\mathring{C}_k$ given by the sum of weights of directed paths in $\Gamma_{C_*}^{\mathcal{M}}$ to all critical $v'\in\mathring{C}_{k-1}$. For more details, we refer the reader to the three articles above (which specify the homotopy equivalence), or [7] for a quick introduction and formulation.

2. Subcomplexes

For an ordered set $M=\{(a_1,b_1),\ldots,(a_k,b_k)\}\subseteq\{(i,j);1\le i<j\le n\}$ we denote $e_M=e_{a_1b_1}\ldots e_{a_kb_k}$. For $M_i:=\{x;(i,x)\in M\}$ we have $e_M=\wedge_{i=1}^{n-1}e_{\{i\}\times M_i}$. We define the *weight* vector $\widetilde{w}(e_M)=(\widetilde{w}_1,\ldots,\widetilde{w}_n)$ by $\widetilde{w}_i=|\{x;(x,i)\in M\}|-|\{y;(i,y)\in M\}|$, i.e. the number of times i appears on the right in e_M minus the number of times i appears on the left in e_M. Then $\sum_{i=1}^n\widetilde{w}_i=0$. Every summand in $\partial(e_M)$ has the same weight as e_M. Therefore a submodule $[\widetilde{w}]$ of $\Lambda^*\mathfrak{nil}_n$, spanned by the basis elements with weight $\widetilde{w}$, forms a chain subcomplex which is a direct summand.

Most equalities will be described more conveniently using the *modified weight* $w(e_M)=(1,\ldots,n)-\widetilde{w}(e_M)=(1-\widetilde{w}_1,\ldots,n-\widetilde{w}_n)$. Then $\sum_{i=1}^n w_i=\binom{n+1}{2}$ and $i-n\le\widetilde{w}_i\le i-1$ implies $1\le w_i\le n$ for all i. We denote $[\![w]\!]=[(1,\ldots,n)-w]$ and let $[\![w]\!]_k$ be the complex $[\![w]\!]$ dimensionally shifted by k. Let $\mathcal{S}_n:=\big\{(w_1,\ldots,w_n)\in\{1,\ldots,n\}^n;\,w_1+\ldots+w_n=\binom{n+1}{2}\big\}$, so that $\Lambda^*\mathfrak{nil}_n=\bigoplus_{w\in\mathcal{S}_n}[\![w]\!]$. Notice that $[\![w_1,\ldots,w_{n-1},n]\!]=[\![w_1,\ldots,w_{n-1}]\!]$ and $[\![1,w_2,\ldots,w_n]\!]\cong[\![w_2-1,\ldots,w_n-1]\!]$, since $w_n(e_M)=n$ (resp. $w_1(e_M)=1$) implies $\{x;(x,n)\in M\}=\emptyset$ (resp. $\{y;(1,y)\in M\}=\emptyset$).

We shall abbreviate $[\![(w_1,\ldots,w_n)]\!]$ to $[\![w_1,\ldots,w_n]\!]$ and $H_k([\![w]\!];\mathbb{Z})$ to $H_k[\![w]\!]$.

Example 2.1. Let us take a look at bracket subcomplexes in $\Lambda^*\mathfrak{nil}_n$ for $n\leq 4$.

Set $\mathcal{S}_2$ consists of permutations of $(1,2)$. Furthermore, there holds $H_k[\![1,2]\!]=H_k\langle\emptyset\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=0\\ 0; & \text{if } k\neq 0\end{smallmatrix}\right.$ and $H_k[\![2,1]\!]=H_k\langle e_{12}\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=1\\ 0; & \text{if } k\neq 1\end{smallmatrix}\right.$.

Set $\mathcal{S}_3$ consists of permutations of $(1,2,3),(2,2,2)$. Furthermore, $H_k[\![1,2,3]\!]=H_k\langle\emptyset\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=0\\ 0; & \text{if } k\neq 0\end{smallmatrix}\right.$, $H_k[\![1,3,2]\!]=H_k\langle e_{23}\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=1\\ 0; & \text{if } k\neq 1\end{smallmatrix}\right.$, $H_k[\![2,1,3]\!]=H_k\langle e_{12}\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=1\\ 0; & \text{if } k\neq 1\end{smallmatrix}\right.$, $H_k[\![2,3,1]\!]=H_k\langle e_{13}e_{23}\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=2\\ 0; & \text{if } k\neq 2\end{smallmatrix}\right.$, $H_k[\![3,1,2]\!]=H_k\langle e_{12}e_{13}\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=2\\ 0; & \text{if } k\neq 2\end{smallmatrix}\right.$, $H_k[\![3,2,1]\!]=H_k\langle e_{12}e_{13}e_{23}\rangle\cong\left\{\begin{smallmatrix}\mathbb{Z}; & \text{if } k=3\\ 0; & \text{if } k\neq 3\end{smallmatrix}\right.$, $H_k[\![2,2,2]\!]=H_k\langle e_{13},e_{12}e_{23}\rangle\simeq 0$.

Set $\mathcal{S}_4$ consists of permutations of (1,1,4,4), (1,2,3,4), (1,3,3,3), (2,2,2,4), (2,2,3,3). The largest non-contractible complexes are

$$\begin{aligned}&[\![2,3,2,3]\!]=\langle e_{14}e_{23},e_{13}e_{24},e_{12}e_{23}e_{24},e_{13}e_{23}e_{34}\rangle \text{ and}\\ &[\![3,2,3,2]\!]=\langle e_{12}e_{14}e_{24},e_{13}e_{14}e_{34},e_{12}e_{14}e_{23}e_{34},e_{12}e_{13}e_{24}e_{34}\rangle,\end{aligned}$$

with $[\![3,2,3,2]\!]\cong[\![2,3,2,3]\!]_1$. See 5 for the complete computation of $H_*\mathfrak{nil}_4$. ◊

Lemma 2.2. $[\![w_1,\ldots,w_n]\!]\cong[\![n{+}1{-}w_n,\ldots,n{+}1{-}w_1]\!]$.

Proof. Define $\tau(e_{ab})=e_{n+1-b,n+1-a}$ and $\tau(\wedge_{i=1}^k e_{a_ib_i})=(-1)^{k+1}\wedge_{i=1}^k\tau(e_{a_ib_i})$. Now $e_M\in[\![w_1,\ldots,w_n]\!]$ implies $\tau(e_M)\in[\![n{+}1{-}w_n,\ldots,n{+}1{-}w_1]\!]$, because

$$\begin{aligned}w_{n+1-i}(\tau(e_{a_1b_1}\ldots e_{a_kb_k}))&=w_{n+1-i}(e_{n+1-b_1,n+1-a_1}\ldots e_{n+1-b_k,n+1-a_k})\\&=n{+}1-i-|\{j;a_j=i\}|+|\{j;b_j=i\}|\\&=n{+}1-(i-|\{j;b_j=i\}|+|\{j;a_j=i\}|)\\&=n{+}1-w_i(e_{a_1b_1}\ldots e_{a_kb_k}).\end{aligned}$$

From $[\tau(e_{ab}),\tau(e_{cd})]=-\tau([e_{ab},e_{cd}])$, we obtain

$$\begin{aligned}\partial\tau(e_{a_1b_1}\ldots e_{a_kb_k})&=(-1)^{k+1}\textstyle\sum_{i<j}(-1)^{i+j}[\tau e_{a_ib_i},\tau e_{a_jb_j}]\ldots\widehat{\tau}(e_{a_ib_i})\ldots\widehat{\tau}(e_{a_jb_j})\ldots\\&=(-1)^{k}\textstyle\sum_{i<j}(-1)^{i+j}\tau[e_{a_ib_i},e_{a_jb_j}]\ldots\widehat{\tau}(e_{a_ib_i})\ldots\widehat{\tau}(e_{a_jb_j})\ldots\\&=\tau\partial(e_{a_1b_1}\ldots e_{a_kb_k}),\end{aligned}$$

so τ is a chain map. Since $\tau\circ\tau=\mathrm{id}$, our τ is an isomorphism of chain complexes. □

Lemma 2.3. $[\![w_1,w_2,\ldots,w_n]\!]\cong[\![w_2,\ldots,w_n,w_1]\!]_{2w_1-n-1}$.

Proof. Define a linear map $\varphi\colon[\![w_1,w_2,\ldots,w_n]\!]\longrightarrow[\![w_2,\ldots,w_n,w_1]\!]_{2w_1-n-1}$ by

$$\varphi\big(\wedge_{i=1}^{n-1}e_{\{i\}\times M_i}\big)=(-1)^{\Sigma M_1}e_{M_1^C\times\{n+1\}}\wedge_{i=2}^{n-1}e_{\{i\}\times M_i},$$

where $M_1^C=\{2,\ldots,n\}\setminus M_1$; it is convenient to have indices in the codomain go from 2 to $n{+}1$ instead of from 1 to n. There holds

$$\begin{aligned}w_i(\varphi(e_M))&=i-|\{x;e_{x,i+1}\in\varphi(e_M)\}|+|\{y;e_{i+1,y}\in\varphi(e_M)\}|\\&=i-\big(|\{x;e_{x,i+1}\in e_M\}|+\left\{\begin{smallmatrix}-1; & e_{1,i+1}\in e_M\\ 0; & e_{1,i+1}\notin e_M\end{smallmatrix}\right.\big)+\big(|\{y;e_{i+1,y}\in e_M\}|+\left\{\begin{smallmatrix}1; & e_{1,i+1}\notin e_M\\ 0; & e_{1,i+1}\in e_M\end{smallmatrix}\right.\big)\\&=i{+}1-|\{x;e_{x,i+1}\in e_M\}|+|\{y;e_{i+1,y}\in e_M\}|=w_{i+1}(e_M)\quad\text{for } i<n\quad\text{and}\\ w_n(\varphi(e_M))&=n-|\{x;e_{x,n+1}\in\varphi(e_M)\}|+|\{y;e_{n+1,y}\in\varphi(e_M)\}|\\&=n-(n{-}1{-}|M_1|)+0=1-0+|M_1|=w_1(e_M).\end{aligned}$$

Length difference of e_M and $\varphi(e_M)$ is $(n-1-|M_1|)-|M_1| = n-1-2(w_1-1) = n+1-2w_1$. Thus φ is a well-defined bijection. By denoting $M\backslash x\cup y := (M\backslash\{x\})\cup\{y\}$, we have

$$\begin{aligned}
&\varphi\partial(e_{\{1\}\times M_1}e_N) = \varphi\big(\textstyle\sum_{x\in M_1,\, y\in N_x\backslash M_1}\varepsilon_{xy}e_{\{1\}\times(M_1\backslash x\cup y)}e_{N\backslash\{(x,y)\}} + (-1)^{|M_1|}e_{\{1\}\times M_1}\partial e_N\big)\\
&= \sum_{x\in M_1,\, y\in N_x\backslash M_1}(-1)^{y-x+\Sigma M_1}\varepsilon_{xy}e_{(M_1\backslash x\cup y)^C\times\{n+1\}}e_{N\backslash\{(x,y)\}} + (-1)^{|M_1|+\Sigma M_1}e_{M_1^C\times\{n+1\}}\partial e_N,\\
&\partial\varphi(e_{\{1\}\times M_1}e_N) = \partial\big((-1)^{\Sigma M_1}e_{M_1^C\times\{n+1\}}e_N\big)\\
&= \sum_{y\in N_x\cap M_1^C,\, x\notin M_1^C}(-1)^{\Sigma M_1}(-\varepsilon'_{xy})e_{(M_1^C\backslash y\cup x)\times\{n+1\}}e_{N\backslash\{(x,y)\}} + (-1)^{\Sigma M_1+|M_1^C|}e_{M_1^C\times\{n+1\}}\partial e_N\\
&= \sum_{x\in M_1,\, y\in N_x\backslash M_1}(-1)^{1+\Sigma M_1}\varepsilon'_{xy}e_{(M_1\backslash x\cup y)^C\times\{n+1\}}e_{N\backslash\{(x,y)\}} + (-1)^{n-1+|M_1|+\Sigma M_1}e_{M_1^C\times\{n+1\}}\partial e_N.
\end{aligned}$$

for $\varepsilon_{xy},\varepsilon'_{xy}\in\{1,-1\}$. Since $[e_{y,n+1},e_{x,y}] = -e_{x,n+1}$, there is a minus before ε'_{xy}. We must show that $(-1)^{y-x}\varepsilon_{xy} = (-1)^n\varepsilon'_{xy}$: if $\alpha =$(position of x in M_1), $\beta =$(position of (x,y) in N), $\gamma =$(position of y in $M_1\backslash x\cup y$), then y in M_1^C has position $y-\gamma-1$ and x in $M_1^C\backslash y\cup x$ has position $x-\alpha$, so $(-1)^{y-x}\varepsilon_{xy} = (-1)^{y-x+\alpha+(|M_1|+\beta)+(\gamma-1)} = (-1)^{n+(y-\gamma-1)+(n-1-|M_1|+\beta)+(x-\alpha-1)} = (-1)^n\varepsilon'_{xy}$. Therefore $\varphi\partial = (-1)^{n-1}\partial\varphi$, hence $\overline{\varphi}(e_M) := \begin{cases}\varphi(e_M)(-1)^{n-1}; & \text{if } |M|\in 2\mathbb{N}\\ \varphi(e_M); & \text{if } |M|\notin 2\mathbb{N}\end{cases}$ is an isomorphism of chain complexes. □

Lemma 2.4. *$[\![w_1,\ldots,w_{k-1},n,w_{k+1},\ldots,w_n]\!] \cong [\![w_1,\ldots,w_{k-1},w_{k+1},\ldots,w_n]\!]_{n-k}$ and $[\![w_1,\ldots,w_{k-1},1,w_{k+1},\ldots,w_n]\!] \cong [\![w_1-1,\ldots,w_{k-1}-1,w_{k+1}-1,\ldots,w_n-1]\!]_{k-1}$.*

Proof. We can identify $[\![w_1,\ldots,w_{n-1},n]\!]$ with $[\![w_1,\ldots,w_{n-1}]\!]$. By Lemma 2.3,

$$\begin{aligned}
[\![w_1,\ldots,w_{k-1},n,w_{k+1},\ldots,w_n]\!] &\cong [\![w_{k+1},\ldots,w_n,w_1,\ldots,w_{k-1},n]\!]_{-2\sum_{i=k+1}^n w_i+(n-k)(n+1)}\\
&\cong [\![w_{k+1},\ldots,w_n,w_1,\ldots,w_{k-1}]\!]_{-2\sum_{i=k+1}^n w_i+(n-k)(n+1)}\\
&\cong [\![w_1,\ldots,w_{k-1},n,w_{k+1},\ldots,w_n]\!]_{(n-k)\cdot 1}.
\end{aligned}$$

We can identify $[\![1,w_2,\ldots,w_n]\!]$ with $[\![w_2-1,\ldots,w_n-1]\!]$. By Lemma 2.3,

$$\begin{aligned}
[\![w_1,\ldots,w_{k-1},1,w_{k+1},\ldots,w_n]\!] &\cong [\![1,w_{k+1},\ldots,w_n,w_1,\ldots,w_{k-1}]\!]_{2\sum_{i=1}^{k-1}w_i-(k-1)(n+1)}\\
&\cong [\![w_{k+1}-1,\ldots,w_n-1,w_1-1,\ldots,w_{k-1}-1]\!]_{2\sum_{i=1}^{k-1}w_i-(k-1)(n+1)}\\
&\cong [\![w_1-1,\ldots,w_{k-1}-1,w_{k+1}-1,\ldots,w_n-1]\!]_{2(k-1)-(k-1)}.
\end{aligned}$$

This establishes the first and second parts of the claim. □

If all elements in $w = (w_1,\ldots,w_n)$ are distinct, then by applying Lemma 2.4 n times, we see that $[\![w]\!]$ has only one generator, namely

$$e_\pi := \textstyle\bigwedge_{i<j,w_i>w_j} e_{ij}, \tag{2.1}$$

the wedge of inversions of the permutation $w = \pi$ of $1,\ldots,n$. Indeed, $w_k(e_\pi) = k-|\{i; e_{ik}\in e_\pi\}|+|\{j; e_{kj}\in e_\pi\}| = k-|\{i; i<k, w_i>w_k\}|+|\{j; k<j, w_k>w_j\}| = 1+|\{i; i<k, w_i<w_k\}|+|\{j; k<j, w_k>w_j\}| = 1+|\{r; r\neq k, w_r<w_k\}| = w_k$.

Let $\mathcal{F}_n = \{(w_1,\ldots,w_n)\in\mathcal{S}_n;\ w_i\neq w_j \text{ for } i\neq j\} = \{\pi(1,\ldots,n); \pi\in S_n\}$. In Lemma 4.2, we show that for $w\notin\mathcal{F}_n$ the homology of $[\![w]\!]$ has only torsion. Therefore the free part is $FH_*(\mathfrak{nil}_n) = \bigoplus_{w\in\mathcal{F}_n}[\![w]\!]$. Thus we obtain what was already known to Kostant [5, 2], who used the Laplacian method (the fact that $\dim H_k(C_*,\partial_*) = \dim\operatorname{Ker}(\partial_{k+1}\partial_{k+1}^t+\partial_k^t\partial_k)$ over $\mathbb{Q}$) to obtain part (a) of the following result:

Theorem 2.5. ***(a)*** *$FH_k(\mathfrak{nil}_n)\cong\mathbb{Z}^{T(n,k)}$, where $T(n,k)$ is the Mahonian number.*
(b) *$H^*(\mathfrak{nil}_n;\mathbb{Q})\cong\langle x_\pi;\ \pi\in S_n\rangle \leq \Lambda_{\mathbb{Q}}[x_{ij}; 1\leq i<j\leq n]$, where $x_\pi = \bigwedge_{i<j,\pi_i>\pi_j}x_{ij}$.*

In words, the cohomology algebra over $\mathbb{Q}$ is isomorphic to the subalgebra of the polynomial exterior algebra, spanned by the inversions of all permutations.

Proof. **(a)** By definition (OEIS A008302), $T(n,k)$ is the number of permutations of $[n]$ which have k inversions, so the result follows from (2.1) and Lemma 4.2.
(b) For any $\alpha\in H^i(\mathfrak{g})$ and $\beta\in H^j(\mathfrak{g})$, the cup product is given by
$$(\alpha\smile\beta)(x_1\cdots x_{i+j}) = \sum_{\pi\in S_{i+j},\,\pi_1<\ldots<\pi_i,\pi_{i+1}<\ldots<\pi_{i+j}} \operatorname{sgn}\pi\,\alpha(x_{\pi_1}\cdots x_{\pi_i})\,\beta(x_{\pi_{i+1}}\cdots x_{\pi_{i+j}}).$$
In our case, $H^*(\mathfrak{nil}_n;\mathbb{Q})$ is spanned by the duals $\{x_\pi := e^*_\pi;\ \pi\in S_n\}$. Furthermore, $(e^*_\pi\smile e^*_{\pi'})(e_M) = \begin{cases}1; & \text{if } e_M=e_\pi e_{\pi'}\\ 0; & \text{if } e_M\neq e_\pi e_{\pi'}\end{cases}$, hence $e^*_\pi\smile e^*_{\pi'} = (e_\pi\wedge e_{\pi'})^*$. □

3. Filtrations

Let $w=(2,w_2,\ldots,w_n)$, so every wedge in $[\![w]\!]$ contains exactly one e_{1*}. There is a natural filtration of $[\![w]\!]$ by subcomplexes: if F^w_k is spanned by $\{e_{1i}e_M; i\geq k\}$, then $0=F^w_{n+1}\leq F^w_n\leq\ldots\leq F^w_2=[\![w]\!]$. The quotient F^w_k/F^w_{k+1} has generators $\{[e_{1k}e_M];\ e_{1k}e_M\in F^w_k\}$ and boundary $\partial[e_{1k}e_M]=-[e_{1k}\,\partial e_M]$. Therefore
$$\begin{aligned}F^w_k/F^w_{k+1} &\cong [\![1,w_2,\ldots,w_{k-1},w_k{+}1,w_{k+1},\ldots,w_n]\!]_1\\ &\cong [\![w_2{-}1,\ldots,w_{k-1}{-}1,w_k,w_{k+1}{-}1,\ldots,w_n{-}1]\!]_1.\end{aligned}\tag{3.1}$$

Lemma 3.1. *If $w_r=w_s=w_t\in\{2,n{-}1\}$ for distinct r,s,t, then $H_*[\![w_1,\ldots,w_n]\!]\cong 0$.*

Proof. By Lemmas 2.3 and 2.2, we may assume that $r=1$ and $w_r=w_s=w_t=2$. For any $i\notin\{s,t\}$ there holds $F^w_i/F^w_{i+1}\cong[\![w_2{-}1,\ldots,1,\ldots,w_i,\ldots,1,\ldots,w_n{-}1]\!]_1\cong[\![w_2{-}2,\ldots,0,\ldots,w_i{-}1,\ldots,w_n{-}2]\!]_t\cong 0$, by (3.1) and Lemma 2.4. Thus we have $0=F^w_n=\ldots=F^w_{t+1}<F^w_t=\ldots=F^w_{s+1}<F^w_s=\ldots=[\![w]\!]$ and a long exact sequence of a pair $\ldots\to H_{k+1}\frac{[\![w]\!]}{F^w_t}\overset{\chi}{\to}H_kF^w_t\to H_k[\![w]\!]\to H_k\frac{[\![w]\!]}{F^w_t}\overset{\chi}{\to}H_{k-1}F^w_t\to\ldots$. To prove $H_*[\![w]\!]\cong 0$, it suffices to show that χ is an isomorphism, where $\chi(x{+}F^w_t)=[\partial(x)]$.

Let $x\in[\![w]\!]/F^w_t=F^w_s/F^w_{s+1}$, so $x=e_{1s}\cdots$. By $w_t=2$, $x=e_{1s}e_{\{2,\ldots,s,\ldots,t-1\}\times\{t\}}\cdots$. By $w_s=2$, $x=[e_{1s}e_{\{2,\ldots,s-1\}\times\{s\}}e_{\{2,\ldots,s,\ldots,t-1\}\times\{t\}}e_M]$ with no indices s and t in M.

Let $y\in F^w_t=F^w_t/F^w_{t+1}$, so $y=e_{1t}\cdots$. Since $w_s=2$, $y=e_{1t}e_{\{2,\ldots,s-1\}\times\{s\}}\cdots$. Since $w_t=2$, $y=e_{1t}e_{\{2,\ldots,s-1\}\times\{s\}}e_{\{2,\ldots,\hat{s},\ldots,t-1\}\times\{t\}}e_M$ with no indices s and t in M.

Since $H_k\frac{[\![w]\!]}{F^w_t}=\frac{\operatorname{Ker}\partial}{\operatorname{Im}\partial}$, its elements are sent by ∂ to F^w_t, so in x the only multiplication is $[e_{1s},e_{st}]=e_{1t}$. Thus χ sends $x\mapsto y$ and it is bijective. □

Lemma 3.2. $[\![\ldots,2,2,\ldots]\!]\simeq 0$ *and* $[\![\ldots,n{-}1,n{-}1,\ldots]\!]\simeq 0$.

Proof. By Lemmas 2.3 and 2.2, it suffices to show that $[\![w]\!]:=[\![2,2,w_3,\ldots,w_n]\!]\simeq 0$. Now $[\![w]\!]$ consists of $e_{1i}e_M$ and $e_{12}e_{2i}e_M$, where $i\geq 3$ and 2 is not an index in M. Hence $\mathcal{M}=\{e_{12}e_{2i}e_M\to e_{1i}e_M;\ e_{1i}e_M\in[\![w]\!]\}$ is a Morse matching with $\mathring{\mathcal{M}}=\emptyset$. □

Lemma 3.3. *Let $w=(2,w_2,w_3,\ldots,w_n)$ and $w'=(2,w_3,\ldots,w_n,w_2)$. Then $F^w_3\cong[\![w']\!]_{2w_2-n-2}/F^{w'}_n$. If $H_*[\![w_2,w_3{-}1,\ldots,w_n{-}1]\!]\cong 0$, then $H_*[\![w]\!]\cong H_*[\![w']\!]_{2w_2-n-2}$.*

Proof. Define a linear map $\varphi\colon F^w_3\to[\![w']\!]_{2w_2-n-2}/F^{w'}_n$ by
$$\varphi\big(e_{1b}\wedge_{i=2}^{n-1}e_{\{i\}\times M_i}\big) = (-1)^{\Sigma M_2}[e_{1b}\,e_{M_2^C\times\{n+1\}}\wedge_{i=3}^{n-1}e_{\{i\}\times M_i}],$$
where $M_2^C=\{3,\ldots,n\}\setminus M_2$ and indices in the codomain are $1,3,\ldots,n{+}1$. Our φ is a bijection. The proof that it is a chain map is similar to the one in Lemma 2.3.

Let $H_*[\![w_2, w_3-1, \dots, w_n-1]\!] \cong 0$, which by (3.1) is $H_*(F_2^w/F_3^w)$. By the long exact sequence and first part, $H_*[\![w]\!] = H_* F_2^w \cong H_* F_3^w \cong H_*([\![w']\!]_{2w_2-n-2}/F_n^{w'})$. By (3.1), $H_* F_n^{w'} \cong H_*[\![w_3-1, \dots, w_n-1, w_2]\!] \cong H_*[\![w_2, w_3-1, \dots, w_n-1]\!]_{n-2w_2} \cong 0$, so by the long exact sequence, $H_*([\![w']\!]/F_n^{w'}) \cong H_*[\![w']\!]$ and the result follows. □

Recall that any chain map $\varphi\colon B_* \to C_*$ induces a chain complex $D_* = \operatorname{Cone}\varphi$, where $D_n = B_{n-1} \oplus C_n$ and $\partial(b,c) = \big(\partial(b), \varphi(b) - \partial(c)\big)$. Furthermore, there is an exact sequence $\ldots \to H_{k+1}D_* \to H_k B_* \overset{\varphi_*}{\to} H_k C_* \to H_k D_* \to H_{k-1} B_* \overset{\varphi_*}{\to} \ldots$.

Lemma 3.4. *Let* $w = (2, w_2, \dots, w_k, 3, 3, w_{k+3}, \dots, w_n)$ *and* $w' = (w_2-2, \dots, w_k-2, 3, w_{k+3}-2, \dots, w_n-2)$. *Then* $H_*[\![w]\!] \cong H_*\operatorname{Cone}\big([\![w']\!]_k \xrightarrow{\cdot 2} [\![w']\!]_k\big)$.

Proof. Let $k=1$, so $w = (2,3,3,\dots)$. By (3.1) and Lemma 3.2, $F_i^w/F_{i+1}^w \simeq 0$ for $i \geq 4$, so $H_* F_4^w \cong 0$ and $H_*[\![w]\!] \cong H_*([\![w]\!]/F_4^w)$. There are 4 types of generators in $[\![w]\!]/F_4^w$:

- $A = \{[e_{12}e_{23}e_{2a}e_{3b}e_M]$; all indices in M are $\geq 4\}$,
- $B = \{[e_{13}e_{23}e_{3a}e_{3b}e_M]$; all indices in M are $\geq 4\}$,
- $C = \{[e_{12}e_{2a}e_{2b}e_M]$; all indices in M are $\geq 4\}$,
- $D = \{[e_{13}e_{2a}e_{3b}e_M]$; all indices in M are $\geq 4\}$.

The set $\mathcal{M} = \{A \ni e_{12}e_{23}e_{2a}e_M \to e_{13}e_{2a}e_M \in D\}$ is a Morse matching, with critical elements $\mathring{\mathcal{M}} = B \cup C$. Its nontrivial zig-zag paths go from B to C and come in pairs:

$$\begin{array}{ccc} [e_{13}e_{23}e_{3a}e_{3b}e_M] \xrightarrow{\;1\;} [e_{13}e_{2a}e_{3b}e_M] & & [e_{13}e_{23}e_{3a}e_{3b}e_M] \xrightarrow{\;-1\;} [e_{13}e_{2b}e_{3a}e_M] \\ \qquad\qquad\qquad {\swarrow}{\scriptstyle -1} & & \qquad\qquad\qquad {\swarrow}{\scriptstyle -1} \\ [e_{12}e_{23}e_{2a}e_{3b}e_M] \xrightarrow{\;1\;} [e_{12}e_{2a}e_{2b}e_M] & \text{and} & [e_{12}e_{23}e_{2b}e_{3a}e_M] \xrightarrow{\;-1\;} [e_{12}e_{2a}e_{2b}e_M], \end{array}$$

which add up to $\cdot 2$. We have $\langle\mathring{\mathcal{M}}\rangle/\langle C\rangle \cong [\![w']\!]_2$ (omit $e_{13}e_{23}$ and indices $1,2$) and $\langle C\rangle \cong [\![w']\!]_1$ (omit e_{12} and indices $1,3$), so $H_*[\![w]\!] \cong H_*\langle\mathring{\mathcal{M}}\rangle \cong H_*\operatorname{Cone}\big([\![w']\!]_1 \overset{\cdot 2}{\to} [\![w']\!]_1\big)$.

Finally, if $k \geq 2$, then $H_*[\![w]\!] \cong H_*[\![2, 3, 3, w_{k+3}, \dots, w_n, w_2, \dots, w_k]\!]_{\sum_{i=2}^k (2w_i-n-2)}$ $\cong H_* \operatorname{Cone}\big(\cdot 2 \circlearrowright [\![3, w_{k+3}-2, \dots, w_n-2, w_2-2, \dots, w_k-2]\!]_{1+\sum_{i=2}^k (2w_i-n-2)}\big) \cong H_* \operatorname{Cone}$ $\big(\cdot 2 \circlearrowright [\![w_2-2, \dots, w_k-2, 3, w_{k+3}-2, \dots, w_n-2]\!]_{1+\sum_{i=2}^k(2w_i-n-2)-\sum_{i=2}^k(2(w_i-2)-(n-1))}\big) \cong$ $H_*\operatorname{Cone}\big([\![w']\!]_k \xrightarrow{\cdot 2} [\![w']\!]_k\big)$ by Lemmas 3.2, 3.3, 2.3, so the job is done. □

Lemma 3.5. *Let* $w = (2, w_2, \dots, w_{k-1}, 2, w_{k+1}, \dots, w_n)$.

(1) Let $A = \{e_{\{1,\dots,k-1\}\times\{k\}} e_{ka} e_M \in [\![w]\!]\}$ *and* $B = \{e_{1a} e_{\{2,\dots,k-1\}\times\{k\}} e_M \in [\![w]\!];\ a > k\}$.
There exists a Morse matching $\mathcal{M}$ *for* $[\![w]\!]$, *such that* $\mathring{\mathcal{M}} = A \cup B$, $\mathring{\partial}|_B = \partial|_B$,

$$\mathring{\partial}|_A\colon e_{\{1,\dots,k-1\}\times\{k\}} e_{ka} e_M \mapsto (-1)^{k+1}\big(e_{\{1,\dots,k-1\}\times\{k\}}\partial(e_{ka}e_M) + n_M e_{1a} e_{\{2,\dots,k-1\}\times\{k\}} e_M\big) + \textstyle\sum_{(b,c)\in X} (-1)^{\epsilon_{bc}+k+1} e_{1c} e_{\{2,\dots,k-1\}\times\{k\}} e_{ba} e_{M\setminus\{(b,c)\}},$$

where $n_M = |\{b \in \{1, \dots, k-1\};\ (b,a) \notin M\}|$, $\epsilon_{bc} =$ *(position of* (b,c) *in* M*), and* $X = \{(b,c) \in M;\ b < k < c, (b,a) \notin M\}$.

(2) $H_*[\![w]\!] \cong H_* \operatorname{Cone}\varphi$ *for some chain map* $\varphi\colon F_{k+1}^w \to F_{k+1}^w$.

(3) $H_*[\![2, w_2, \dots, w_{n-2}, 2, w_n]\!] \cong H_* \operatorname{Cone}\big(\cdot(w_n-1) \circlearrowright [\![w_2-1, \dots, w_{n-2}-1, 1, w_n]\!]_1\big)$.

Proof. (1): There are four types of generators in $[\![w]\!]$, namely A, B,
$C = \{e_{1a} e_{\{2,\dots,k-1\}\times\{k\}} e_M;\ a < k,$ there is no index 1 or k in $M\}$,
$D = \{e_{\{1,\dots,\hat{a},\dots,k-1\}\times\{k\}} e_M;\ 1 < a < k,$ there is no index 1 or k in $M\}$.

The set $\mathcal{M} = \big\{c \to d;\ c \in C, d \in D\big\}$ is a Morse matching, with $\mathring{\mathcal{M}} = A \cup B$. The zig-zag paths that start in B are arrows and end in B, so $\mathring{\partial}|_B = \partial|_B$. The zig-zag paths that start in A are $e_{\{1,\dots,k-1\}\times\{k\}} e_{ka} e_M \xrightarrow{(-1)^{k+1}} e_{1a} e_{\{2,\dots,k-1\}\times\{k\}} e_M$ and

$$e_{\{1,\dots,k-1\}\times\{k\}}e_{ka}e_M \xrightarrow{(-1)^b} e_{\{1,\dots,\hat{b},\dots,k-1\}\times\{k\}}e_{ba}e_M$$

$$e_{1b}e_{\{2,\dots,k-1\}\times\{k\}}e_{ba}e_M \xrightarrow{(-1)^{b+1}} e_{\{1,\dots,\hat{b},\dots,k-1\}\times\{k\}}e_{ba}e_M$$

$$e_{1b}e_{\{2,\dots,k-1\}\times\{k\}}e_{ba}e_M \xrightarrow{(-1)^{k+1}} e_{1a}e_{\{2,\dots,k-1\}\times\{k\}}e_M \quad \text{for } n_M \text{ choices,}$$

$$e_{1b}e_{\{2,\dots,k-1\}\times\{k\}}e_{ba}e_M \xrightarrow{(-1)^{\epsilon_{bc}+k+1}} e_{1c}e_{\{2,\dots,k-1\}\times\{k\}}e_{ba}e_{M\setminus\{(b,c)\}} \quad \text{for } (b,c)\in X.$$

(2): Follows from *(1)*, because $\langle B\rangle = F^w_{k+1}$ and $\langle\mathring{\mathcal{M}}\rangle/\langle B\rangle \cong \langle B\rangle_1$ (we mod out B, so 2nd and 3rd summand in $\mathring{\partial}|_A$ are 0, thus $e_{\{1,\dots,k-1\}\times\{k\}}e_{ka}e_M \mapsto e_{1a}e_{\{2,\dots,k-1\}\times\{k\}}e_M$ is a chain isomorphism). Ergo φ is the part of $\mathring{\partial}|_A$ that goes to B.
(3): Follows from *(2)*, since $F^w_{k+1} = F^w_n \cong [\![w']\!]_1$ (omit e_{1n} and index 1) and $X=\emptyset$ and $n_M = |\{b\in\{1,\dots,n-2\};\ (b,n)\notin M\}| = n-1-(n-w_n) = w_n-1$. □

Dwyer [1] reports how Kunkel proved that $H_*\mathfrak{nil}_n$ has p-torsion for prime $p<n-1$. Now we can easily see that $H_*\mathfrak{nil}_n$ also has p^m-torsion for every $p^m<n-1$, which proves Conjecture 1.16.(1) in [4, p.141] due to Jöllenbeck:

Example 3.6. Let $q=p^m=n-2$ and $w=(2,3,\dots,q+1,2,q+1)$. By Lemma 3.5, $H_*[\![w]\!] \cong H_*\,\mathrm{Cone}([\![w']\!]_1 \xrightarrow{\cdot q} [\![w']\!]_1)$. Since $w'=(2,\dots,q,1,q+1)$ is a permutation of $(1,\dots,q+1)$ and $|\{(i,j);\ i<j, w'_i>w'_j\}| = q-1$, we have $H_k[\![w']\!] \cong \begin{cases}\mathbb{Z}; & \text{if } k=q-1\\ 0; & \text{if } k\neq q-1\end{cases}$, so $H_k[\![w]\!] \cong \begin{cases}\mathbb{Z}_q; & \text{if } k=q\\ 0\ ; & \text{if } k\neq q\end{cases}$. If $q<n-2$, then $H_*[\![w,q+3,\dots,n]\!] \cong H_*[\![w]\!]$. ◊

In [1], Dwyer proved that there is no p-torsion in $H_*\mathfrak{nil}_n$ for any prime $p\geq n-1$. The next example shows that $H_*\mathfrak{nil}_n$ can have p^m-torsion for some $p^m\geq n-1$:

Example 3.7. Let $w=(2{,}4{,}7{,}5{,}4{,}2{,}5{,}7)$. Its complex $[\![w]\!]$ is spanned by 192 wedges. Using Lemma 3.5, $H_*[\![w]\!] \cong H_*\mathrm{Cone}(\varphi\circlearrowright F^w_7)$. By (3.1), $F^w_8 \cong [\![3{,}6{,}4{,}3{,}1{,}4{,}7]\!]_1 \cong [\![3{,}6{,}4{,}3{,}1{,}4]\!]_1 \cong [\![2{,}5{,}3{,}2{,}3]\!]_5 \cong [\![2{,}3{,}2{,}3]\!]_8$, so $H_{10}F^w_8 \cong \mathbb{Z}_2$ is generated by $[e_{18}e_M =: a]$ for $M=\{(2,7),(4,5)\}\cup\{(i,6); 2\leq i\leq 5\}\cup\{(3,i); i=4,5,7\}$. By Lemmas 2.4, 2.2, 3.5,

$$\begin{aligned} H_*F^w_7/F^w_8 &\cong H_*[\![3{,}6{,}4{,}3{,}1{,}5{,}6]\!]_1 \cong H_*[\![2{,}5{,}3{,}2{,}4{,}5]\!]_5 \cong \\ &\cong H_*[\![2{,}3{,}5{,}4{,}2{,}5]\!]_5 \cong H_*\mathrm{Cone}\big(\cdot 4\circlearrowright [\![2{,}4{,}3{,}1{,}5]\!]_6\big). \end{aligned}$$

Since $H_*[\![2{,}4{,}3{,}1{,}5]\!] \cong \mathbb{Z}$ is generated by $[e_{14}e_{23}e_{24}e_{34}]$, we have $H_*[\![2{,}3{,}5{,}4{,}2{,}5]\!] \cong \mathbb{Z}_4$ is generated by $[e_{16}e_N]$ where $N=\{(2,3),(2,4),(2,5),(3,4)\}$. Then $H_5[\![2{,}5{,}3{,}2{,}4{,}5]\!]$ is generated by $[e_{16}e_{N'}]$ for $N'=\{(7-y,7-x); (x,y)\in N\}$, and $H_{10}F^w_7/F^w_8$ is generated by $[e_{17}e_{28}e_P =: b]$ for $P=\{(3{,}4),(3{,}5),(3{,}7),(4{,}5)\}\cup\{(i{,}6); i=2,\dots,5\}$.

The exact sequence $\ldots \to H_{k+1}\frac{F^w_7}{F^w_8} \xrightarrow{\chi} H_kF^w_8 \xrightarrow{\iota_*} H_kF^w_7 \xrightarrow{\pi_*} H_k\frac{F^w_7}{F^w_8} \xrightarrow{\chi} H_{k-1}F^w_8 \to \ldots$ implies $H_kF^w_7\cong 0$ for $k\neq 10$ and $H_{10}F^w_7 = (\text{extension of } \mathbb{Z}_2 \text{ by } \mathbb{Z}_4) \cong (\mathbb{Z}_2\times\mathbb{Z}_4 \text{ or } \mathbb{Z}_8)$ is generated by $[a]$ and $[b]$. Since $\partial\big(\sum_{i\in\{3,4,5,7\}} e_{17}e_{2i}e_{i8}e_P\big) = 4e_{17}e_{28}e_P + e_{18}e_M$ $(*)$ in F^w_7, we have $0=4[b]+[a]$ in $H_{10}F^w_7$, hence $[b]$ itself is a generator and $H_{10}F^w_7\cong\mathbb{Z}_8$.

Let us compute φ_*. In the proof of Lemma 3.5 for our case, $a,b\in\langle B\rangle$ and $x := e_{16}e_{67}e_{28}e_P = e_{\{1,\dots,5\}\times\{6\}}e_{67}e_R \in \langle A\rangle \cong \langle B\rangle_1$ for $R=\{(2{,}8),(3{,}4),(3{,}5),(3{,}7),(4{,}5)\}$;

$$\varphi(x) = \mathring{\partial}(x) = \pm n_R e_{17}e_{\{2,\dots,5\}\times\{6\}}e_R \pm e_{18}e_{\{2,\dots,5\}\times\{6\}}e_{27}e_{R\setminus\{(2,8)\}} = \pm 4b \pm a.$$

By $(*)$, $\varphi_*(x)$ is 0 or $\pm 8[b]$. In both cases, $\varphi_*=0$. The exactness of $\ldots\to H_{k+1}[\![w]\!]\to H_kF^w_7 \xrightarrow{\varphi_*} H_kF^w_7 \to H_k[\![w]\!] \to H_{k-1}F^w_7 \xrightarrow{\varphi_*} \ldots$ then implies $H_k[\![w]\!] \cong \begin{cases}\mathbb{Z}_8; & \text{if } k\in\{10,11\}\\ 0\ ; & \text{if } k\notin\{10,11\}\end{cases}$. ◊

4. Free part of homology

We can generalise the filtration from the previous section to an arbitrary complex $[\![w_1,\ldots,w_n]\!]=[\![w]\!]$. In every $e_M\in[\![w]\!]$, there are exactly $t:=w_1-1$ occurrences of e_{1*}. Thus $F^w_k:=\langle e_{1i_1}\ldots e_{1i_t}e_M;\ i_1+\ldots+i_t\geq k,\ 1\text{ is not in }M\rangle\leq[\![w]\!]$ is a subcomplex.

Define $w(i_1,\ldots,i_t)\in\mathcal{S}_{n-1}$ as $(w'_2,\ldots,w'_n)$, where $w'_j=\left\{\begin{smallmatrix}w_j & ; & \text{if } j\in\{i_1,\ldots,i_t\}\\ w_j-1; & & \text{if } j\notin\{i_1,\ldots,i_t\}\end{smallmatrix}\right.$. Then

$$F^w_k/F^w_{k+1}\cong\textstyle\bigoplus_{i_1+\ldots+i_t=k}[\![w(i_1,\ldots,i_t)]\!]_t. \tag{4.1}$$

Example 4.1. Let us compute $H_*[\![3,3,3,3,3]\!]$. By Lemmas 3.2, 3.5, (4.1), $F^w_{10}=0$, $F^w_9/F^w_{10}\cong[\![2,2,3,3]\!]_2\simeq 0$, $H_*F^w_8/F^w_9\cong H_*[\![2,3,2,3]\!]_2\cong\mathbb{Z}_2$ is generated by $[e_{13}e_{15}\,e_{24}e_{35}]$, $F^w_7/F^w_8\cong[\![3,2,2,3]\!]_2\oplus[\![2,3,3,2]\!]_2\simeq 0$, $H_*F^w_6/F^w_7\cong H_*[\![3,2,3,2]\!]_2\cong H_*[\![2,3,2,3]\!]_3\cong\mathbb{Z}_2$ is generated by $[e_{12}e_{14}\,e_{23}e_{25}e_{35}]$, $F^w_5/F^w_6\cong[\![3,3,2,2]\!]_2\simeq 0$, $F^w_5=[\![w]\!]$. Thus $H_*[\![w]\!]\cong H_*F_6$ and $H_*F^w_7\cong H_*F^w_8\cong H_*F^w_8/F^w_9\cong\mathbb{Z}_2$ is generated by $[e_{13}e_{15}\,e_{24}e_{35}]$. In the exact sequence $\ldots\to H_{k+1}\frac{F^w_6}{F^w_7}\xrightarrow{\chi}H_kF^w_7\to H_kF^w_6\to H_k\frac{F^w_6}{F^w_7}\xrightarrow{\chi}H_{k-1}F^w_7\to\ldots$, our χ sends

$$\begin{aligned}[e_{12}e_{14}\,e_{23}e_{25}e_{35}]\mapsto{}&[e_{13}e_{14}e_{25}e_{35}+e_{14}e_{15}e_{23}e_{35}]=\\&[-e_{13}e_{15}e_{24}e_{35}-\partial(e_{13}e_{15}e_{23}e_{34}e_{35})+\partial(e_{13}e_{14}e_{25}e_{34}e_{45})].\end{aligned}$$

It is an isomorphism, hence by exactness, $H_*[\![3,3,3,3,3]\!]\cong H_*F^w_6\cong 0$. ♢

Lemma 4.2. *For $w\in\mathcal{S}_n\setminus\mathcal{F}_n=:\mathcal{T}_n$, the homology of $[\![w]\!]$ has only torsion.*

Proof. We use induction on n. Computing over $\mathbb{Q}$, by universal coefficient theorem it suffices to show that $H_*[\![w]\!]\cong 0$. The claim is trivial for $n=2$ since $\mathcal{T}_2=\emptyset$. Let $n>2$ and $w\in\mathcal{T}_n$. By Lemma 2.4 and induction hypothesis, we may assume that $2\leq w_1,\ldots,w_n\leq n-1$. By Lemma 2.3 we may assume that $w_1\leq w_i$ for all i. If $w_1\geq 3$, all elements of the sequence $w(i_1,\ldots,i_{w_1-1})$ are at least 2, so $w(i_1,\ldots,i_{w_1-1})\in\mathcal{T}_{n-1}$. By (4.1) and induction, $H_*F^w_k/F^w_{k+1}\cong 0$ for all k, hence $H_*[\![w]\!]\cong 0$.

Let $w_1=2$. If $w(i)\in\mathcal{T}_{n-1}$ for all i, then by (4.1) and induction, $H_*F^w_i/F^w_{i+1}\cong 0$ for all i, hence $H_*[\![w]\!]\cong 0$. Suppose there exists $i\in\{2,\ldots,n\}$ with $w(i)\notin\mathcal{T}_{n-1}$. Then $w_i=n-1$, $w_1=2$, and other elements of w form a permutation of $(2,\ldots,n-1)$, so there are exactly two numbers $j<i$ such that $w(j),w(i)\in\mathcal{F}_{n-1}$. In the filtration $0=F^w_{n+1}\leq\ldots\leq F^w_{i+1}\leq F^w_i\leq\ldots\leq F^w_{j+1}\leq F^w_j\leq\ldots\leq F^w_2=[\![w]\!]$, by (3.1) and (2.1) we have $H_kF^w_i/F^w_{i+1}\cong\left\{\begin{smallmatrix}\mathbb{Q}; & \text{if } k=1+|I_i|\\ 0\,; & \text{if } k\neq 1+|I_i|\end{smallmatrix}\right.$, $H_kF^w_j/F^w_{j+1}\cong\left\{\begin{smallmatrix}\mathbb{Q}; & \text{if } k=1+|I_j|\\ 0\,; & \text{if } k\neq 1+|I_j|\end{smallmatrix}\right.$, and $H_*F^w_r/F^w_{r+1}\cong 0$ for $r\notin\{i,j\}$, where $I_i=\{\text{inversions of } w(i)\}$ and $I_j=\{\text{inversions of } w(j)\}$. Therefore $0\cong H_*F_{n+1}\cong\ldots\cong H_*F_{i+1}$, $H_*F_i\cong\left\{\begin{smallmatrix}\mathbb{Q}; & \text{if } k=1+|I_i|\\ 0\,; & \text{if } k\neq 1+|I_i|\end{smallmatrix}\right.$, $H_*F^w_i\cong\ldots\cong H_*F^w_{j+1}$, $H_*F^w_j\cong\ldots\cong H_*[\![w]\!]$. Consequently, it suffices to show that in the exact sequence

$$\begin{array}{ccccccccccccccc}\ldots\to & H_{k+2}\frac{F^w_j}{F^w_{j+1}} & \xrightarrow{\chi} & H_{k+1}F^w_{j+1} & \to H_{k+1}F^w_j\to & H_{k+1}\frac{F^w_j}{F^w_{j+1}} & \xrightarrow{\chi} & H_kF^w_{j+1} & \to H_kF^w_j\to & H_k\frac{F^w_j}{F^w_{j+1}} & \xrightarrow{\chi} & H_{k-1}F^w_{j+1} & \to\ldots\\ & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & \\ & 0 & & 0 & & \mathbb{Q} & & \mathbb{Q} & & 0 & & 0 & \end{array}$$

the morphism χ_{k+1} for $k=|I_j|$ is bijective. Notice that $H_{k+1}\frac{F^w_j}{F^w_{j+1}}$ and $H_kF^w_{j+1}\cong H_k\frac{F^w_i}{F^w_{i+1}}$ are generated by $[e_{1j}\bigwedge_{(a,b)\in I_j}e_{a+1,b+1}]=:[x]$ and $[e_{1i}\bigwedge_{(a,b)\in I_i}e_{a+1,b+1}]=:[y]$. Since $I_j=I_i\cup\{(j-1,i-1)\}$, we get $\chi([x])=[\partial(x)]=[\pm y]$, which concludes the argument. □

Lemma 4.3. $H_k[\![w_n,\ldots,w_1]\!]\cong H_{\binom{n}{2}-k-1}[\![w_1,\ldots,w_n]\!]$ *for* $(w_1,\ldots,w_n)\in\mathcal{T}_n$.

Proof. Let (C^*,δ^*) be the dual of the complex (C_*,∂_*) of $\mathfrak{nil}_n$, let f_M be the dual of a basis element e_M, and $N=\binom{n}{2}$. Define $\tau_*: C_*\to C^{N-*}$ by $\tau(e_M)=\varepsilon_M f_{M^C}$, where

ε_M is the sign of the permutation (M, M^C) of $\{(i,j);\, 1\le i<j\le n\}$. By [3, p.640], τ_* is a chain isomorphism, i.e. $\tau_{k-1}\partial_k = \delta^{N-k}\tau_k$. For $e_M\in[\![w_1,\ldots,w_n]\!]$, we have

$$\begin{aligned} w_i\big(\tau(e_M)\big) &= i-|\{x;(x,i)\in M^C\}|+|\{x;(i,x)\in M^C\}| \\ &= i-\big(i-1-|\{x;(x,i)\in M\}|\big)+\big(n-i-|\{x;(i,x)\in M\}|\big) = n+1-w_i, \end{aligned}$$

hence $\tau([\![w_1,\ldots,w_n]\!])=[\![n+1-w_1,\ldots,n+1-w_n]\!]^*\cong[\![w_n,\ldots,w_1]\!]^*$ by Lemma 2.2. Now the result follows from Lemma 4.2 and the universal coefficient theorem. □

5. Computations

We have $H_*\mathfrak{nil}_n \cong \big(\bigoplus_{w\in\mathcal{F}_n} H_*[\![w]\!]\big)\oplus\big(\bigoplus_{w\in\mathcal{T}_n} H_*[\![w]\!]\big)$. The free part $\bigoplus_{w\in\mathcal{F}_n} H_*[\![w]\!]$ is known from Theorem 2.5. For the torsion part, we use Lemma 2.4:

$$\begin{aligned} TH_*(\mathfrak{nil}_n) &\cong \big(\oplus_{1,n\notin w\in\mathcal{T}_n} H_*[\![w]\!]\big) \oplus \frac{\big(\oplus_{1\in w\in\mathcal{T}_n} H_*[\![w]\!]\big)\oplus\big(\oplus_{n\in w\in\mathcal{T}_n} H_*[\![w]\!]\big)}{\oplus_{1,n\in w\in\mathcal{T}_n} H_*[\![w]\!]} \\ &\cong \big(\oplus_{1,n\notin w\in\mathcal{T}_n} H_*[\![w]\!]\big) \oplus \frac{\oplus_{w\in\mathcal{T}_{n-1}}\oplus_{k=1}^{n} H_*[\![w]\!]_{k-1}\oplus H_*[\![w]\!]_{n-k}}{\oplus_{w\in\mathcal{T}_{n-2}}\oplus_{i\in[n-1],j\in[n]} H_*[\![w]\!]_{i-1+n-j}} \\ &\cong \big(\oplus_{1,n\notin w\in\mathcal{T}_n} H_*[\![w]\!]\big) \oplus \frac{\oplus_{k=0}^{n-1} TH_{*+k}(\mathfrak{nil}_{n-1})^2}{\oplus_{i=0}^{2n-3} TH_{*+i}(\mathfrak{nil}_{n-2})^{\min\{i+1,2n-2-i\}}}. \end{aligned}$$

By induction and Lemmas 3.1, 3.2, it suffices to calculate only $H_*[\![w]\!]$ coming from $\widetilde{\mathcal{T}}_n := \big\{w\in\mathcal{T}_n;\, 1,n\notin w,\, \nexists i\colon w_i=w_{i+1}\in\{2,n-1\},\, \nexists i<j<k\colon w_i=w_j=w_k\in\{2,n-1\}\big\}$. Define maps $\alpha,\beta,\gamma\colon \widetilde{\mathcal{T}}_n\to\widetilde{\mathcal{T}}_n$ by $\alpha(w_1,\ldots,w_n)=(w_2,\ldots,w_n,w_1)$, $\beta(w_1,\ldots,w_n)=(n+1-w_n,\ldots,n+1-w_1)$, $\gamma(w_1,\ldots,w_n)=(w_n,\ldots,w_1)$. Let $\sim$ be the smallest equivalence relation on $\widetilde{\mathcal{T}}_n$ with $w\sim\xi(w)$ for $\xi\in\{\alpha,\beta,\gamma\}$. By Lemmas 2.3, 2.2, 4.3, we need to compute H_* only for one complex in each equivalence class.

<u>Case $n=4$</u>: The set $\mathcal{S}_4$ consists of all permutations of (1,1,4,4), (1,2,3,4), (1,3,3,3), (2,2,2,4), (2,2,3,3), whilst $\widetilde{\mathcal{T}}_4/_\sim = \{(2,3,2,3)\}$. Now $[\![3,2,3,2]\!]\cong[\![2,3,2,3]\!]_1$ has only 4 generators, so H_* can be computed directly, but let us use Lemma 3.5: $H_*[\![2,3,2,3]\!]\cong H_*\,\mathrm{Cone}([\![2,1,3]\!]_1 \overset{\cdot 2}{\to} [\![2,1,3]\!]_1)$. Since $[\![2,1,3]\!]=\langle e_{12}\rangle$, we conclude that $H_k[\![2,3,2,3]\!]\cong\begin{cases}\mathbb{Z}_2; & \text{if } k=2\\ 0\ ; & \text{if } k\neq 2\end{cases}$. Hence the torsion part is $TH_k(\mathfrak{nil}_4)\cong\begin{cases}\mathbb{Z}_2; & \text{if } k\in\{2,3\}\\ 0\ ; & \text{if } k\notin\{2,3\}\end{cases}$.

<u>Case $n=5$</u>: Set $\widetilde{\mathcal{T}}_5/_\sim$ consists of $a=(2,3,4,2,4)$, $b=(2,3,3,3,4)$, $c=(2,3,3,4,3)$, $d=(3,3,3,3,3)$. By Lemma 3.5, $H_*[\![a]\!]\cong H_*\,\mathrm{Cone}([\![2,3,1,4]\!]_1\overset{\cdot 3}{\to}[\![2,3,1,4]\!]_1)\cong \begin{cases}\mathbb{Z}_3; & \text{if } k=3\\ 0\ ; & \text{if } k\neq 3\end{cases}$. By Lemma 3.4, $H_*[\![b]\!]\cong H_*\,\mathrm{Cone}([\![1,3,2]\!]_2\overset{\cdot 2}{\to}[\![1,3,2]\!]_2)\cong\begin{cases}\mathbb{Z}_2; & \text{if } k=3\\ 0\ ; & \text{if } k\neq 3\end{cases}$, and $H_*[\![c]\!]\cong H_*\,\mathrm{Cone}([\![3,2,1]\!]_1\overset{\cdot 2}{\to}[\![3,2,1]\!]_1)\cong\begin{cases}\mathbb{Z}_2; & \text{if } k=4\\ 0\ ; & \text{if } k\neq 4\end{cases}$. By Example 4.1, $H_*[\![d]\!]\cong 0$. Because $\beta(a)=(2,4,2,3,4)=\alpha^3(a)$, $\beta(b)=(2,3,3,3,4)=b$, $\beta(c)=(3,2,3,3,4)=\alpha^4(c)$, and $\gamma(x)\neq\alpha^i(x)$ for all $x\in\{a,b,c\}$ and all i, we conclude that

$$\bigoplus_{w\in\widetilde{\mathcal{T}}_n} H_k[\![w]\!] = \bigoplus_{x\in\{a,b,c\}}\ \bigoplus_{i\in\{0,\ldots,4\}} \big(H_k[\![\alpha^i(x)]\!]\oplus H_k[\![\gamma\alpha^i(x)]\!]\big) = \begin{cases}\mathbb{Z}_2^4\oplus\mathbb{Z}_3^2; & k=3,\\ \mathbb{Z}_2^6\oplus\mathbb{Z}_3^3; & k=4,\\ \mathbb{Z}_2^6\oplus\mathbb{Z}_3^3; & k=5,\\ \mathbb{Z}_2^4\oplus\mathbb{Z}_3^2; & k=6.\end{cases}$$

<u>Case $n=6$</u> is still doable by hand. Set $\widetilde{\mathcal{T}}_6/_\sim$ has 28 elements: 9 cases are done by Lemma 3.4, 6 by Lemma 3.5, and the rest by examining their filtration. There are only 3 classes containing no 2 or n–1: $(3,3,3,4,4,4)$, $(3,3,4,3,4,4)$, $(3,4,3,4,3,4)$.

<u>Cases $n=7,8$</u> require a computer. The set $\widetilde{\mathcal{T}}_7/_\sim$ has 250 elements, and $\widetilde{\mathcal{T}}_8/_\sim$ has 3485 elements. See the table below for the homology of $\mathfrak{nil}_7$ and $\mathfrak{nil}_8$.

<u>Cases $n\geq 9$</u>: The set $\widetilde{\mathcal{T}}_9/_\sim$ has 59 102 elements. We have not been able to compute, among other things, the homology of the complex $[\![5,5,5,5,5,5,5,5,5]\!]$.

These results are displayed in the two tables below (one for elementary divisors, the other for invariant factors). We see that $\mathbb{Z}_{120}$ appears 128 times in $H_*(\mathfrak{nil}_8)$, not $2\frac{8!}{4!}=3360$ times. This disproves Conjecture 1.16.(2) by Jöllenbeck in [4, p.141].

6. Afterword

6.1. **Conclusion.** Methods, designed for specific Lie algebras, where we partition the problem into smaller pieces and solve only the nontrivial nonequivalent parts, can enable us to compute more than twice as much, compared with usual techniques.

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$k\backslash n$	7	8
0	0^{1}	0^{1}
1	0^{6}	0^{7}
2	$0^{20}2^{4}$	$0^{27}2^{5}$
3	$0^{49}2^{35}3^{6}$	$0^{76}2^{57}3^{8}$
4	$0^{98}2^{124}3^{27}4^{6}$	$0^{174}2^{253}3^{45}4^{9}$
5	$0^{169}2^{303}3^{78}4^{28}5^{4}$	$0^{343}2^{793}3^{168}4^{53}5^{8}$
6	$0^{259}2^{635}3^{168}4^{65}5^{17}$	$0^{602}2^{2132}3^{479}4^{164}5^{47}$
7	$0^{359}2^{1122}3^{275}4^{112}5^{38}$	$0^{961}2^{4880}3^{1050}4^{380}5^{145}$
8	$0^{455}2^{1674}3^{384}4^{160}5^{56}$	$0^{1415}2^{9882}3^{1927}4^{730}5^{309}8^{1}$
9	$0^{531}2^{2096}3^{481}4^{196}5^{63}$	$0^{1940}2^{17721}3^{3178}4^{1200}5^{524}8^{5}$
10	$0^{573}2^{2238}3^{522}4^{210}5^{64}$	$0^{2493}2^{27826}3^{4781}4^{1728}5^{766}8^{12}$
11	$0^{573}2^{2096}3^{481}4^{196}5^{63}$	$0^{3017}2^{38810}3^{6504}4^{2253}5^{1007}8^{18}$
12	$0^{531}2^{1674}3^{384}4^{160}5^{56}$	$0^{3450}2^{48576}3^{7902}4^{2720}5^{1219}8^{17}$
13	$0^{455}2^{1122}3^{275}4^{112}5^{38}$	$0^{3736}2^{54457}3^{8614}4^{3011}5^{1351}8^{11}$
14	$0^{359}2^{635}3^{168}4^{65}5^{17}$	$0^{3836}2^{54457}3^{8614}4^{3011}5^{1351}8^{11}$
15	$0^{259}2^{303}3^{78}4^{28}5^{4}$	$0^{3736}2^{48576}3^{7902}4^{2720}5^{1219}8^{17}$
16	$0^{169}2^{124}3^{27}4^{6}$	$0^{3450}2^{38810}3^{6504}4^{2253}5^{1007}8^{18}$
17	$0^{98}2^{35}3^{6}$	$0^{3017}2^{27826}3^{4781}4^{1728}5^{766}8^{12}$
18	$0^{49}2^{4}$	$0^{2493}2^{17721}3^{3178}4^{1200}5^{524}8^{5}$
19	0^{20}	$0^{1940}2^{9882}3^{1927}4^{730}5^{309}8^{1}$
20	0^{6}	$0^{1415}2^{4880}3^{1050}4^{380}5^{145}$
21	0^{1}	$0^{961}2^{2132}3^{479}4^{164}5^{47}$
22		$0^{602}2^{793}3^{168}4^{53}5^{8}$
23		$0^{343}2^{253}3^{45}4^{9}$
24		$0^{174}2^{57}3^{8}$
25		$0^{76}2^{5}$
26		0^{27}
27		0^{7}
28		0^{1}

Table 2. Elementary divisor decomposition of $H_*(\mathfrak{nil}_n;\mathbb{Z})$

$k\backslash n$	4	5	6	7	8
2	2^1	2^2	2^3	2^4	2^5
3	2^1	$2^6 6^2$	$2^{16} 6^4$	$2^{29} 6^6$	$2^{49} 6^8$
4		$2^7 6^3$	$2^{37} 6^{10} 12^3$	$2^{103} 6^{21} 12^6$	$2^{217} 6^{36} 12^9$
5		$2^7 6^3$	$2^{62} 6^{17} 12^9$	$2^{253} 6^{50} 12^{24} 60^4$	$2^{678} 6^{115} 12^{45} 60^8$
6		$2^6 6^2$	$2^{95} 6^{23} 12^{12}$	$2^{532} 6^{103} 12^{48} 60^{17}$	$2^{1817} 6^{315} 12^{117} 60^{47}$
7		2^2	$2^{114} 6^{24} 12^{12}$	$2^{959} 6^{163} 12^{74} 60^{38}$	$2^{4210} 6^{670} 12^{235} 60^{145}$
8			$2^{95} 6^{23} 12^{12}$	$2^{1450} 6^{224} 12^{104} 60^{56}$	$2^{8686} 6^{1196} 12^{422} 60^{308} 120^1$
9			$2^{62} 6^{17} 12^9$	$2^{1811} 6^{285} 12^{133} 60^{63}$	$2^{15748} 6^{1973} 12^{681} 60^{519} 120^5$
10			$2^{37} 6^{10} 12^3$	$2^{1926} 6^{312} 12^{146} 60^{64}$	$2^{24785} 6^{3041} 12^{974} 60^{754} 120^{12}$
11			$2^{16} 6^4$	$2^{1811} 6^{285} 12^{133} 60^{63}$	$2^{34577} 6^{4233} 12^{1264} 60^{989} 120^{18}$
12			2^3	$2^{1450} 6^{224} 12^{104} 60^{56}$	$2^{43411} 6^{5165} 12^{1518} 60^{1202} 120^{17}$
13				$2^{959} 6^{163} 12^{74} 60^{38}$	$2^{48865} 6^{5592} 12^{1671} 60^{1340} 120^{11}$
14				$2^{532} 6^{103} 12^{48} 60^{17}$	$2^{48865} 6^{5592} 12^{1671} 60^{1340} 120^{11}$
15				$2^{253} 6^{50} 12^{24} 60^4$	$2^{43411} 6^{5165} 12^{1518} 60^{1202} 120^{17}$
16				$2^{103} 6^{21} 12^6$	$2^{34577} 6^{4233} 12^{1264} 60^{989} 120^{18}$
17				$2^{29} 6^6$	$2^{24785} 6^{3041} 12^{974} 60^{754} 120^{12}$
18				2^4	$2^{15748} 6^{1973} 12^{681} 60^{519} 120^5$
19					$2^{8686} 6^{1196} 12^{422} 60^{308} 120^1$
20					$2^{4210} 6^{670} 12^{235} 60^{145}$
21					$2^{1817} 6^{315} 12^{117} 60^{47}$
22					$2^{678} 6^{115} 12^{45} 60^8$
23					$2^{217} 6^{36} 12^9$
24					$2^{49} 6^8$
25					2^5

TABLE 3. Invariant factor decomposition of the torsion of $H_*(\mathfrak{nil}_n;\mathbb{Z})$

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