

Rack invariants of links in $L(p, 1)$

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Abstract

We describe a presentation for the augmented fundamental rack of a link in the lens space $L(p, 1)$. Using this presentation, the (enhanced) counting rack invariants that have been defined for the classical links are applied to the links in $L(p, 1)$. In this case, the counting rack invariants also include the information about the action of $\pi_1(L(p, 1))$ on the augmented fundamental rack of a link.

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1 Introduction & Background

While being an interesting algebraic object by themselves, racks and quandles have been extensively used in the study of classical knots and links [9]. Any link in a 3-manifold has an (augmented) fundamental rack, whose presentation may be obtained from a link diagram D , and in [4] it has been shown that the fundamental rack is a complete invariant for irreducible framed links in a 3-manifold. In [8], Sam Nelson defined computable rack invariants of classical links, which are based on counting the rack homomorphisms from the fundamental rack of a link to a fixed finite rack.

In this paper we describe the augmented fundamental rack $R(D)$ of a framed link in the lens space $L(p, 1)$. We study the set of rack homomorphisms from $R(D)$ to a finite rack X , based on which we define the counting rack invariants of links in $L(p, 1)$. Using a `Phyton` code, we are able to compute the rack counting invariants of knots and links in $L(p, 1)$. The set of rack homomorphisms $Hom(R(D), X)$ also displays the action of $\pi_1(L(p, 1))$ on the fundamental rack $R(D)$. The information about this action may be used as an additional tool to distinguish links in $L(p, 1)$.

The paper is organized as follows. In subsection 1.1 we review the basic rack terms which will be needed in the rest of the paper. In subsection 1.2 we define the augmented fundamental

rack of a link in a 3-manifold, and recall the presentation for the fundamental rack of a classical link. In section 2 we describe a general presentation of the augmented fundamental rack $R(D)$ of a link in $L(p, 1)$, based on a link diagram D . We simplify this presentation and transform it to a finite primary rack presentation. In subsection 2.1, we study the set of rack homomorphisms from $R(D)$ to a finite rack X . By classifying those homomorphisms, we are able to compute the homomorphism set $Hom(R(D), X)$ for a given finite rack X . In section 3, we define the counting rack invariants of links in $L(p, 1)$. We demonstrate the calculation of these invariants in some examples. In the final section 4, we use the calculation of $Hom(R(D), X)$ to study the action of $\pi_1(L(p, 1))$ on the augmented fundamental rack $R(D)$.

1.1 The definition of a rack

Definition 1.1. A **rack** is an algebraic structure comprising a nonempty set R with a binary operation $\triangleright: R \times R \rightarrow R$ satisfying two axioms:

1. For any $x, y \in R$ there is a unique $z \in R$ such that $x = z \triangleright y$.
2. For any $x, y, z \in R$ the formula $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ holds.

It follows from the first rack axiom that the map $f_y: R \rightarrow R$ defined by $f_y(t) = t \triangleright y$ is a bijection. We denote its inverse by $f_y^{-1}(t) = t \triangleright \bar{y}$.

We say that two elements $x, y \in R$ are **operator equivalent** (which we denote by $x \equiv y$) if the equality $z \triangleright x = z \triangleright y$ holds for every $z \in R$.

Lemma 1.2. If y is an element of a rack R , then $y \triangleright y \equiv y$.

Proof. Let $x \in R$ be any element of the rack. By the first rack axiom, there is a unique $z \in R$ such that $x = z \triangleright y$. Applying the second rack axiom we compute $x \triangleright (y \triangleright y) = (z \triangleright y) \triangleright (y \triangleright y) = (z \triangleright y) \triangleright y = x \triangleright y$. \square

If the underlying set of a rack is finite, then the rack operation may be encoded in a square matrix with integer entries as follows [8]. Let R be a rack with n elements y_1, \dots, y_n . The **rack matrix** of R is the matrix $M_R \in \mathbb{Z}^{n \times n}$ defined by $M_R^{(i,j)} = k$ where $y_i \triangleright y_j = y_k$ for $i, j = 1, 2, \dots, n$. Observe that reordering the elements of R changes the rack matrix, so M_R is only defined up to a simultaneous permutation of its rows and columns. Denote by Z_n the set $\{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$. The following lemma may be compared to [6, Lemma 1].

Lemma 1.3. An $(n \times n)$ integer matrix M with entries from the set Z_n is a rack matrix if and only if the following conditions are satisfied:

- a) Each column of M is a permutation of the set Z_n .
- b) For any indices $i, j, k \in Z_n$ we have $M^{(M^{(i,j)}, k)} = M^{(M^{(i,k)}, M^{(j,k)})}$.

Proof. Let M be an $(n \times n)$ rack matrix with the underlying set $\{y_1, \dots, y_n\}$. Then the first rack axiom implies the condition a), and the second rack axiom implies the condition b).

Now suppose we are given the matrix M with entries from Z_n , satisfying the conditions a) and b). Define a binary operation \triangleright on the set Z_n by $i \triangleright j = M^{(i,j)}$. If the conditions a) and b) are satisfied, it follows that (Z_n, \triangleright) is a rack with the rack matrix M . \square

We recall the following definition from [8].

Definition 1.4. Let R be a rack. For any $x \in R$ and $n \in \mathbb{N}$ define $x^{\triangleright n}$ recursively by $x^{\triangleright 1} = x \triangleright x$ and $x^{\triangleright(k+1)} = x^{\triangleright k} \triangleright x^{\triangleright k}$. The **rack rank** of x is the minimal number $N(x) \in \mathbb{N}$ such that $x^{\triangleright N(x)} = x$, or $N(x) = \infty$ if $x^{\triangleright N} \neq x$ for all $N \in \mathbb{N}$. The **rack rank** of R is then

$$N(R) = \text{lcm}\{N(x) \mid x \in R\}.$$

A rack whose rack rank equals 1 is called a **quandle**.

If R is a finite rack, then the rack rank $N(R)$ may be obtained from the rack matrix of R , as remarked in [8]: the diagonal of the rack matrix is a permutation $\pi: R \rightarrow R$, given by $\pi(x) = x \triangleright x$, and the rack rank of R equals the order of $\pi \in S_{|R|}$.

Example 1.5. The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$ is a rack matrix of a rack R with $N(R) = 2$.

The concept of rack may be generalized by making the operator group explicit. We recall the following definition from [4, page 355].

Definition 1.6. Let G be a group, acting on itself by conjugation as $g^h := h^{-1}gh$ for any $g, h \in G$. An **augmented rack** (X, G) is a set X with an action by the group G , written as $(x, g) \mapsto x^g$ and a function $\partial: X \rightarrow G$, satisfying the augmentation identity:

$$\partial(x^g) = g^{-1}(\partial x)g \text{ for all } x \in X, g \in G.$$

The rack operation of X on itself is defined by $x \triangleright y := x^{\partial(y)}$.

Definition 1.7. Let (X, G) be an augmented rack and let Y be a rack. A function $f: X \rightarrow Y$ is a **rack homomorphism** if the following two conditions are satisfied for any $x_1, x_2 \in X$:

1. $f(x_1 \triangleright x_2) = f(x_1) \triangleright f(x_2)$
2. $f(x_1) = f(x_2) \Leftrightarrow f(x_1^g) = f(x_2^g)$ for any $g \in G$.

Thus, a rack homomorphism f of an augmented rack (X, G) induces an action on the image $f(X)$ by $(f(x), g) \mapsto f(x^g)$, which is well defined by the second condition above.

Definition 1.8. Let R be a rack. An equivalence relation \sim on R is called a **congruence** if the condition $(x \sim y, z \sim w \Rightarrow x \triangleright z \sim y \triangleright w)$ is valid for all $x, y, z, w \in R$.

We introduce the general rack presentations, following the definition in [4, page 376].

Definition 1.9. For two sets A and B , denote by $F(A \cup B)$ the free group on $A \cup B$. The **extended free rack** $FR(A, B)$ is the augmented rack $(A \times F(A \cup B), F(A \cup B))$, where the map $\partial: A \times F(A \cup B) \rightarrow F(A \cup B)$ is given by $\partial(a^w) := w^{-1}aw$. The augmentation identity implies that the rack operation is given by

$$(a^w) \triangleright (c^z) = (a^w)^{\partial(c^z)} = a^{wz^{-1}cz}$$

for any $a, c \in A$ and $w, z \in F(A \cup B)$.

The **free rack** on the set A is then defined as $FR(A) := FR(A, \{\})$.

A general rack presentation is given by the following sets:

- the set of the primary generators S_P ,
- the set of the operator generators S_O ,
- the set of the primary relations R_P and
- the set of the operator relations R_O .

Definition 1.10. The rack generated by the presentation $[S_P, S_O : R_P, R_O]$ is the quotient $\frac{FR(S_P, S_O)}{\sim}$, where \sim is the smallest congruence on $FR(S_P, S_O)$ containing

- $x \sim y$ if $(x = y) \in R_P$
- $z \triangleright x \sim z \triangleright y$ for each $z \in FR(S_P, S_O)$ if $(x = y) \in R_P$
- $z^u \sim z^v$ for each $z \in FR(S_P, S_O)$ if $u \equiv v \in R_O$.

A general rack presentation without operator generators ($S_O = \emptyset$) is called a primary rack presentation.

1.2 The fundamental rack of a link in a 3-manifold

We briefly summarize the following from [4, page 358] for the reader's convenience. Let Q be a closed connected orientable 3-manifold and let L be a link in Q . The link is **framed** if a cross section $\lambda: L \rightarrow \partial N(L)$ of the normal disk bundle is chosen. We call this cross section a **framing** and denote its image by $L^+ = \lambda(L)$.

Let $Q_0 = \text{closure}(Q - N(L))$ and choose a basepoint $x_0 \in Q_0$. Denote

$$\Gamma(L) = \{\text{homotopy classes of paths in } Q_0 \text{ from a point in } L^+ \text{ to } x_0\}.$$

During the homotopy the initial point may move around on L^+ , while the final point is kept fixed.

The fundamental group $\pi_1(Q_0, x_0)$ acts on the set $\Gamma(L)$: let $a \in \Gamma(L)$ be represented by a path α and $g \in \pi_1(Q_0, x_0)$ be represented by a loop γ , then $a^g := [\alpha \cdot \gamma]$, where \cdot denotes concatenation of paths.

Using this action, the set $\Gamma(L)$ may be equipped with a structure of an augmented rack. Any $p \in L^+$ lies on a unique meridian circle of the normal circle bundle and we denote by m_p the loop based at p which follows around the meridian in the positive direction.

Definition 1.11. *The **augmented fundamental rack** of the framed link L is the augmented rack $(\Gamma(L), \pi_1(Q_0, x_0))$ as above with the function $\partial: \Gamma(L) \rightarrow \pi_1(Q_0, x_0)$ defined as follows. Given two classes $a, b \in \Gamma$, which are represented by the paths α and β respectively, define $\partial(b) = \bar{\beta} \cdot m_{\beta(0)} \cdot \beta$, where \cdot denotes the concatenation of paths. This produces the rack operation*

$$a \triangleright b = a^{\partial(b)} = [\alpha \cdot \bar{\beta} \cdot m_{\beta(0)} \cdot \beta] .$$

Example 1.12. *The **fundamental rack of a classical link**. Let L be a framed link in S^3 , given by a blackboard-framed link diagram D . Label the arcs of D by y_1, \dots, y_n . For each arc y_i of the diagram D , choose a path x_i from a point on $\lambda(y_i)$ to the basepoint, where $\lambda: L \rightarrow \partial N(L)$ is the chosen framing. Then the homotopy classes of the paths x_1, \dots, x_n are the generators of the fundamental rack of L . The augmentation map $\partial: \Gamma(L) \rightarrow \pi_1(S^3 \setminus N(L), *)$ is given by*

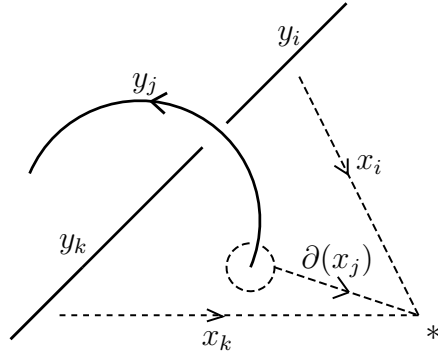


Figure 1: The illustration of the crossing relation

$\partial(x_i) = \bar{x}_i \cdot m_{x_i(0)} \cdot x_i$ and the fundamental group $\pi_1(S^3 \setminus N(L), *)$ is actually generated by the images $\partial(x_1), \dots, \partial(x_n)$. Each crossing of the diagram D consists of an overcrossing arc y_j and two arcs y_i and y_k of the undercrossing strand. If we see the arc y_i on the right (and y_k on the left) of y_j when passing along y_j , then we may see the homotopy between $x_i^{\partial(x_j)}$ and x_k in the Figure 1. This homotopy implies the crossing relation $x_i \triangleright x_j = x_k$ in the fundamental rack of L . The fundamental rack of L then has a primary presentation

$$[x_1, \dots, x_n : \text{crossing relations of } D] .$$

2 The fundamental augmented rack of a link in $L(p, 1)$

Let L be a framed link in the lens space $L(p, 1)$. The lens space is obtained by integral surgery on an unknot U in S^3 . We specify the framing coefficient p of the surgery along U by twisting the unknot p times in the positive direction. Thus there are p positive self-crossings of U in the diagram and the framing is given as the blackboard framing. We draw the framed link L inside the diagram containing the surgery curve $U \subset S^3$ and denote the resulting diagram by D . Let m be the number of crossings where both the underarc and the overarc belong to the link L . Label the arcs of the link L in the diagram by x_1, \dots, x_{m+d} and label the arcs of the surgery curve U by a, a_1, \dots, a_{d-1} as shown in the Figure 2. The general presentation of the

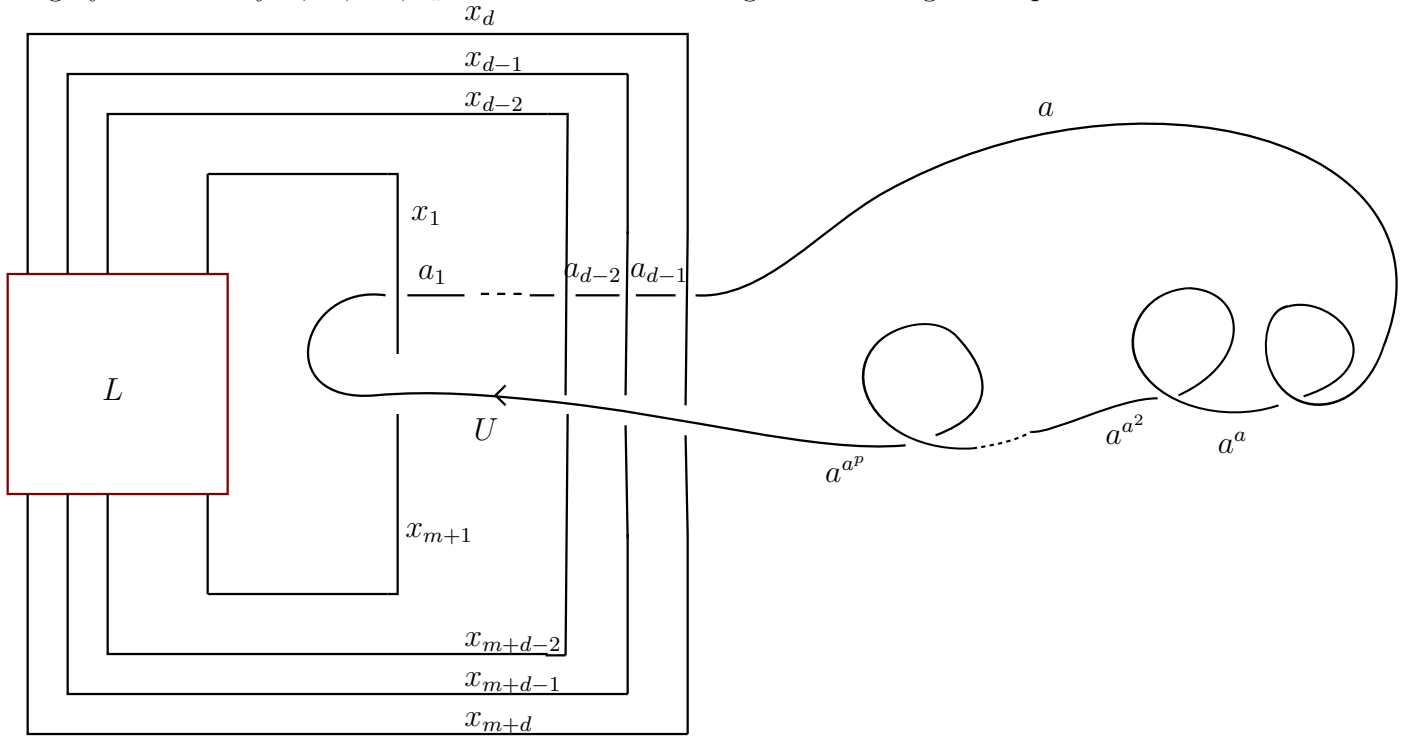


Figure 2: The diagram D of a framed link in $L(p, 1)$

augmented fundamental rack of L is obtained from the diagram D as follows [4, page 379]:

- (i) The arcs of the link L represent the primary generators x_1, \dots, x_{m+d}
- (ii) The arcs of the surgery curve U represent the operator generators a, a_1, \dots, a_{d-1}
- (iii) At each crossing where the underarc belongs to the link L , the crossing relation gives a primary relator
- (iv) At each crossing where the underarc belongs to the surgery curve U , the crossing relation gives an operator relator. Finally we obtain another operator relator by reading around the surgery curve and noting the (signed) undercrossings.

We obtain the following general presentation:

$$\begin{aligned} & [\{x_1, \dots, x_{m+d}\}, \{a, a_1, \dots, a_{d-1}\} : \{\text{crossing relations of } L, x_i^{a^{(a^p)}} = x_{m+i} \text{ for } i = 1, \dots, d\}, \\ & \{\bar{x}_1^{\epsilon_1} a_1 x_1^{\epsilon_1} \equiv a^{(a^p)}, \bar{x}_2^{\epsilon_2} a_2 x_2^{\epsilon_2} \equiv a_1, \dots, \bar{x}_d^{\epsilon_d} a x_d^{\epsilon_d} \equiv a_{d-1}, a a^a \dots a^{(a^{p-1})} \bar{x}_1^{\epsilon_1} \bar{x}_2^{\epsilon_2} \dots \bar{x}_d^{\epsilon_d} \equiv 1\}] . \end{aligned}$$

where $\epsilon_i = 1$ if the arc x_i follows the arc x_{m+i} in the chosen orientation of L , and $\epsilon_i = -1$ otherwise. Using the operator relations, the operator generators a_1, \dots, a_{d-1} are expressed in terms of the operator generator a and thus may be omitted from the presentation. The remaining relation is calculated as

$$a^{(a^p)} \equiv \bar{x}_1^{\epsilon_1} a_1 x_1^{\epsilon_1} \equiv \bar{x}_1^{\epsilon_1} \bar{x}_2^{\epsilon_2} a_2 x_2^{\epsilon_2} x_1^{\epsilon_1} \equiv \dots \equiv \bar{x}_1^{\epsilon_1} \bar{x}_2^{\epsilon_2} \dots \bar{x}_d^{\epsilon_d} a x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1} .$$

By the definition of an augmented rack 1.6, the action of the operator group on itself is taken to be conjugation. It follows that $a^{(a^k)} \equiv (a^k)^{-1} a (a^k) \equiv a$ for any $k \in \mathbb{N}$. The presentation thus simplifies to

$$\begin{aligned} & [\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L, x_i^a = x_{m+i} \text{ for } i = 1, \dots, d\}, \\ & \{a^p \equiv x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}\}] \end{aligned} \quad (1)$$

There is a class of links in $L(p, 1)$ which we need to consider separately:

Definition 2.1. *A link $L \subset L(p, 1)$ is **affine** (sometimes also called *local*) if it is contained in a 3-ball $B^3 \subset L(p, 1)$.*

If L is an affine link, then its diagram D may be drawn without intersections between L and the surgery curve U . Thus $d = 0$ and the presentation of $R(D)$ is very simple: apart from the crossing relations of L , there is only one operator relation $a^p \equiv 1$.

We summarize our results in the following proposition.

Proposition 2.2. *Let L be a framed link in $L(p, 1)$, given by the diagram in the Figure 2. Then the augmented fundamental rack of L is given by a presentation*

$$[\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L, x_i^a = x_{m+i} \text{ for } i = 1, \dots, d\}, \{a^p \equiv x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}\}]$$

if $d > 0$, and

$$[\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L\}, \{a^p \equiv 1\}]$$

otherwise.

Remark 2.3. *Erasing the surgery curve U in our diagram, we obtain an ordinary diagram of a classical link in S^3 . The presentation of the fundamental rack of this classical link may be given as*

$$[\{x_1, \dots, x_{m+d}\} : \{\text{crossing relations of } L, x_i = x_{m+i} \text{ for } i = 1, \dots, d\}] ,$$

to see our presentation as a generalization of the classical case. By removing the unnecessary generators x_{m+1}, \dots, x_{m+d} we obtain the presentation of Example 1.12.

If a framed link in $L(p, 1)$ is given by the diagram D , then its fundamental rack $R(D)$ has the general presentation, given in the proposition 2.2. The operator generator a is intrinsic to the ambient space $L(p, 1)$ and is not itself an element of $R(D)$; rather it defines an action $A: R(D) \rightarrow R(D)$ by $A(x) = x^a$. Define a mapping $F: R(D) \rightarrow R(D)$ by

$$F(x) = \begin{cases} x \triangleright x_d^{\epsilon_d} \triangleright \dots \triangleright x_1^{\epsilon_1} & \text{if } d > 0, \\ x & \text{if } d = 0. \end{cases}$$

Lemma 2.4. *Let $R(D)$ be given by the general presentation in the proposition 2.2. Then A and F are rack automorphisms of $R(D)$ for which $A^p = F$.*

Proof. By the definition of the rack operation in the extended free rack 1.9, we have

$$A(x) \triangleright A(y) = (x^a) \triangleright (y^a) = x^{aa^{-1}ya} = x^{ya} = A(x \triangleright y).$$

By the second rack axiom we have $F(x \triangleright y) = F(x) \triangleright F(y)$. It follows from the first rack axiom that F is a bijection, whose inverse for $d > 0$ is given by $F^{-1}(x) = x \triangleright (\bar{x}_1^{\epsilon_1} \dots \bar{x}_d^{\epsilon_d})$, while A is a bijection with the inverse $A^{-1}(x) = x^{\bar{a}}$.

If $d = 0$, then $F = A^p = id_{R(D)}$ by the proposition 2.2, thus the remaining statements of the lemma are obvious.

Now suppose that $d > 0$. The operator relation $a^p \equiv x_d^{\epsilon_d} \dots x_1^{\epsilon_1}$ implies that $A^p(x) = x^{a^p} = F(x)$ for any $x \in R(D)$. \square

The operator relations of the presentation (1) imply that the fundamental augmented rack $R(D)$ actually has a finite primary presentation.

Proposition 2.5. *Let $S = \{(x_i, k) \mid 1 \leq i \leq m + d, 0 \leq k \leq p - 1\} = \{x_1, x_2, \dots, x_{m+d}\} \times \mathbb{Z}_p$. The fundamental augmented rack $R(D)$ has a primary presentation*

$$\begin{aligned} [S : (x_i, k) \triangleright (x_j, k) &= (x_l, k) \text{ for every crossing relation } x_i \triangleright x_j = x_l \text{ and } k = 0, \dots, p - 1, \\ (x_{m+i}, k) &= (x_i, k + 1), \text{ for } i = 1, \dots, d \text{ and } k = 0, \dots, p - 2, \\ (x_{m+i}, p - 1) &= (x_i, 0) \triangleright (x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright (x_1^{\epsilon_1}, 0) \text{ for } i = 1, \dots, d] \end{aligned}$$

if $d > 0$, and

$$[S : (x_i, k) \triangleright (x_j, k) = (x_l, k) \text{ for every crossing relation } x_i \triangleright x_j = x_l \text{ and } k = 0, \dots, p - 1]$$

if $d = 0$.

Proof. $R(D)$ has a general presentation, given by the proposition 2.2. Every element $x \in R(D)$ is of the form x_i^w for some word $w \in F(\{x_1, \dots, x_{m+d}, a\})$. Thus x may be written as a finite product of the elements $x_j^{(a^k)}$ for $1 \leq j \leq m + d$ and $0 \leq k \leq p - 1$, since the second operator relation implies that $x_j^{(a^p)} = x_j \triangleright x_d^{\epsilon_d} \triangleright \dots \triangleright x_1^{\epsilon_1}$. Denote by $(x_i, k) \in \{x_1, x_2, \dots, x_{m+d}\} \times \mathbb{Z}_p$ the generator of $R(D)$, corresponding to the element $x_i^{(a^k)}$.

We have shown in the lemma 2.4 that A is a rack automorphism, thus we have $A(x \triangleright z) = A(x) \triangleright A(z)$ for any $x, z \in S$. Each primary relation of (1) therefore induces p relations of the primary presentation (2). Every crossing relation $x_i \triangleright x_j = x_l$ induces relations $(x_i, k) \triangleright (x_j, k) = (x_l, k)$ for $k = 0, \dots, p-1$. For $i = 1 \dots, d$, the relation $x_{m+i} = x_i^a$ of the presentation (1) induces relations $(x_{m+i}, k) = (x_i, k+1)$ for $k = 0, \dots, p-2$ and the final relation $(x_{m+i}, p-1) = (x_i, 0) \triangleright (x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright (x_1^{\epsilon_1}, 0)$. \square

Remark 2.6. *If the fundamental rack $R(D)$ is given by the presentation (2), then the action of the rack automorphism $A: R(D) \rightarrow R(D)$ is given by*

$$A(x_i, k) = \begin{cases} (x_i, k+1) & \text{for } 0 \leq k < (p-1), \\ (x_i, 0) \triangleright (x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright (x_1^{\epsilon_1}, 0) & \text{for } k = p-1. \end{cases}$$

2.1 Labeling the homomorphisms from $R(D)$ to a finite rack

Let X be a finite rack. We will study the set of homomorphisms $Hom(R(D), X)$, where $R(D)$ denotes the augmented fundamental rack of a framed link given by the diagram D .

If L is a classical link, then its fundamental rack is a quotient of the free rack $FR(\{x_1, \dots, x_m\})$ whose generators correspond to the arcs of the diagram D . Thus, any rack homomorphism $f: R(D) \rightarrow X$ is completely specified by the images $f(x_i)$, $i = 1, \dots, m$. This is usually thought of as coloring each arc x_i by a color from X . A function $f: \{x_1, \dots, x_m\} \rightarrow X$ defines a coloring if and only if it preserves the crossing relation at every crossing of the diagram D .

Now let L be a framed link in $L(p, 1)$, whose diagram D is labeled as shown in the Figure 2. By the proposition 2.5, the fundamental rack $R(D)$ is a quotient of the free rack

$$FR(\{(x_i, k) \mid 1 \leq i \leq m+d, 0 \leq k \leq p-1\}) ,$$

which implies that any rack homomorphism $f: R(D) \rightarrow X$ is completely determined by the images $f((x_i, k))$ for $i = 1, \dots, m+d$ and $k = 0, 1, \dots, p-1$. We may think of this as choosing for each arc x_i of the diagram D a p -tuple of colors from X , or, equivalently, as choosing p colorings of the diagram D by the colors from X . Denote again the generating set of $R(D)$ by $S = \{x_1, \dots, x_{m+d}\} \times \mathbb{Z}_p$. A function $f: S \rightarrow X$ defines a vector function $(f_0, \dots, f_{p-1}): \{x_1, \dots, x_{m+d}\} \rightarrow X^p$ by $f_k(x_i) = f((x_i, k))$. When will this function define a coloring?

Lemma 2.7. *If a map $f: R(D) \rightarrow X$ satisfies the condition $f(x \triangleright y) = f(x) \triangleright f(y)$ for any $x, y \in R(D)$, then for any pair of elements $z, w \in R(D)$ the following implication holds:*

$$f(A^k(z)) = f(A^k(w)) \text{ for } 0 \leq k \leq p-1 \quad \Rightarrow \quad f(A^k(z)) = f(A^k(w)) \text{ for every } k \in \mathbb{N} .$$

Proof. Suppose that $f(A^k(z)) = f(A^k(w))$ for every $0 \leq k \leq p-1$. For $K \geq p$, we may write $K = p+r$ for some $r < K$ and use the lemma 2.4 to calculate

$$\begin{aligned} f(A^K(z)) &= f(A^r(F(z))) = f(F(A^r(z))) = f(A^r(z)) \triangleright f(x_d^{\epsilon_d}) \triangleright \dots \triangleright f(x_1^{\epsilon_1}) \\ f(A^K(w)) &= f(A^r(F(w))) = f(F(A^r(w))) = f(A^r(w)) \triangleright f(x_d^{\epsilon_d}) \triangleright \dots \triangleright f(x_1^{\epsilon_1}) \end{aligned}$$

This completes the proof by induction. \square

Lemma 2.8. *If $f: R(D) \rightarrow X$ is a rack homomorphism, then we have*

$$f(A^k(x_i, j)) = \begin{cases} f_{j+k}(x_i) & \text{if } j+k \leq (p-1), \\ f_{(j+k) \bmod p}(x_i) \triangleright f_{(j+k) \bmod p}(x_d^{\epsilon_d}) \triangleright \dots \triangleright f_{(j+k) \bmod p}(x_1^{\epsilon_1}) & \text{otherwise.} \end{cases}$$

for any $1 \leq i \leq m+d$ and $0 \leq j, k \leq p-1$.

Proof. The equality follows from the lemma 2.4 and the definition of f . \square

Proposition 2.9. *Let X be a finite rack. Let L be a framed affine link in $L(p, 1)$, whose diagram D is labeled as shown in the Figure 2 (for $d = 0$). A function $f: S \rightarrow X$ defines a rack homomorphism from $R(D)$ to X if and only if the following conditions are satisfied:*

(i) $f_k(x_i) \triangleright f_k(x_j) = f_k(x_l)$ for every crossing relation $x_i \triangleright x_j = x_l$ of the link L and for $k = 0, \dots, p-1$.

(ii) $f(x) = f(y) \Leftrightarrow f(A^k(x)) = f(A^k(y))$ for any $x, y \in S$ and for any $0 \leq k \leq p-1$.

Proof. By the proposition 2.5, the augmented fundamental rack $R(D)$ has the generating set $S = \{(x_i, k) \mid 1 \leq i \leq m+d, 0 \leq k \leq p-1\}$ and a presentation

$$[S : (x_i, k) \triangleright (x_j, k) = (x_l, k) \text{ for every crossing relation } x_i \triangleright x_j = x_l \text{ and } k = 0, \dots, p-1].$$

Let $f: S \rightarrow X$ be a function that satisfies the conditions (i) and (ii). The function f defines a rack homomorphism from the free rack $FR(S)$ to X by

$$f((x_i, r) \triangleright (x_j, s)) := f(x_i, r) \triangleright f(x_j, s)$$

for any $(x_i, r), (x_j, s) \in S$. Since f satisfies the condition (i), it follows that $f(x_i, k) \triangleright f(x_j, k) = f_k(x_i) \triangleright f_k(x_j) = f_k(x_l) = f(x_l, k)$ for every crossing relation $x_i \triangleright x_j = x_l$ and $k = 0, \dots, p-1$. Therefore f preserves all the relations of the above presentation. By definition 1.7, it remains to show that f preserves the action of the rack automorphism A . Let $f(x_i, r) = f(x_j, s)$ for some $(x_i, r), (x_j, s) \in S$. Then it follows from (ii) that $f(A^k(x_i, r)) = f(A^k(x_j, s))$ for $0 \leq k \leq p-1$. Observe that by lemma 2.4 we have $A^p = F = Id_{R(D)}$ and thus

$$f(A^{k+p}(x_i, r)) = f(A^k(F(x_i, r))) = f(A^k(x_i, r)) = f(A^k(x_j, s)) = f(A^{k+p}(x_j, s))$$

for $k = 0, \dots, p-1$. It follows by induction that $f(A^n(x_i, r)) = f(A^n(x_j, s))$ for any $n \in \mathbb{Z}$, therefore f preserves the action of the automorphism A and thus defines a rack homomorphism from $R(D)$ to X .

Conversely, let $f: S \rightarrow X$ be a function that defines a rack homomorphism from $R(D)$ to X . Then f preserves all the relations of the above presentation, which implies (i), and f preserves the action of A , which implies (ii). \square

Proposition 2.10. *Let X be a finite rack. Let L be a framed link in $L(p, 1)$, which is not affine and whose diagram D is labeled as shown in the Figure 2. A function $f: S \rightarrow X$ defines a rack homomorphism from $R(D)$ to X if and only if the following conditions are satisfied:*

(i) $f_k(x_i) \triangleright f_k(x_j) = f_k(x_l)$ for every crossing relation $x_i \triangleright x_j = x_l$ of the link L and for $k = 0, \dots, p-1$.

(ii) $f_0(x_{m+i}) = f_1(x_i)$ for $i = 1, \dots, d$.

(iii) $f(x) = f(y) \Leftrightarrow f(A^k(x)) = f(A^k(y))$ for any $x, y \in S$ and for any $0 \leq k \leq p-1$.

Proof. By the proposition 2.5, the augmented fundamental rack $R(D)$ has a presentation

$$\begin{aligned} [S : & (x_i, k) \triangleright (x_j, k) = (x_l, k) \text{ for every crossing relation } x_i \triangleright x_j = x_l \text{ and } k = 0, \dots, p-1, \\ & (x_{m+i}, k) = (x_i, k+1), \text{ for } i = 1, \dots, d \text{ and } k = 0, \dots, p-2, \\ & (x_{m+i}, p-1) = (x_i, 0) \triangleright (x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright (x_1^{\epsilon_1}, 0) \text{ for } i = 1, \dots, d] \end{aligned}$$

Let $f: S \rightarrow X$ be a function that satisfies the conditions (i)-(iii). The function f defines a rack homomorphism from the free rack $FR(S)$ to X by

$$f((x_i, r) \triangleright (x_j, s)) := f(x_i, r) \triangleright f(x_j, s)$$

for any $(x_i, r), (x_j, s) \in S$. Since f satisfies the condition (i), it follows that $f(x_i, k) \triangleright f(x_j, k) = f_k(x_i) \triangleright f_k(x_j) = f_k(x_l) = f(x_l, k)$ for every crossing relation $x_i \triangleright x_j = x_l$ and $k = 0, \dots, p-1$. Since f satisfies the condition (ii), we have $f(x_{m+i}, 0) = f_0(x_{m+i}) = f_1(x_i) = f(x_i, 1)$, and by the condition (iii) it follows that $f(x_{m+i}, k) = f(A^k(x_{m+i}, 0)) = f(A^k(x_i, 1)) = f(x_i, k+1)$ for $k = 0, \dots, p-2$ and

$$\begin{aligned} f(x_{m+i}, p-1) &= f(A^{p-1}(x_{m+i}, 0)) = f(A^{p-1}(x_i, 1)) = f((x_i, 0) \triangleright (x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright (x_1^{\epsilon_1}, 0)) = \\ &= f(x_i, 0) \triangleright f(x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright f(x_1^{\epsilon_1}, 0) \end{aligned}$$

for $i = 1, \dots, d$. Therefore f preserves all the relations of the above presentation. To see that f preserves the action of the rack automorphism A , let $f(x_i, r) = f(x_j, s)$ for some $(x_i, r), (x_j, s) \in S$. It follows from (iii) that $f(A^k(x_i, r)) = f(A^k(x_j, s))$ for $0 \leq k \leq p-1$. By lemma 2.4 we have

$$\begin{aligned} f(A^{k+p}(x_i, r)) &= f(F(A^k(x_i, r))) = f(A^k(x_i, r) \triangleright (x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright (x_1^{\epsilon_1}, 0)) = \\ &= f(A^k(x_i, r)) \triangleright f(x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright f(x_1^{\epsilon_1}, 0) = f(A^k(x_j, s)) \triangleright f(x_d^{\epsilon_d}, 0) \triangleright \dots \triangleright f(x_1^{\epsilon_1}, 0) = \\ &= f(A^{k+p}(x_j, s)) \end{aligned}$$

for $k = 0, \dots, p-1$. It follows by induction that $f(A^n(x_i, r)) = f(A^n(x_j, s))$ for any $n \in \mathbb{Z}$, therefore f preserves the action of A and defines a rack homomorphism from $R(D)$ to X .

Conversely, let $f: S \rightarrow X$ be a function that defines a rack homomorphism from $R(D)$ to X . Then f preserves all the relations of the above presentation, which implies (i) and (ii), and f preserves the action of A , which implies (iii). \square

Let L be a link in $L(p, 1)$, given by the diagram on the Figure 2. The surgery curve U bounds an obvious disk \mathcal{D} in S^3 . Cutting the link L along the disk \mathcal{D} , we obtain a tangle t_L in S^3 . The fundamental rack of a tangle is obtained in a similar way as the fundamental rack of a (classical) link, see [3, page 210]. The fundamental rack of the tangle t_L is given by

$$R(t_L) = [x_1, \dots, x_{m+d} | \text{crossing relations of } L] .$$

Observe that by adding identifications on the generators, corresponding to the open arcs of t_L , we obtain the fundamental rack of the classical link L' in S^3 that corresponds to L :

$$R(L') = [x_1, \dots, x_{m+d} | \text{crossing relations of } L, x_i = x_{m+i} \text{ for } i = 1, \dots, d] .$$

Corollary 2.11. *Let X be a finite rack.*

1. *If $f: S \rightarrow X$ defines a coloring of $R(D)$, then $f_k: \{x_1, \dots, x_{m+d}\} \rightarrow X$ is a coloring of $R(t_L)$ for $k = 0, \dots, p-1$.*
2. *Suppose L is an affine link in $L(p, 1)$. If $f_k: \{x_1, \dots, x_{m+d}\} \rightarrow X$ is a coloring of $R(t_L)$ for $k = 0, \dots, p-1$, such that the map $f = (f_0, \dots, f_{p-1})$ satisfies the condition (ii) of proposition 2.9, then f is a coloring of $R(D)$.*
3. *Suppose L is a link in $L(p, 1)$ which is not affine. If $f_k: \{x_1, \dots, x_{m+d}\} \rightarrow X$ is a coloring of $R(t_L)$ for $k = 0, \dots, p-1$, such that the map $f = (f_0, \dots, f_{p-1})$ satisfies the conditions (ii) and (iii) of proposition 2.10, then f is a coloring of $R(D)$.*

Proof. 1. If f defines a coloring of $R(D)$, then the propositions 2.9(i) and 2.10(i) imply that f_k preserves all the crossing relations of L . It follows that f_k defines a coloring of $R(t_L)$ for $k = 0, \dots, p-1$.

2. If $f_k: \{x_1, \dots, x_{m+d}\} \rightarrow X$ is a coloring of $R(t_L)$ for $k = 0, \dots, p-1$, then f_k preserves all the crossing relations of L . The map $f = (f_0, \dots, f_{p-1})$ therefore satisfies the condition (i) of the proposition 2.9. If it also satisfies the condition (ii), then it defines a coloring of $R(D)$.
3. If $f_k: \{x_1, \dots, x_{m+d}\} \rightarrow X$ is a coloring of $R(t_L)$ for $k = 0, \dots, p-1$, then f_k preserves all the crossing relations of L . The map $f = (f_0, \dots, f_{p-1})$ therefore satisfies the condition (i) of the proposition 2.9. If it also satisfies the conditions (ii) and (iii), then it defines a coloring of $R(D)$.

□

Sam Nelson wrote a `Python` code which computes all the rack homomorphisms from a fundamental rack of a classical link L to a finite rack X . The input of his function `pdhomlist(PD, MX)` are the planar diagram PD of the link L and the rack matrix M_X of the rack X . The output are all the colorings of the diagram PD with colors from X which satisfy the crossing relations.

Since a representation of the fundamental rack of a (classical) tangle is obtained from its planar diagram in the same way as the representation of the fundamental rack of a classical link, the function also works for tangles.

In our case, L is a link in the lens space $L(p, 1)$ and by the corollary 2.11, every coloring of $R(D)$ defines a p -tuple of colorings of $R(t_L)$. We may use the function $pdhomlist(PD, M_X)$ to find the set of all possible colorings of $R(t_L)$. Any p -tuple from this set will satisfy the condition (i) from the Propositions 2.9 and 2.10. We must then choose the p -tuple (f_0, \dots, f_{p-1}) in such a way that it will also satisfy the conditions (ii) and (iii) to define a rack homomorphism from $R(D) \rightarrow X$.

3 The counting rack invariants of links in $L(p, 1)$

In this section we recall the counting rack invariants of classical links, defined by Sam Nelson in [8]. We show that those invariants may be generalized and applied to the links in the lens spaces $L(p, 1)$.

Recall the following result about the diagram equivalence for links in $L(p, q)$:

Theorem 3.1 ([7],[5]). *Two links L_1 and L_2 in $L(p, q)$ with diagrams D_1 and D_2 are isotopic if and only if there exists a finite sequence of moves $\Omega_1, \Omega_2, \Omega_3$ and $SL_{p,q}$ that transform one diagram to the other.*

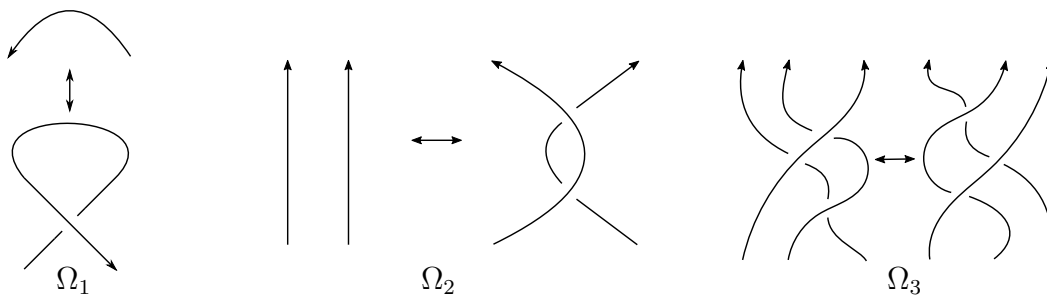


Figure 3: The moves Ω_1, Ω_2 and Ω_3

Let L be an n -component link in $L(p, 1)$, and let X be a finite rack with rack rank N . Each framing of L is given by a writhe vector $\mathbf{w} \in \mathbb{Z}^n$. Denote the blackboard-framed diagram of L with writhe vector $\mathbf{w} \in \mathbb{Z}^n$ by (D, \mathbf{w}) . For each framing \mathbf{w} , the diagram (D, \mathbf{w}) of the framed link defines its own fundamental rack $R(D, \mathbf{w})$ with its own homomorphism set $Hom(R(D, \mathbf{w}), X)$. Luckily, since X is a finite rack, there are only finitely many framings which produce different homomorphism sets, by the following results of [8]:

Definition 3.2. [8, Definition 4] *Let $N \in \mathbb{N}$. Two blackboard-framed oriented link diagrams are N -phone cord equivalent if one may be obtained from the other by a finite sequence of*

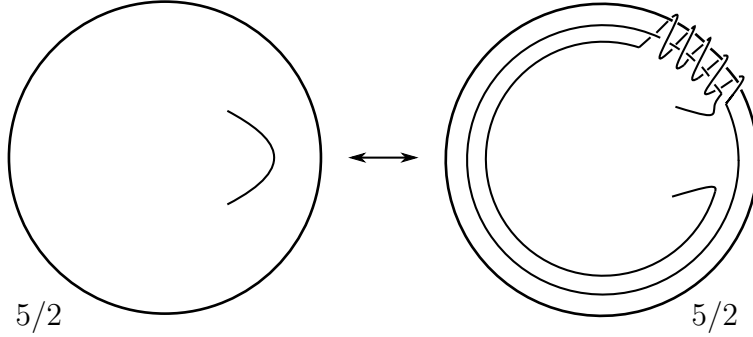


Figure 4: The slide move $SL_{5,2}$

Ω_2 and Ω_3 moves and the N -**phone cord move** in the Figure 5, where N is the number of loops.

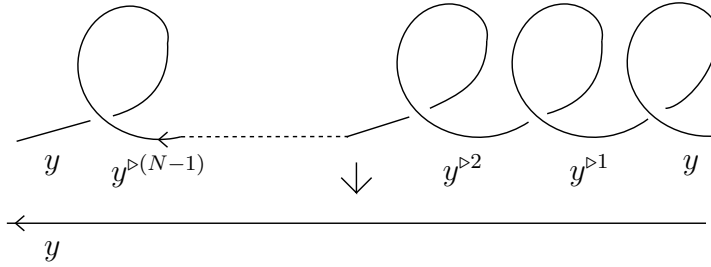


Figure 5: N -phone cord move

Proposition 3.3. [8, Proposition 4] Let X be a finite rack with rack rank N . If two link diagrams D and D' are N -phone cord isotopic, then $|\text{Hom}(R(D), X)| = |\text{Hom}(R(D'), X)|$.

Proof. Suppose the D' is obtained from D by an N -phone cord move as shown on the Figure. Every rack homomorphism $f: R(D) \rightarrow X$ is given by the p -tuples $(f_0(x), \dots, f_{p-1}(x)) \in X^p$ for every arc x of the diagram D . The same coloring also defines a rack homomorphism from $R(D') \rightarrow X$: the colors of the additional arcs are given by $f_i(y^{\triangleright k}) = f_i(y)^{\triangleright k}$ for $k = 1, \dots, N-1$ and $f_i(y^{\triangleright N}) = f_i(y)^{\triangleright N} = f_i(y)$ for $i = 0 \dots, p-1$. \square

It follows from the Proposition 3.3 that two writhe vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ define the same counting rack invariant if $\mathbf{u} \equiv \mathbf{v} \pmod N$ (meaning that $u_i \equiv v_i \pmod N$ for $i = 1, \dots, n$). The set of writhes of L modulo N can be indexed by the set $(\mathbb{Z}_N)^n$. Let (D, \mathbf{w}) denote the blackboard-framed diagram of L with writhe vector $\mathbf{w} \in (\mathbb{Z}_N)^n$.

Definition 3.4. [8, Definition 5] Let L be an n -component link in $L(p, 1)$ and let X be a finite rack with rack rank N . The **integral rack counting invariant** of L with respect to X is

$$\Phi_X^{\mathbb{Z}}(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^n} |\text{Hom}(R(D, \mathbf{w}), X)|.$$

The integer rack counting invariant is an integer, counting the cardinality of the set of rack homomorphisms. By decorating each rack homomorphism by the writhe vector of its associated framing, we may enhance this invariant and replace the integer by a polynomial. For a writhe vector $\mathbf{w} = (w_1, \dots, w_n) \in (\mathbb{Z}_N)^n$, denote the product $\prod_{k=1}^n q_1^{w_1} q_2^{w_2} \dots q_n^{w_n}$ by the formal variable $q^{\mathbf{w}}$. Then we may define

Definition 3.5. [8, Definition 6] Let L be an n -component link in $L(p, 1)$ with the writhe vector $\mathbf{w} \in (\mathbb{Z}_N)^n = W$. Let X be a finite rack with rack rank N . The **writhe-enhanced rack counting invariant** of L with respect to X is

$$\Phi_X^W(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^n} |\text{Hom}(R(D, \mathbf{w}), X)| q^{\mathbf{w}} .$$

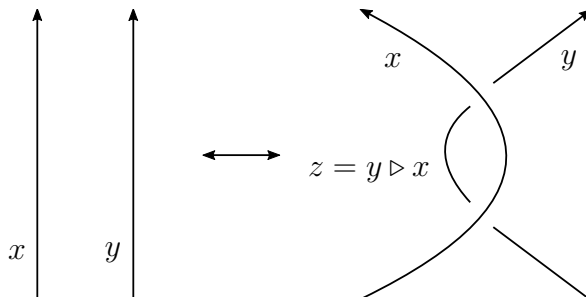
Proposition 3.6. Φ_X^Z and Φ_X^W are invariants of links in $L(p, 1)$ for any finite rack X .

Proof. We need to show that

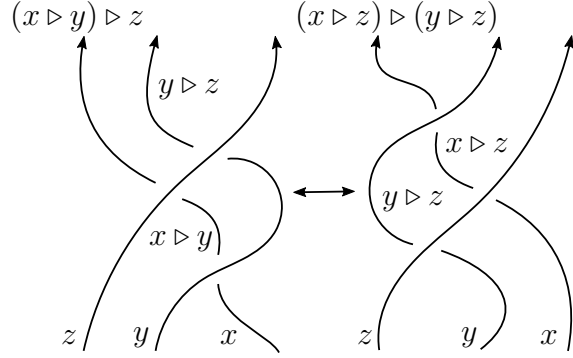
$$\Phi_X^Z(L_1) = \Phi_X^Z(L_2) \text{ and } \Phi_X^W(L_1) = \Phi_X^W(L_2) \tag{3}$$

for any two isotopic links L_1, L_2 in $L(p, 1)$. Let D_i be a diagram of L_i for $i = 1, 2$. By the Theorem 3.1, it is enough to check the equalities (3) in the case when D_1 and D_2 differ by any one of the moves $\Omega_1, \Omega_2, \Omega_3$ and $SL_{p,1}$.

First observe that the functions Φ_X^Z and Φ_X^W are defined as a sum over all possible framings \mathbf{w} which produce different values of $|\text{Hom}(R(D, \mathbf{w}), X)|$ (see proposition 3.3) and thus are invariant of a given framing of the diagram D . We may deduce that if D_1 and D_2 differ by an Ω_1 move, then the equalities (3) hold.



Secondly, suppose that the diagram D_2 is obtained from D_1 by applying an Ω_2 move during which the arc x overcrosses the arc y . Then the presentation of $R(D_1)$ contains the generators x and y , and the presentation of $R(D_2)$ is obtained from the presentation of $R(D_1)$ by adding a new generator z and a new relation $z = y \triangleright x$. Since the new generator is written as a product of the previous generators, both presentations present the same rack: $R(D_1) = R(D_2)$ and as $\Phi_X^Z(L_i)$ and $\Phi_X^W(L_i)$ are both functions of the fundamental rack $R(D_i)$, the equalities (3) follow.



Now suppose that D_2 is obtained from D_1 by applying an Ω_3 move. This means there are three arcs x, y, z representing generators of $R(D_1)$, whose intercrossings change so that a generator $(x \triangleright y) \triangleright z$ in $R(D_1)$ becomes the generator $(x \triangleright z) \triangleright (y \triangleright z)$ in $R(D_2)$. By the second rack axiom we have $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) \in R(D_i)$, therefore $R(D_1) = R(D_2)$ and again the equalities (3) follow.

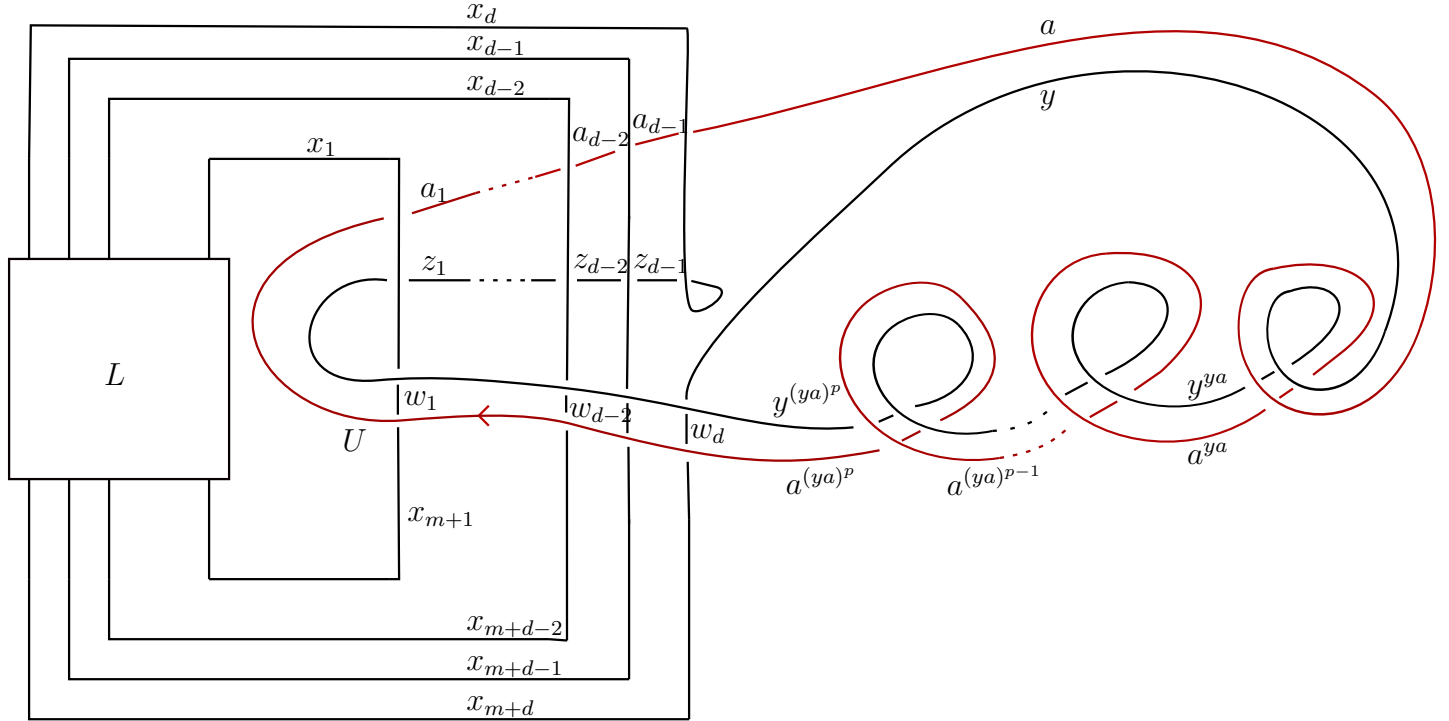


Figure 6: The diagram D_2 , obtained by the move $SL_{p,1}$ on a link in $L(p, 1)$

Finally, suppose that the diagram D_2 is obtained from D_1 by applying an $SL_{p,1}$ move. Let D_1 be labeled as shown on the Figure 2. Applying the move $SL_{p,1}$, a strand of the link L winds

once along the blackboard framing of the surgery curve U . In doing so, the presentation

$$R(D_1) = [\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L, x_i^a = x_{m+i} \text{ for } i = 1, \dots, d\}, \\ \{a^p \equiv x_d^{\epsilon_d} x_{d-1}^{\epsilon_{d-1}} \dots x_1^{\epsilon_1}\}]$$

of the fundamental rack $R(D_1)$ changes to a new presentation

$$R(D_2) = [\{x_1, \dots, x_{m+d}, y, z_1, \dots, z_{d-1}, w_1, \dots, w_d\}, \{a, a_1, \dots, a_{d-1}\} : \{\text{crossing relations of } L, \\ x_i^{y^{(ya)^p}} = w_i \text{ for } i = 1, \dots, d-1, y^{y^{(ya)^p}} = w_d, w_i^{a^{(ya)^p}} = x_{m+i} \text{ for } i = 1, \dots, d, \\ z_1^{x_1^{\epsilon_1}} = y^{(ya)^p}, z_2^{x_2^{\epsilon_2}} = z_1, \dots, z_{d-1}^{x_{d-1}^{\epsilon_{d-1}}} = z_{d-2}, x_d^{x_d^{\epsilon_d}} = z_{d-1}\}, \\ \{\bar{x}_1^{\epsilon_1} a_1 x_1^{\epsilon_1} \equiv a^{(ya)^p}, \bar{x}_2^{\epsilon_2} a_2 x_2^{\epsilon_2} \equiv a_1, \dots, \bar{x}_{d-1}^{\epsilon_{d-1}} a_{d-1} x_{d-1}^{\epsilon_{d-1}} \equiv a_{d-2}, \bar{x}_d^{\epsilon_d} a x_d^{\epsilon_d} \equiv a_{d-1}, (ya)^p \equiv x_d^{\epsilon_d} x_{d-1}^{\epsilon_{d-1}} \dots x_1^{\epsilon_1}\}]$$

obtained from the diagram D_2 on Figure 6. It is easy to see that the new generators z_1, \dots, z_{d-1} and w_1, \dots, w_d , as well as the operator generators a_1, \dots, a_{d-1} may be omitted from the presentation. The primary relations imply that $x_d^{x_d^{\epsilon_d} x_{d-1}^{\epsilon_{d-1}} \dots x_1^{\epsilon_1}} = y^{(ya)^p}$ and thus $x_d = y^{(ya)^p \bar{x}_1^{\epsilon_1} \dots \bar{x}_d^{\epsilon_d}}$, therefore $x_d = y$ by the last operator relation. The presentation simplifies to

$$R(D_2) = [\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L, x_i^{x_i^{da}} = x_{m+i} \text{ for } i = 1, \dots, d\}, \\ \{(x_d a)^p \equiv x_d^{\epsilon_d} \dots x_1^{\epsilon_1}\}],$$

which is equivalent to the presentation of $R(D_1)$. This shows the equalities (3) are valid. \square

Example 3.7. Let K_0, K_1 and K_2 be the knots in $L(3, 1)$, given in the Figure 7, linking the exceptional fibre of $L(3, 1)$ zero times, once and twice respectively. Let R be the rack, given by

$$\text{the rack matrix } M_R = \begin{bmatrix} 1 & 3 & 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 & 2 & 2 \\ 2 & 1 & 3 & 3 & 3 & 3 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}. \text{ Since the rack rank of } R \text{ equals 2, we need to count}$$

the colorings of two different diagrams of K_i , one with an odd writhe and one with an even writhe. We calculate the counting rack invariants of K_i with respect to R as

$$\begin{aligned} \Phi_R^{\mathbb{Z}}(K_0) &= 22 & \Phi_R^W(K_0) &= 12 + 10q \\ \Phi_R^{\mathbb{Z}}(K_1) &= 20 & \Phi_R^W(K_1) &= 10 + 10q \\ \Phi_R^{\mathbb{Z}}(K_2) &= 10 & \Phi_R^W(K_2) &= 6 + 4q \end{aligned}$$

Example 3.8. Consider the two links L_1, L_2 in $L(3, 1)$, given in the Figure 8. Let R be the

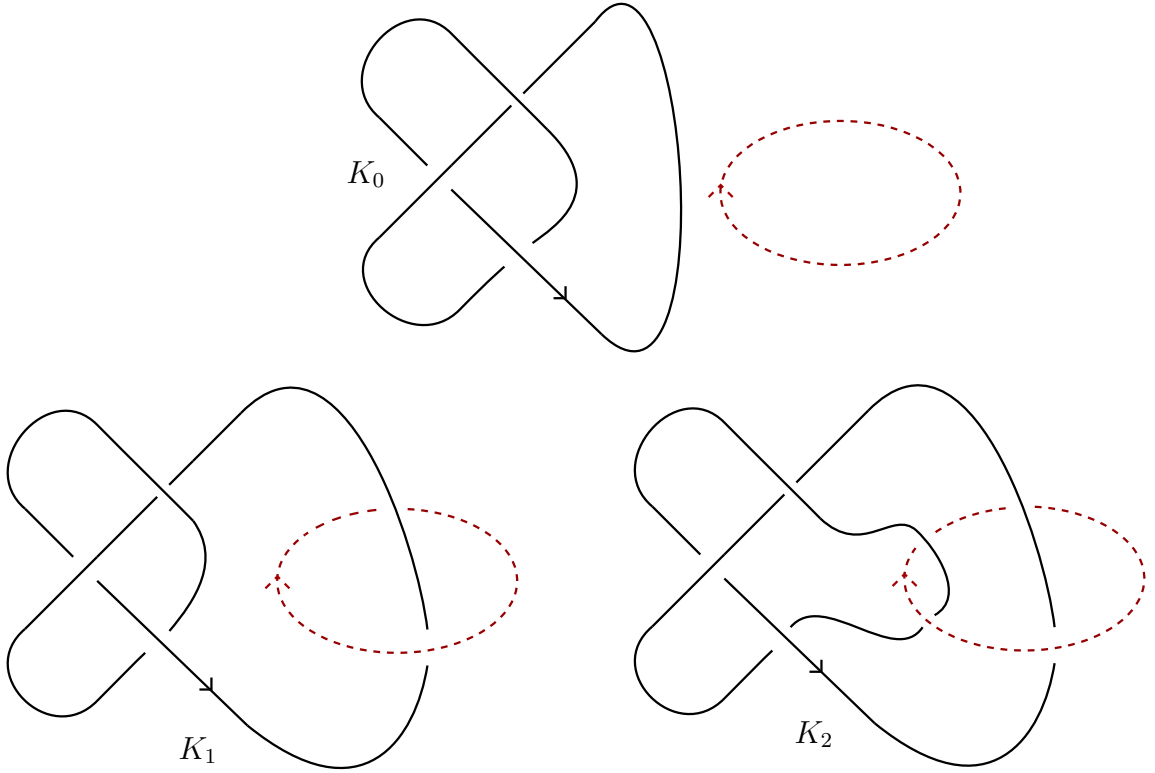


Figure 7: The knots K_0, K_1 and K_2 in the example 3.7

rack of rank 2 with the rack matrix $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix}$. The counting rack invariants of the links

L_1 and L_2 with respect to R are calculated as

$$\Phi_R^{\mathbb{Z}}(L_1) = 12$$

$$\Phi_R^W(L_1) = 1 + 3q_1 + q_2 + 7q_1q_2$$

$$\Phi_R^{\mathbb{Z}}(L_2) = 12$$

$$\Phi_R^W(L_2) = 5 + 3q_1 + q_2 + 3q_1q_2,$$

thus the writhe-enhanced counting invariant distinguishes the links, while the integral rack counting invariant doesn't.

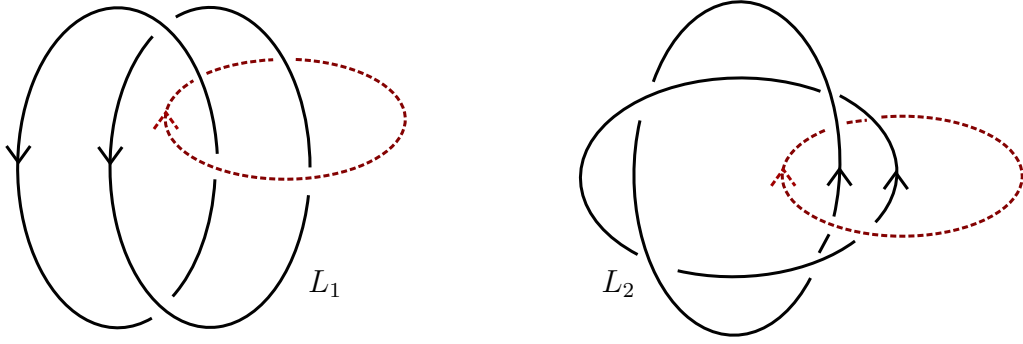


Figure 8: The links L_1 and L_2 in the example 3.8

4 The rack symmetry invariant

In the section 3 we described the idea of the counting rack invariants, which starts by visualizing a rack homomorphism of the fundamental rack of a link as a coloring of its diagram. In the case of non-classical links, however, the augmented fundamental rack $R(D)$ carries an additional structure, which arises as a non-trivial action of the fundamental group.

Lemma 4.1. *The rack automorphism A of the augmented fundamental rack $R(D)$ corresponds to the generator of the fundamental group of $L(p, 1)$.*

Proof. Recall the definition of the augmented fundamental rack of a link in the subsection 1.2. For a framed link L in $Q = L(p, 1)$, denote by $N(L)$ the regular neighbourhood of L and let $Q_0 = \text{closure}(L(p, 1) - N(L))$.

By the proposition 2.2, $R(D)$ is given by the general presentation

$$[\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L, x_i^a = x_{m+i} \text{ for } i = 1, \dots, d\}, \\ \{(x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}) a \equiv a (x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}), a^p \equiv x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}\} \text{ if } d > 0, \text{ and}$$

$$[\{x_1, \dots, x_{m+d}\}, \{a\} : \{\text{crossing relations of } L\}, \{a^p \equiv 1\}]$$

otherwise. By [4, page 379], the presentation of $\pi_1(Q_0)$ may then be obtained as

$$\langle x_1, \dots, x_{m+d}, a : \text{Wirtinger relations of } L, a^{-1}x_i a = x_{m+i} \text{ for } i = 1, \dots, d, \\ (x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}) a = a (x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1}), a^p = x_d^{\epsilon_d} \dots x_2^{\epsilon_2} x_1^{\epsilon_1} \rangle \text{ if } d > 0, \text{ and}$$

$\langle x_1, \dots, x_{m+d}, a : \text{Wirtinger relations of } L, a^p = 1 \rangle$ otherwise. Note that in the presentation of $R(D)$, the generator x_i represents a homotopy class of a path γ , while in the presentation of $\pi_1(Q_0)$, we denote by x_i the homotopy class of the loop $\bar{\gamma} \cdot m_{\gamma(0)} \cdot \gamma$ - this is a loop which encircles the arc x_i in the diagram of L .

Consider the following part of the long exact sequence of the pair (Q, Q_0) :

$$\dots \longrightarrow \pi_2(Q) \longrightarrow \pi_2(Q, Q_0) \xrightarrow{\partial} \pi_1(Q_0) \xrightarrow{i} \pi_1(Q) \longrightarrow \dots$$

Every generator x_i of $\pi_1(Q_0)$ represents a boundary of a meridinal disc of the link L , thus $x_i = \partial d_i$ for some $d_i \in \pi_2(Q, Q_0)$ and consequently $i(x_i) = 0 \in \pi_1(Q)$. The operator generator a (which represents the rack automorphism A), however, is not trivial in $Q = L(p, 1)$: it represents the boundary of a meridinal disc of the exceptional fiber of $L(p, 1)$, thus $i(a) \neq 0$ and $i(a)^p = 0$. \square

We have shown that the rack automorphism A originates from the action of the fundamental group $\pi_1(L(p, 1))$ on the augmented fundamental rack $R(D)$. By the definition 1.7, every rack homomorphism $f: R(D) \rightarrow X$ has an induced action on the image $f(R(D)) \subset X$. If X is a finite rack, this action may be given as an element of the symmetric group $S_{|X|}$.

Let X be a finite rack with the underlying set $\{1, 2, \dots, m\}$ and let L be a link in $L(p, 1)$ with the fundamental rack $R(D)$. Any rack homomorphism $f: R(D) \rightarrow X$ defines a permutation $\sigma_f \in S_m$ by

$$\sigma_f(k) = \begin{cases} f(A(x)) & \text{if } k = f(x), \\ k & \text{if } k \notin f(R(D)). \end{cases}$$

This permutation is well defined by the definition of the rack homomorphism 1.7. Denote by $\text{ord}(\sigma_f)$ the order of the permutation $\sigma_f \in S_m$. Instead of just counting the number of homomorphisms, we might count the number of permutations of a given order that these homomorphisms define. In this way we may further enhance the rack counting invariant of the fundamental rack of a link in $L(p, 1)$.

Definition 4.2. *Let X be a finite rack with rack rank N and let L be an n -component link in $L(p, 1)$. The **rack symmetry invariant** of L with respect to X is*

$$\Phi_X^{\text{Sym}}(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^n} \left(\sum_{f \in \text{Hom}(R(D, \mathbf{w}), X)} x^{\text{ord}(\sigma_f) - 1} \right).$$

The **writhe-enhanced rack symmetry invariant** of L with respect to X is

$$\Phi_X^{W, \text{Sym}}(L) = \sum_{\mathbf{w} \in (\mathbb{Z}_N)^n} \left(\sum_{f \in \text{Hom}(R(D, \mathbf{w}), X)} x^{\text{ord}(\sigma_f) - 1} \right) q^{\mathbf{w}}.$$

Example 4.3. *Consider the knots K_0 and K_1 in $L(2, 1)$, whose diagrams are shown in the*

Figure 9. Let R be the rack with the rack matrix $M_R = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{bmatrix}$. The rack rank of R

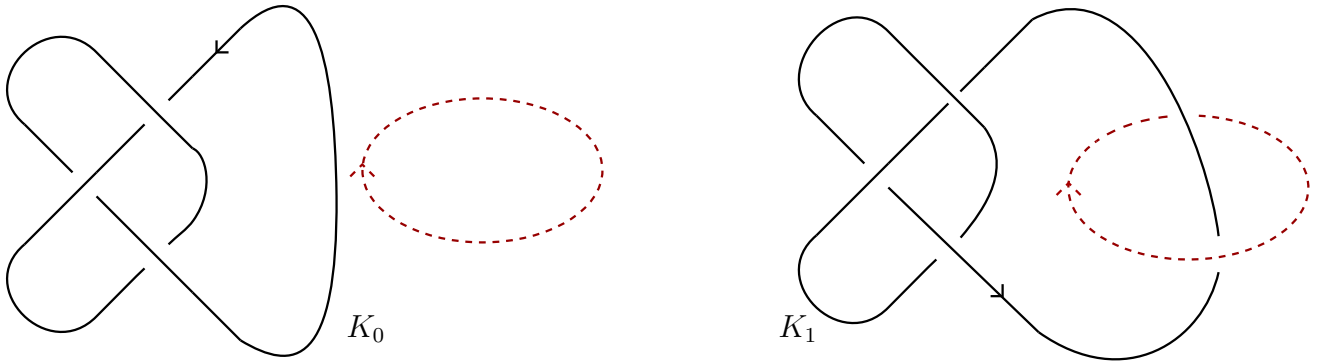


Figure 9: Knots K_0 and K_1 in the example 4.3

equals 1, which means R is a quandle. The rack counting invariant with respect to R does not distinguish K_1 from K_0 , since $\phi_R^{\mathbb{Z}}(K_0) = \phi_R^{\mathbb{Z}}(K_1) = 16$. Calculating the rack symmetry invariant, however, gives

$$\phi_R^{Sym}(K_0) = 16, \quad \phi_R^{Sym}(K_1) = 4 + 12x^2,$$

thus the rack symmetry invariant distinguishes K_1 from K_0 .

Another refinement of the rack counting invariant are the rack cocycle invariants. These have been defined for the classical links in [8], [2]. The construction might also be generalized by defining the rack cocycle invariant of links in $L(p, 1)$.

The `Python` code for computing the rack counting invariants of links in the lens spaces $L(p, 1)$ will be available to any interested reader upon request.

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