

## THE ALEXANDER POLYNOMIAL FOR CLOSED BRAIDS IN LENS SPACES

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ABSTRACT. We present a reduced Burau-like representation for the mixed braid group on one strand representing links in lens spaces and show how to calculate the Alexander polynomial of a link directly from the mixed braid.

### 1. INTRODUCTION

It is widely known that if a knot  $K$  is the closure of a braid  $\beta$ , an element of the Artin braid group  $B_n$ , the Alexander polynomial of  $K$  is given by the formula

$$(1) \quad \Delta_K(t) = \frac{1-t}{1-t^n} \det(I - \beta_*),$$

where  $\beta_*$  is the reduced Burau representation of  $\beta$ . In [18] Morton extended this formula and introduced a colored Burau representation of the braid group that enables us to compute the multivariable Alexander polynomial of a link by similar algebraic means.

On the other hand, we know by the Lickorish-Wallace theorem [17] that every closed, orientable, connected 3-manifold  $M$  can be obtained by performing Dehn surgery on a link in  $S^3$ . When studying links in  $M$ , we take a disjoint union of a surgery link, used to construct  $M$ , and the link in  $M$  itself, to obtain a so-called mixed link, see Figure 1(a) (see also [4, 7, 9, 8] and for alternative approaches [2, 19, 20, 21]). The corresponding mixed braid group [13, 16] enables us to represent a link in  $M$  as a closure of a mixed braid.

Just as the braid group plays an important role in (classical) knot theory in  $S^3$ , the mixed braid group plays an important role in the theory of knots and links in other 3-manifolds. An increasing number of topological and algebraical tools are being developed in the ongoing investigation of constructing and generalizing the classical knot invariant to those of knots in 3-manifolds (e.g. via Markov trace functions on the associated algebras, computations of skein modules, Chern-Simons theories, ...). In these studies lens spaces are of special interest, since, by the Lickorish-Wallace theorem, they play the role of constructing blocks of c.c.o. 3-manifolds.

It was recently shown in [10] how the Alexander polynomial of a link  $L$  in  $S^3$  changes when we think of  $L$  as a mixed link and perform rational surgery on some of its components. In particular, an explicit formula was given that computes the Alexander polynomial of a link inside a lens space directly from the mixed link diagram.

In this paper we introduce a Burau-like representation of the mixed braid group on one strand  $B_{1,n}$  [12, 14], which enables us to generalize Formula (1) to lens spaces, i.e. it allows us to compute the Alexander polynomial of a link in the lens space  $L(p, q)$  directly from the mixed braid group representative.

The paper is structured as follows. In Section 2, we recall the definition of the mixed braid group on one strand. In Section 3, we recall the definitions of the Alexander polynomial in  $S^3$  and the Alexander polynomial(s) in  $L(p, q)$ . In Section 4, we recall Morton's results, introduce the Burau-like representation for the mixed braid group on one strand, and state our main result (Theorem 4.2), the algebraic formula for computing the Alexander polynomial.

### 2. THE MIXED BRAID GROUP ON ONE STRAND

The lens space  $L(p, q)$  is the manifold obtained by performing Dehn surgery on the unknot  $\widehat{I}$  with surgery coefficients  $-p/q$ , where we assume  $0 < q < p$  are two coprime integers. Following [13, 4], we fix  $\widehat{I}$  pointwise and represent a link  $L$  in  $L(p, q)$  by the link  $\widehat{I} \cup L \subset S^3$ , which we call a *mixed link*, composed of the *fixed component*  $\widehat{I}$  and the *moving component*  $L$ . When appropriate, we emphasize that surgery

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Date: August 24, 2019.

2010 *Mathematics Subject Classification*. Primary: 57M27. Secondary: 20F36, 57M07.

*Key words and phrases*. Burau representation, Alexander polynomial, links in lens spaces, mixed braids, mixed braid group.

has been performed on the fixed component and denote the link  $L \subset L(p, q)$  as  $\widehat{I}^{-p/q} \cup L$ . Taking a regular projection of  $\widehat{I} \cup L$  to the plane of  $\widehat{I}$ , we obtain a *mixed link diagram*, as in Figure 1(a).

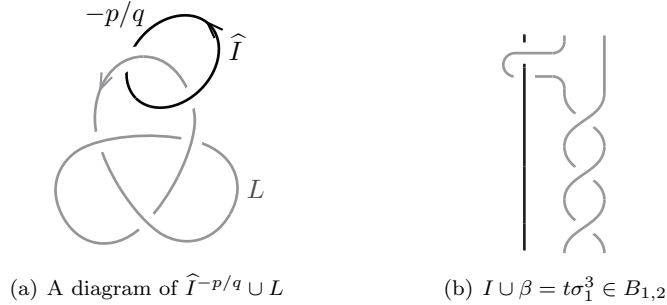


FIGURE 1. A mixed link diagram (a) and a mixed braid (b).

By the Alexander theorem, we can represent  $\widehat{I} \cup L$  as the closure of a braid  $I \cup \beta$  in the braid group  $B_{1+n}$ , where the strand  $I$ , called the *fixed strand*, belongs to  $\widehat{I}$ , while the  $n$  strands of  $\beta$  are called *moving strands* and belong to  $L$ . By the parting process described in [16] and [4], we can assume that the fixed strand begins and ends on the left, while all crossings belonging to the moving components are pushed to the right and may occasionally make a simple wind around the fixed strand as in Figure 1(b).

Fixing the vertical left strand  $I$ , we can form the *mixed braid group on one strand*  $B_{1,n}$ , a subgroup of  $B_{1+n}$ , with the following presentation [12]:

$$(2) \quad \begin{aligned} B_{1,n} = \langle t, \sigma_1, \dots, \sigma_{n-1} \mid & \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1, \\ & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i < n - 1, \\ & t \sigma_i = \sigma_i t \text{ for } i \geq 2, \\ & t \sigma_1 t \sigma_1 = \sigma_1 t \sigma_1 t \rangle, \end{aligned}$$

where the generators  $t$  and  $\sigma_i$  are illustrated in Figure 2.

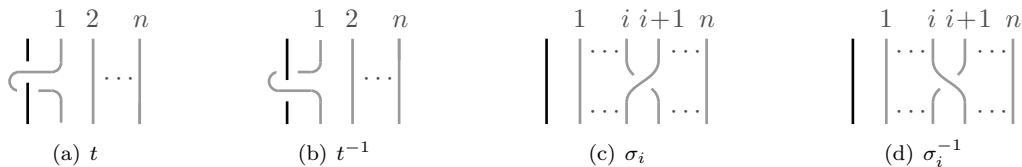


FIGURE 2. Generators of  $B_{1,n}$  and their inverses.

We advise the reader to see [16] for more details on this and more general constructions related to braiding mixed links.

### 3. THE ALEXANDER POLYNOMIAL FOR LINKS IN LENS SPACES

In this Section we describe a Torres-type formula [23], constructed in [10], which relates the two-variable Alexander polynomial of a mixed link in  $S^3$  to the corresponding Alexander polynomial of a link in  $L(p, q)$ .

We briefly recall the algebraic definition of the Alexander polynomial of a link in  $S^3$ , based on the Fox construction (see [24, 11], cf. [25]).

Given a group  $\pi$  with a finite presentation

$$\pi = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle,$$

denote by  $H = \pi/\pi'$  its abelianization and by  $F = \langle x_1, \dots, x_n \mid \rangle$  the corresponding free group. Apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}\pi \xrightarrow{\alpha} \mathbb{Z}H,$$

where  $\frac{\partial}{\partial x}$  denotes the Fox differential,  $\gamma$  is the quotient map by the relations  $r_1, \dots, r_m$  and  $\alpha$  is the abelianization map.

The *Alexander-Fox matrix* of the presentation of  $\pi$  is the matrix  $A = [\alpha(\gamma(\frac{\partial r_i}{\partial x_j}))]_{1 \leq i \leq m, 1 \leq j \leq n}$ . The *first elementary ideal*  $E_1(\pi)$  is the ideal of  $\mathbb{Z}H$ , generated by the determinants of all the  $(n-1)$  minors of  $A$ .

For a link  $L$  in  $S^3$ , the abelianization of  $\pi = \pi_1(S^3 \setminus L, *)$  is a free abelian group, whose generators correspond to the components of  $L$ . For a  $\nu$ -component link, we have  $\mathbb{Z}H \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_\nu^{\pm 1}]$ .

Let  $E_1(\pi)$  be the first elementary ideal, obtained from a presentation of  $\pi = \pi_1(S^3 \setminus L, *)$ . The *Alexander polynomial*  $\Delta_L(t_1^{\pm 1}, \dots, t_n^{\pm 1}) \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  of a link  $L$  is the generator of the smallest principal ideal containing  $E_1(\pi)$ .

We are only interested in distinguishing the variable, corresponding to the fixed component, from the variables, corresponding to the moving link components. In the above construction, we thus replace the map  $\alpha$  by the map  $\eta \circ \alpha$ , where  $\eta: \mathbb{Z}H \rightarrow \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$  is the canonical projection, defined by

$$\eta(t_i) = \begin{cases} s, & \text{if } t_i \text{ corresponds to the fixed component,} \\ t, & \text{if } t_i \text{ corresponds to a moving component.} \end{cases}$$

We obtain a two-variable Alexander polynomial  $\Delta_L(s, t)$ , which can be viewed as the Alexander polynomial of a link in the solid torus.

We are now ready to define the Alexander polynomial of a link  $L$  in  $L(p, q)$ . Given a mixed link  $\widehat{I} \cup L \subset S^3$ , the following proposition allows us to describe the link group of  $\widehat{I}^{-p/q} \cup L$  (the fundamental group of  $L(p, q) \setminus L$ ).

**Proposition 3.1** ([22]). *Let  $\pi_1(S^3 \setminus (\widehat{I} \cup L), *) = \langle x_1, \dots, x_n \mid w_1, \dots, w_n \rangle$  be the presentation of the link group of  $\widehat{I} \cup L$ . The presentation of the link group of  $\widehat{I}^{-p/q} \cup L$  is given by*

$$(3) \quad \pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n \mid w_1, \dots, w_n, m^p \cdot l^{-q} \rangle,$$

where  $m$  (resp.  $l$ ) denote the meridian (resp. longitude) of the regular neighbourhood of  $S^3 \setminus \widehat{I}$ .

The abelianization of the fundamental group of a link in  $L(p, q)$  may also contain torsion, see [10, Corollary 2.10]. In this case we need the notion of a twisted Alexander polynomial. We recall the following from [1] (see also [3]).

Let  $\pi$  be a finitely presented group and denote by  $H = \pi/\pi'$  its abelianization. Every homomorphism  $\sigma: Tors(H) \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$  determines a twisted Alexander polynomial  $\Delta^\sigma(\pi)$  as follows. Choose a splitting  $H = Tors(H) \times K$ , where  $K \cong H/Tors(H)$  is the free part of  $H$ . The map  $\sigma$  induces a ring homomorphism  $\sigma: \mathbb{Z}H \rightarrow \mathbb{C}[K]$  sending  $(f, g) \in Tors(H) \times K$  to  $\sigma(f)g$ . We apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}\pi \xrightarrow{\alpha} \mathbb{Z}H \xrightarrow{\sigma} \mathbb{C}[K]$$

and obtain the  $\sigma$ -twisted Alexander matrix  $A^\sigma = [\sigma(\alpha(\gamma(\frac{\partial r_i}{\partial x_j})))]_{i,j}$ . The *twisted Alexander polynomial* is defined by  $\Delta^\sigma(\pi) = \gcd(\sigma(E_1(\pi)))$ .

If we replace the twisted map  $\sigma$  by the canonical projection  $\tau: \mathbb{Z}H \rightarrow \mathbb{Z}K$ , which sends the torsion part of  $H$  to 1, we obtain the *Alexander polynomial*  $\Delta(\pi)$ . In order to obtain the one-variable polynomial  $\Delta(\widehat{I}^{-p/q} \cup L)(t)$  of a  $\nu$ -component link  $L$ , we compose the projection  $\tau$  by the canonical projection  $\mathbb{Z}K \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_\nu^{\pm 1}] \rightarrow \mathbb{Z}[t^{\pm 1}]$ , that sends each  $t_i$  to  $t$ .

We continue by describing how to obtain the Alexander polynomial for  $\widehat{I}^{-p/q} \cup L$  from the Alexander polynomial of  $\widehat{I} \cup L \subset S^3$ .

Let  $D$  be the disk, bounded by  $\widehat{I}$ . We may assume that  $L$  intersects  $D$  transversely in  $k$  intersection points with intersection signs  $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$ . Denote by  $[L] = \sum_{i=1}^k \epsilon_i$  the homology class of  $L$  in  $H_1(S^3 \setminus \widehat{I}) \cong \mathbb{Z}$ .

By Proposition 3.1, the presentation of  $\pi_1(L(p, q) \setminus L, *)$  is obtained from the presentation of the link group  $\pi_1(S^3 \setminus (\widehat{I} \cup L), *)$  by adding one relation. The Alexander-Fox matrices are thus closely related and consequently so are the Alexander polynomials, as the following theorem states.

**Theorem 3.2** ([10]). *Let  $p' = \frac{p}{\gcd\{p, [L]\}}$  and  $[L]' = \frac{[L]}{\gcd\{p, [L]\}}$ . The Alexander polynomial of  $\widehat{I}^{-p/q} \cup L$  and the two-variable Alexander polynomial of the classical link  $\widehat{I} \cup L$  are related by*

$$(4) \quad \Delta_{\widehat{I}^{-p/q} \cup L}(t) = \begin{cases} \frac{t-1}{t^{[L]}'-1} \Delta_{\widehat{I} \cup L}(t^{q[L]'}, t^{p'}) & \text{if } [L]' \neq 0 \\ \Delta_{\widehat{I} \cup L}(t^q, t) & \text{if } [L]' = 0. \end{cases}$$

It is also shown in [10] that a normalized version of the Alexander polynomial in lens spaces, denoted by  $\nabla_L(t)$ , respects a skein relation

$$\nabla_{L_+}(t) - \nabla_{L_-}(t) = (t^{\frac{p'}{2}} - t^{-\frac{p'}{2}})\nabla_{L_0}(t),$$

where  $L_+, L_-$ , and  $L_0$  is a skein triple in  $L(p, q)$ .

#### 4. THE BURAU REPRESENTATION

In [18], Morton showed how to express the multivariable Alexander polynomial of a closed braid  $\widehat{\beta}$  directly from the braid  $\beta$  itself by the following construction.

Take the reduced Burau representation

$$B_n \rightarrow GL_{n-1}(\mathbb{Z}[a^{\pm 1}]),$$

given by

$$\sigma_i \mapsto \overline{C}_i(a) := \begin{matrix} & & 1 & & i & & n-1 \\ 1 & \left( \begin{array}{cccccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & a & -a & 1 & \\ & & & & 1 & \\ & & & & & \ddots \\ n-1 & & & & & & 1 \end{array} \right) & & \end{matrix},$$

where the above matrix differs from the identity matrix solely at three places in the row  $i$ . In the case  $i = 0$  or  $i = n-1$ , the matrix is truncated appropriately. We label each strand of the braid  $\beta$  by  $t_1, \dots, t_n$  by putting the label  $t_j$  on the strand that starts from the  $j$ -th position at the bottom as in Figure 3.

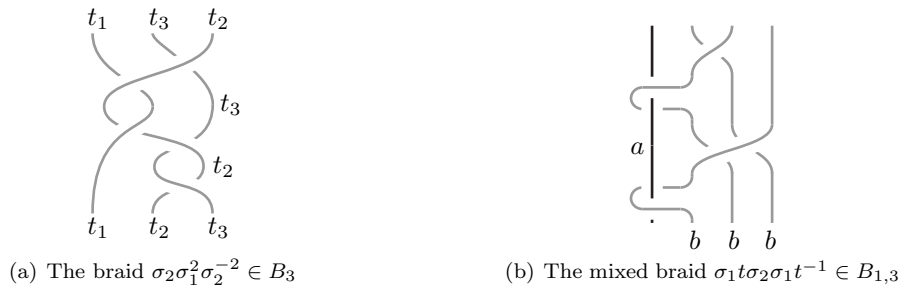


FIGURE 3. A labelled braid (a) and a mixed braid (b)

We assign to the braid

$$\beta = \prod_{r=1}^l \sigma_{i_r}^{\epsilon_r} \in B_n$$

the *coloured reduced Burau matrix*

$$(5) \quad \overline{B}_\beta(t_1, \dots, t_n) := \prod_{r=1}^l (\overline{C}_{i_r}(a_r))^{\epsilon_r},$$

where the variable  $a_r$  denotes the label of the undercrossing strand at crossing  $r$ , counted from top of the braid. Recall that a braid  $\beta$  determines a permutation  $\pi \in S_n$ , such that any strand in  $\beta$  connects position  $j$  at the bottom to the position  $\pi(j)$  at the top.

Denoting by  $A$  the axis of the braid  $\beta$ , we consider the multivariable Alexander polynomial  $\Delta_{\widehat{\beta \cup A}}(t_1, \dots, t_\nu, x)$ , where  $x$  denotes the variable, corresponding to the braid axis. Morton proved the following result.

**Theorem 4.1** ([18]). *The multivariable Alexander polynomial  $\Delta_{\widehat{\beta \cup A}}$ , where  $A$  is the axis of the closed  $n$ -braid  $\widehat{\beta}$ , is given by the polynomial  $\det(I - x\overline{B}_\beta(t_1, \dots, t_\nu))$  with the identifications  $t_{\pi(j)} = t_j$ .*

On the other hand, the Torres-Fox formula obtained in [23] tells us how the Alexander polynomial of a link changes when we remove one of its components. For a two component link  $L = L_1 \cup L_2$ , we have

$$\Delta_{L_1}(t_1) = \frac{1-t_1}{1-t_1^l} \Delta_L(t_1, 1),$$

where  $l$  is the linking number of  $L_1$  and  $L_2$ , and for a  $\nu$ -component link  $L = L_1 \cup \dots \cup L_\nu$ , where  $\nu > 2$ , the formula states

$$\Delta_{L_1 \cup \dots \cup L_{\nu-1}}(t_1, \dots, t_{\nu-1}) = \frac{1}{1-t_1^{l_1} t_2^{l_2} \dots t_{\nu-1}^{l_{\nu-1}}} \Delta_L(t_1, \dots, t_{\nu-1}, 1),$$

where  $l_i$  denotes the linking number of  $L_i$  and  $L_\nu$ .

Once we suppress the axis  $A$  in Theorem 4.1 by taking  $x = 1$ , we can directly apply the Torres-Fox formula and obtain the following equality:

$$(6) \quad \frac{\det(I - \overline{B}_\beta(t_1, \dots, t_\nu))}{1-t_1 \dots t_\nu} \Big|_{t_{\pi(j)}=t_j} = \begin{cases} \Delta_L(t_1, \dots, t_\nu), & \text{if } \nu > 1, \\ \frac{\Delta_L(t_1)}{1-t_1}, & \text{if } \nu = 1, \end{cases}$$

where  $\nu$  is the number of components of the link  $\widehat{\beta}$ .

Following Morton's construction, we define a representation of the mixed braid group on one strand

$$\rho : B_{1,n} \rightarrow GL_n(\mathbb{Z}[a^{\pm 1}, b^{\pm 1}])$$

by

$$(7) \quad \rho(t) = \begin{matrix} & \begin{matrix} 1 & 2 & & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \\ n \end{matrix} & \begin{pmatrix} ab & 1-b & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{matrix}, \quad \rho(\sigma_i) = \begin{matrix} & \begin{matrix} 1 & & & i+1 & & n \end{matrix} \\ \begin{matrix} 1 \\ \\ \\ n \end{matrix} & \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & a & -a & 1 & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{matrix},$$

where the matrices above differ from the identity matrix solely at the first two places in the first row for  $\rho(t)$  and the three places in the  $(i+1)$ -th row for  $\rho(\sigma_i)$ .

For example, a representation of the mixed braid group  $B_{1,4}$  is given by

$$t \mapsto \begin{pmatrix} ab & 1-b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & -a & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & -a & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & -a \end{pmatrix}.$$

We are now ready to state our main theorem.

**Theorem 4.2.** *Let  $I \cup \beta \in B_{1,n}$  be a mixed braid on one strand, such that  $\widehat{I}^{-p/q} \cup \widehat{\beta}$  represents a link  $\widehat{\beta}$  in  $L(p, q)$ . Let  $[\widehat{\beta}]$  be the sum of the exponents of the generator  $t$  appearing in  $I \cup \beta$ . Denote  $p' = \frac{p}{\gcd\{p, [\widehat{\beta}]\}}$  and  $[\widehat{\beta}]' = \frac{[\widehat{\beta}]}{\gcd\{p, [\widehat{\beta}]\}}$ . The Alexander polynomial of the link  $L = \widehat{I}^{-p/q} \cup \widehat{\beta}$  is given by*

$$(8) \quad \Delta_L(t) = \begin{cases} \frac{t-1}{(t^{[\widehat{\beta}]'}-1)(1-t^{np'+q[\widehat{\beta}]'})} \det(I - \rho(I \cup \beta)(t^{q[\widehat{\beta}]'}, t^{p'})), & \text{if } [\widehat{\beta}] \neq 0 \\ \frac{1}{1-t^{n+q}} \det(I - \rho(I \cup \beta)(t^q, t)), & \text{if } [\widehat{\beta}] = 0 \end{cases}$$

*Proof.* Define a map  $i : B_{1,n} \rightarrow B_{1+n}$  by  $i(t) = \sigma_1^2$  and  $i(\sigma_i) = \sigma_{i+1}$ . It is easy to check that  $i$  is a group monomorphism. We now label the first strand  $I$  by  $a$  and the rest of the strands of  $\beta$  by  $b$ . Observe in Equations (7) that  $\rho(t) = \overline{C}_1(a)\overline{C}_1(b)$  and  $\rho(\sigma_i) = \overline{C}_{i+1}(b)$ , thus the (bi)coloured reduced Burau matrix satisfies  $\overline{B}_{i(I \cup \beta)}(a, b) = \rho(I \cup \beta)$ . By Theorem 4.1, the 2-variable Alexander polynomial of  $\widehat{I} \cup \widehat{\beta}$  is given by

$$\Delta_{\widehat{I} \cup \widehat{\beta}}(a, b) = \frac{\det(I - \rho(I \cup \beta)(a, b))}{1 - ab^n}.$$

By Theorem 3.2, we can use Equation (4) to obtain the Alexander polynomial of the link  $L$  in  $L(p, q)$ . If  $[\widehat{\beta}] \neq 0$ , we have

$$\Delta_{\widehat{\Gamma}_{-p/q} \cup \widehat{\beta}}(t) = \frac{t-1}{t^{[\widehat{\beta}]'} - 1} \Delta_{\widehat{\Gamma} \cup \widehat{\beta}}(t^{q[\widehat{\beta}]'}, t^{p'}) = \frac{(t-1) \det(I - \rho(I \cup \beta)(t^{q[\widehat{\beta}]'}, t^{p'}))}{(t^{[\widehat{\beta}]'} - 1)(1 - t^{n_{p'} + q[\widehat{\beta}]'})}$$

and if  $[\widehat{\beta}] = 0$  we have

$$\Delta_{\widehat{\Gamma}_{-p/q} \cup \widehat{\beta}}(t) = \Delta_{\widehat{\Gamma} \cup \widehat{\beta}}(t^q, t) = \frac{\det(I - \rho(I \cup \beta)(t^q, t))}{1 - t^{n+q}}.$$

□

**Example 4.3.** It has been calculated in [10] that the Alexander polynomial for the knot in Figure 1(a) is equal to  $t^{2p} - t^p + 1$ . The braid representative of this knot is represented in Figure 1(b). We have

$$\rho(t\sigma_1^3) = \begin{pmatrix} ab & 1-b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & -a \end{pmatrix} = \begin{pmatrix} -ba^3 + a^3 + ba^2 - a^2 + a & a^3b - a^3 \\ a^3 - a^2 + a & -a^3 \end{pmatrix},$$

$$\det(I - \rho(t\sigma_1^3)(t^q, t^p)) = -t^{2p+q} + t^{3p+q} - t^{4p+q} + t^{2p} - t^p + 1.$$

Since  $[t\sigma_1^3] = 1$ , Equation (8) yields

$$\Delta_L(t) = \frac{-t^{2p+q} + t^{3p+q} - t^{4p+q} + t^{2p} - t^p + 1}{1 - t^{2p+q}} = t^{2p} - t^p + 1.$$

**Remark 4.4.** It follows from the proof of Theorem 4.2 that the 2-variable Alexander polynomial of a link in the solid torus, seen as a mixed link on one fixed strand  $\widehat{I} \cup \widehat{\beta}$ , is given by

$$\Delta_{\widehat{I} \cup \widehat{\beta}}(a, b) = \frac{\det(I - \rho(I \cup \beta)(a, b))}{1 - ab^n}.$$

*Acknowledgments.* The first author was supported by the Slovenian Research Agency grants J1-8131, J1-7025, N1-0064, and P1-0292. The second author was supported by the Slovenian Research Agency grant N1-0083.

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