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NONLINEAR SINGULAR PROBLEMS WITH INDEFINITE POTENTIAL TERM

N.S. PAPAGEORGIOU, V.D. RĂDULESCU, AND D.D. REPOVŠ

Abstract. We consider a nonlinear Dirichlet problem driven by a nonhomogeneous differential operator plus an indefinite potential. In the reaction we have the competing effects of a singular term and of concave and convex nonlinearities. In this paper the concave term is parametric. We prove a bifurcation-type theorem describing the changes in the set of positive solutions as the positive parameter λ varies. This work continues our research published in arXiv:2004.12583, where $\xi \equiv 0$ and in the reaction the parametric term is the singular one.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous parametric singular problem:

$$(P_\lambda) \quad \left\{ \begin{array}{l} -\mathrm{div}\, a(Du(z)) + \xi(z)u(z)^{p-1} = \vartheta(u(z)) + \lambda u(z)^{q-1} + f(z,u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ \lambda > 0, \ 1 < q < p < \infty. \end{array} \right\}$$

The map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the differential operator of (P_λ) is strictly monotone, continuous (hence maximal monotone, too) and satisfies certain other regularity and growth conditions which are listed in hypotheses $H(a)$ below (see Section 2). These conditions are general enough to incorporate in our framework many differential operators of interest such as the p-Laplacian and the (p,q)-Laplacian (that is, the sum of a p-Laplacian and a q-Laplacian). The operator $u \mapsto \mathrm{div}\, a(Du)$ is not homogeneous and this is a source of difficulties in the analysis of problem (P_λ). The potential function $\xi \in L^\infty(\Omega)$ is indefinite (that is, sign changing). So the operator $u \mapsto -\mathrm{div}\, a(Du) + \xi(z)|u|^{p-2}u$ is not coercive and this is one more difficulty in the analysis of problem (P_λ). In the reaction (the right-hand side of (P_λ)), the term $\vartheta(\cdot)$ is singular at $x = 0$, while the perturbation contains the combined effects of a parametric concave term $x \mapsto \lambda x^{q-1}$ $(x \geqslant 0)$ (recall that $q < p$), with $\lambda > 0$ being the parameter and of a Carathéodory function $f(z,x)$ (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$ the mapping $x \mapsto f(z,x)$ is continuous), which is assumed to exhibit $(p-1)$-superlinear growth near $+\infty$, but without satisfying the usual for superlinear problems Ambrosetti-Rabinowitz condition (the AR-condition for short). So in problem (P_λ) we have the competing effects of singular, concave and convex terms.

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Using variational methods related to the critical point theory, combined with suitable truncation, perturbation and comparison techniques, we produce a critical parameter value $\lambda^* > 0$ such that

(i) for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions;
(ii) for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution;
(iii) for all $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.

This work continues our research published in Papageorgiou, Rădulescu & Repovš [16], where $\xi \equiv 0$ and in the reaction the parametric term is the singular one (cf. arXiv:2004.12583). It is also related to the work of Papageorgiou & Smyrlis [17] and Papageorgiou & Winkert [19], where the differential operator is the p-Laplacian, $\xi \equiv 0$ and no concave terms are allowed. Singular p-Laplacian equations with no potential term and reactions of special form were considered by Chu, Gao & Sun [2], Giacomoni, Schindler & Takač [5], Li & Gao [10], Mohammed [12], Perera & Zhang [20], and Papageorgiou, Rădulescu & Repovš [14].

2. Mathematical background and hypotheses

In this section we present the main mathematical tools which we will use in the analysis of problem (P_λ). We also fix our notation and state the hypotheses on the data of the problem.

So, let X be a Banach space, X^* its topological dual, and let $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that
$\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1 + ||u_n||_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence".

This is a compactness-type condition on the functional $\varphi(\cdot)$, which leads to the minimax theory of the critical values of $\varphi(\cdot)$ (see, for example, Papageorgiou, Rădulescu & Repovš [15]). We denote by K_φ the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

The main spaces in the analysis of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. We denote by $||\cdot||$ the norm of $W_0^{1,p}$. By the Poincaré inequality we have

$$||u|| = ||Du||_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\Omega)$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We will also use two additional ordered Banach spaces. The first one is

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This cone is ordered with positive (order) cone

$$K_+ = \{u \in C_0(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} K_+ = \{u \in K_+ : c_u \hat{d} \leqslant u \text{ for some } c_u > 0\},$$

where $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. On account of Lemma 14.16 of Gilbarg & Trudinger [6, p. 355], we have

$$\text{“}c_u \hat{d} \leqslant u \text{ for some } c_u > 0 \text{ if and only if } \hat{c}_u \hat{u}_1 \leqslant u \text{ for some } \hat{c}_u > 0\text{”}, \tag{1}$$

with $\hat{u}_1$ being the positive, L^p-normalized (that is, $||\hat{u}_1||_p = 1$) eigenfunction corresponding to the principal eigenvalue $\hat{\lambda}_1 > 0$ of the Dirichlet p-Laplacian. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski & Papageorgiou [4, pp.737-738]), imply that $\hat{u}_1 \in \operatorname{int} C_+$.

The second ordered space is $C^1(\overline{\Omega})$ with positive (order) cone

$$\hat{C}_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega},\ \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Clearly, this cone has a nonempty interior.

Concerning ordered Banach spaces with an order cone which has a nonempty interior (solid order cone), we have the following result which will be useful in our analysis (see Papageorgiou, Rădulescu & Repovš [15, Proposition 4.1.22]).

Proposition 2.1. *If X is an ordered Banach space with positive (order) cone K, $\operatorname{int} K \neq \emptyset$, and $e \in \operatorname{int} K$, then for every $u \in X$ we can find $\lambda_u > 0$ such that $\lambda_u e - u \in K$.*

Let $l \in C^1(0, \infty)$ with $l(t) > 0$ for all $t > 0$. We assume that

$$\begin{aligned} &0 < \hat{c} \leqslant \frac{l'(t)t}{l(t)} \leqslant c_0, c_1 t^{p-1} \leqslant l(t) \leqslant c_2[t^{s-1} + t^{p-1}] \\ &\text{for all } t > 0, \text{ and some } c_1, c_2 > 0, 1 \leqslant s < p. \end{aligned} \tag{2}$$

Then the conditions on the map $a(\cdot)$ are the following:

$H(a) : a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, +\infty)$, $t \mapsto a_0(t)$ is strictly increasing on $(0, +\infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$ and

$$\lim_{t \to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$ such that

$$|\nabla a(y)| \leqslant c_3 \frac{l(|y|)}{|y|} \text{ for all } y \in \mathbb{R}^N \backslash \{0\};$$

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \dfrac{l(|y|)}{|y|} |\xi|^2$ for all $y \in \mathbb{R}^n \backslash \{0\}$, $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \displaystyle\int_0^t a_0(s) s ds$, then there exists $\tau \in (q, p]$ such that

$$\limsup_{t \to 0^+} \frac{\tau G_0(t)}{t^\tau} \leqslant c^*$$

and $0 \leqslant pG_0(t) - a_0(t)t^2$ for all $t > 0$.

Remark 2.1. *Hypotheses $H(a)(i),(ii),(iii)$ are dictated by the nonlinear regularity theory of Lieberman [10] and the nonlinear maximum principle of Pucci & Serrin* [21]. *Hypothesis $H(a)(iv)$ serves the needs of our problem, but in fact, it is a mild condition and it is satisfied in all cases of interest (see the examples below). These conditions were used by Papageorgiou & Rădulescu* [13] *and by Papageorgiou, Rădulescu & Repovš* [16].

Hypotheses $H(a)$ imply that the primitive $G_0(\cdot)$ is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently $G(\cdot)$ is convex, $G(0) = 0$ and

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N\backslash\{0\},\ \nabla G(0) = 0,$$

that is, $G(\cdot)$ is the primitive of $a(\cdot)$. From the convexity of $G(\cdot)$ we have

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{3}$$

Using hypotheses $H(a)(i),(ii),(iii)$ and (2), we can easily obtain the following lemma, which summarizes the main properties of the map $a(\cdot)$.

Lemma 2.2. *If hypotheses $H(a)(i),(ii),(iii)$ hold, then*

(a) *the map $y \mapsto a(y)$ is continuous, strictly monotone (hence maximal monotone, too);*
(b) $|a(y)| \leqslant c_4(|y|^{s-1} + |y|^{p-1})$ *for some $c_4 > 0$, and all $y \in \mathbb{R}^N$;*
(c) $(a(y), y)_{\mathbb{R}^N} \geqslant \dfrac{c_1}{p-1}|y|^p$ *for all $y \in \mathbb{R}^N$.*

Using this lemma and (3), we obtain the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.3. *If hypotheses $H(a)(i),(ii),(iii)$ hold, then* $\dfrac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1 + |y|^p)$ *for some $c_5 > 0$, and all $y \in \mathbb{R}^N$.*

The examples that follow confirm that the framework provided by hypotheses $H(a)$ is broad and includes many differential operators of interest (see [13]).

Example 2.1. (a) $a(y) = |y|^{p-2}y$ *with* $1 < p < \infty$.
This map corresponds to the p-Laplace differential operator defined by

$$D_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W_0^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$ *with* $1 < q < p < \infty,\ \mu \geqslant 0$.
This map corresponds to the (p,q)-Laplace differential operator defined by

$$D_p u + D_q u \text{ for all } u \in W_0^{1,p}(\Omega).$$

Such operators arise in models of physical processes. We mention the works of Cherfils & Ilyasov [1] *(reaction-diffusion systems) and Zhikov* [22] *(homogenization of composites consisting of two materials with distinct hardening exponents, in elasticity theory).*

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$ *with* $1 < p < \infty$.
This map corresponds to the modified capillary operator.

(d) $a(y) = |y|^{p-2}y\left(1 + \dfrac{1}{1+|y|^p}\right)$ *with* $1 < p < \infty$.

The hypotheses on the potential term $\xi(\cdot)$ and on the singular part $\vartheta(\cdot)$ of the reaction are the following:

$H(\xi)$: $\xi \in L^\infty(\Omega)$.

$H(\vartheta)$: $\vartheta : (0,+\infty) \to (0,+\infty)$ is a locally Lipschitz function such that

(i) for some $\gamma \in (0,1)$ we have
$$0 < c_6 \leqslant \liminf_{x\to 0^+} \vartheta(x)x^\gamma \leqslant \limsup_{x\to 0^+} \vartheta(x)x^\gamma \leqslant c_7;$$

(ii) $\vartheta(\cdot)$ is nonincreasing.

Remark 2.2. *In the literature we almost always encounter the following particular singular term*
$$\vartheta(x) = x^{-\gamma} \text{ for all } x > 0, \text{ with } 0 < \gamma < 1.$$

Of course, hypotheses $H(\vartheta)$ provide a much more general framework and can accomodate also singularities like the ones that follow:
$$\begin{aligned}
&\vartheta_1(x) = x^{-\gamma}\left[1+\ln(1+x)\right], \quad x>0,\\
&\vartheta_2(x) = x^{-\gamma}e^{-x}, \quad x>0\\
&\vartheta_3(x) = \begin{cases} x^{-\gamma}(1-\eta\sin x) & 0<x\leqslant \dfrac{\pi}{2}\\ x^{-\gamma}(1-\eta) & \text{if } \dfrac{\pi}{2}<x\end{cases} \quad \text{with } 0<\gamma<1.
\end{aligned}$$

The following strong comparison principle can be found in Papageorgiou, Rădulescu & Repovš [16, Proposition 6] (see also Papageorgiou & Smyrlis [17, Proposition 4]).

Proposition 2.4. *If hypotheses $H(a), H(\vartheta)$ hold, $\hat{\xi} \in L^\infty(\Omega)$, $\hat{\xi}(z) \geqslant 0$ for almost all $z \in \Omega$, $h_1, h_2 \in L^\infty(\Omega)$ satisfy*
$$0 < c_8 \leqslant h_2(z) - h_1(z) \text{ for almost all } z \in \Omega$$
and $u, v \in C^{1,\alpha}(\overline{\Omega})$ satisfy $0 < u(z) \leqslant v(z)$ for all $z \in \Omega$ and for almost all $z \in \Omega$ we have

- $-\operatorname{div} a(Du(z)) - \vartheta(u(z)) + \xi(z)u(z)^{p-1} = h_1(z)$
- $-\operatorname{div} a(Dv(z) - \vartheta(v(z)) + \xi(z)v(z)^{p-1} = h_2(z)$,

then $v - u \in \operatorname{int} \hat{C}_+$.

In what follows, p^* is the critical Sobolev exponent corresponding to p, that is,
$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p<N.\\ +\infty & \text{if } N\leqslant p.\end{cases}$$

Now we introduce our hypotheses on the nonlinearity $f(z,x)$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}_+$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) $f(z,x) \leqslant a(z)(1+x^{r-1})$ for almost all $z \in \Omega$, and all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii) if $F(z,x) = \displaystyle\int_0^x f(z,s)ds$,

then $\displaystyle\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) there exists $\sigma \in ((r-p)\max\{\frac{N}{p},1\},p^*)$, $\sigma > q$ such that

$$0 < \hat{\beta}_0 \leqslant \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\sigma} \text{ uniformly for almost all } z\in\Omega;$$

(iv) $\limsup_{x\to 0^+} \frac{f(z,x)}{x^{r-1}} \leqslant \eta_0$ uniformly for almost all $z \in \Omega$;

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the function

$$x \mapsto f(z,x) + \hat{\xi}_\rho x^{\rho-1}$$

is nondecreasing on $[0,\rho]$.

Remark 2.3. *Since our aim is to find positive solutions and the above hypotheses concern the positive semiaxis* $\mathbb{R}_+ = [0,+\infty)$*, we may assume that*

$$f(z,x) = 0, \text{ for almost all } z\in\Omega, \text{ and all } x \leqslant 0. \tag{4}$$

Hypotheses $H(f)(ii),(iii)$ imply that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for almost all } z\in\Omega. \tag{5}$$

So, the nonlinearity $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. However, this superlinearity of $f(z,\cdot)$ is not formulated using the AR-condition. We recall that the AR-condition (unilateral version due to (4)), says that there exist $\gamma > p$ and $M > 0$ such that

$$0 < \gamma F(z,x) \leqslant f(z,x)x \text{ for almost all } z\in\Omega, \text{ and all } x \geqslant M, \tag{6a}$$

$$0 < \mathrm{ess\,inf}_\Omega F(\cdot,M). \tag{6b}$$

If we integrate (6a) and use (6b), we obtain the weaker condition

$$\begin{aligned} & c_9 x^\gamma \leqslant F(z,x) \text{ for almost all } z\in\Omega, \text{ all } x\geqslant M, \text{ and some } c_9>0, \\ \Rightarrow\ & c_9 x^{\gamma-1} \leqslant f(z,x) \text{ for almost all } z\in\Omega, \text{ and all } x \geqslant M. \end{aligned} \tag{7}$$

Therefore the AR-condition implies that $f(z,\cdot)$ exhibits at least $(\gamma-1)$-polynomial growth. Evidently, (7) implies the much weaker condition (5). In this work instead of the standard AR-condition, we employ the less restrictive hypothesis $H(f)(iii)$. In this way we incorporate in our framework also $(p-1)$-superlinear terms with "slower" growth near $+\infty$, which fail to satisfy the AR-condition. The following function satisfies hypotheses $H(f)$ but fails to satisfy the AR-condition (for the sake of simplicity we drop the z-dependence)

$$f(x) = x^{p-1}\ln(1+x) \text{ for all } x \geqslant 0.$$

Finally, let us fix the notation which we will use throughout this work. For $x \in \mathbb{R}$ we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for almost all $z \in \Omega$. It follows that

$$u^\pm \in W_0^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

If $u, v \in W_0^{1,p}(\Omega)$ and $u \leqslant v$, then we define.

$$\begin{aligned} [u,v] &= \{y \in W_0^{1,p}(\Omega) : u(z) \leqslant y(z) \leqslant v(z) \text{ for almost all } z\in\Omega\}, \\ [u) &= \{y \in W_0^{1,p}(\Omega) : u(z) \leqslant y(z) \text{ for almost all } z\in\Omega\}. \end{aligned}$$

Also, by $\mathrm{int}_{C_0^1(\overline{\Omega})}[u,v]$ we denote the interior in the $C_0^1(\overline{\Omega})$-norm topology of the set $[u,v]\cap C_0^1(\overline{\Omega})$.

By $A : W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(\frac{1}{p}+\frac{1}{p'}=1)$ we denote the nonlinear operator defined by

$$\langle A(u),h\rangle = \int_\Omega (a(Du),Dh)_{\mathbb{R}^N}dz \text{ for all } u,h\in W_0^{1,p}(\Omega).$$

We know (see Gasinski & Papageorgiou [4]), that $A(\cdot)$ is continuous, strictly monotone (hence maximal monotone, too) and of type $(S)_+$, that is,

"if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle \leqslant 0$,
then $u_n\to u$ in $W_0^{1,p}(\Omega)$."

We introduce the following two sets related to problem (P_λ):

$\mathcal{L} = \{\lambda>0 :$ problem (P_λ) admits a positive solution$\}$,
$S_\lambda =$ the set of positive solutions for problem (P_λ).

We let $\lambda^*=\sup\mathcal{L}$.

3. Positive solutions

We start by considering the following purely singular problem:

(8) $$-\mathrm{div}\,a(Du(z))) + \xi^+(z)u(z)^{p-1} = \vartheta(u(z)) \text{ in } \Omega,\ u|_{\partial\Omega}=0,\ u>0.$$

From Papageorgiou, Rădulescu & Repovš [16, Proposition 10], we have the following property.

Proposition 3.1. *If hypotheses $H(a),H(\xi),H(\vartheta)$ hold, then problem (8) admits a unique positive solution $v\in\mathrm{int}\,C_+$.*

Let $\beta>||\xi||_\infty$. Then hypotheses $H(f)(i),(iv)$ and since $1<q<p<r$, imply that we can find $c_{10},c_{11}>0$ such that

(9) $$\lambda x^{q-1}+f(z,x)\leqslant \lambda c_{10}x^{q-1}+c_{11}x^{r-1}-\beta x^{p-1} \text{ for almost all } z\in\Omega, \text{ and all } x\geqslant 0.$$

Let $k_\lambda(x)=\lambda c_{10}x^{q-1}+c_{11}x^{r-1}-\beta x^{p-1}$ for all $x\geqslant 0$. With $v\in\mathrm{int}\,C_+$ from Proposition 3.1, we consider the following auxiliary Dirichlet problem:

$(10)_\lambda$ $$\left\{\begin{array}{l} -\mathrm{div}\,a(Du(z))+\xi(z)u(z)^{p-1}=\vartheta(v(z))+k_\lambda(u(z)) \text{ in } \Omega \\ u|_{\partial\Omega}=0,\ u>0. \end{array}\right\}$$

For this problem we prove the following result.

Proposition 3.2. *If hypotheses $H(a),H(\xi),H(\vartheta)$ hold, then for all small enough $\lambda>0$ problem $(10)_\lambda$ has a smallest positive solution*

$$\overline{u}_\lambda\in\mathrm{int}\,C_+.$$

Proof. Recall that $v\in\mathrm{int}\,C_+$ (see Proposition 3.1). Hence $v\in\mathrm{int}\,K_+$ (see (1)). For $s>N$ we consider the function $\hat{u}_1^{1/s}\in K_+$. According to Proposition 2.1, we can find $\mu>0$ such that

(11) $$\begin{aligned} &\hat{u}_1^{1/s}\leqslant \mu v,\\ \Rightarrow\quad & v^{-\gamma}\leqslant \mu^\gamma\hat{u}_1^{-\gamma/s}. \end{aligned}$$

From the Lemma in Lazer & McKenna [9, p. 726], we have

$$\begin{aligned} & \hat{u}_1^{-\gamma/s} \in L^s(\Omega), \\ \Rightarrow \quad & v^{-\gamma} \in L^s(\Omega) \text{ (see (11))}. \end{aligned} \tag{12}$$

Hypotheses $H(\vartheta)$ imply that we can find $c_{12} > 0$ and $\delta > 0$ such that

$$0 \leqslant \vartheta(x) \leqslant c_{12}x^{-\gamma} \text{ for all } 0 \leqslant x \leqslant \delta \text{ and } 0 \leqslant \vartheta(x) \leqslant \vartheta(\delta) \text{ for all } x > \delta. \tag{13}$$

It follows from (12), (13) that

$$\vartheta(v(\cdot)) \in L^s(\Omega) \ (s > N).$$

Let $\hat{k}_\lambda(x) = \lambda c_{10}x^{q-1} + c_{11}x^{r-1}$ for all $x \geqslant 0$ and set $\hat{K}_\lambda(x) = \int_0^x \hat{k}_\lambda(s)ds$. We consider the C^1- functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \psi_\lambda(u) &= \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \int_\Omega \hat{K}_\lambda(u^+)dz - \int_\Omega \vartheta(v)u^+ dz \\ &\quad \text{for all } u \in W_0^{1,p}(\Omega) \\ &\geqslant \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \frac{\lambda c_{10}}{q}||u||_q^q - \frac{c_{11}}{r}||u||_r^r - \int_\Omega \vartheta(v)|u|dz \\ &\quad \text{(see Corollary 2.3)} \\ &\geqslant c_{12}||u||^p - c_{13}[\lambda||u|| + ||u||^r] \\ &\quad \text{for some } c_{12}, c_{13} > 0 \text{ and all } 0 < \lambda \leqslant 1 \text{ (recall that } \beta > ||\xi||_\infty \text{ and } 1 < q < r) \\ &= [c_{12} - c_{13}(\lambda||u||^{1-p} + ||u||^{r-p})]||u||^p. \end{aligned} \tag{14}$$

We introduce the function $\Im_\lambda(t) = \lambda t^{1-p} + t^{r-p}$, $t > 0$. Evidently, $\Im_\lambda \in C^1(0,+\infty)$ and since $1 < p < r$, we see that

$$\Im_\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

So, we can find $t_0 > 0$ such that

$$\begin{aligned} & \Im_\lambda(t_0) = \inf\{\Im_\lambda(t) : t > 0\}, \\ \Rightarrow \quad & \Im'_\lambda(t_0) = 0, \\ \Rightarrow \quad & \lambda(p-1)t_0^{-p} = (r-p)t_0^{r-p-1} \\ \Rightarrow \quad & t_0 = \left[\frac{\lambda(p-1)}{r-p}\right]^{\frac{1}{r-1}}. \end{aligned}$$

Since $\frac{p-1}{r-1} < 1$, it follows that

$$\Im_\lambda(t_0) \to 0 \text{ as } \lambda \to 0^+.$$

So, we can find $\lambda_0 \in (0,1]$ such that

$$\Im_\lambda(t_0) \leqslant \frac{c_{12}}{c_{13}} \text{ for all } \lambda \in (0,\lambda_0].$$

For $\rho = t_0$, we see from (14) that

$$\psi_\lambda|_{\partial\overline{B}_\rho} > 0, \tag{15}$$

where $\overline{B}_\rho = \{u \in W_0^{1,p}(\Omega) : ||u|| \leqslant \rho\}$ and $\partial\overline{B}_\rho = \{u \in W_0^{1,p}(\Omega) : ||u|| = \rho\}$.

We fix $\lambda \in (0,\lambda_0]$. Hypothesis $H(a)(iv)$ implies that we can find $c_0^* > c^*$ and $\delta > 0$ such that

$$G(y) \leqslant \frac{c_0^*}{\tau}|y|^\tau \text{ for all } |y| \leqslant \delta.$$

Let $u \in \operatorname{int} C_+$ and choose small enough $t \in (0,1)$ such that

$$t|Du(z)| \leqslant \delta \text{ for all } z \in \overline{\Omega}.$$

Then we have

$$\psi_\lambda(tu) \leqslant \frac{t^\tau c_0^*}{\tau}||Du||_\tau^\tau + \frac{t^p}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \frac{\lambda t^q}{q}||u||_q^q .$$

Since $q < \tau \leqslant p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\psi_\lambda(tu) < 0, \\ \Rightarrow\quad &\inf_{\overline{B}_\rho} \psi_\lambda < 0. \end{aligned} \tag{16}$$

The functional $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous and by the Eberlein-Smulian theorem and the reflexivity of $W_0^{1,p}(\Omega)$, the set $\overline{B}_\rho$ is sequentially weakly compact. So, by the Weierstrass-Tonelli theorem, we can find $\overline{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(\overline{u}) = \inf\{\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\} \quad (\lambda \in (0,\lambda_0]). \tag{17}$$

From (15), (16) and (17) it follows that

$$\begin{aligned} &0 < ||\overline{u}|| < \rho, \\ &\Rightarrow \psi_\lambda'(\overline{u}) = 0 \text{ (see (17))}, \\ &\Rightarrow \langle A(\overline{u}), h\rangle + \int_\Omega [\xi(z)+\beta]|\overline{u}|^{p-2}\overline{u} h dz = \int_\Omega [\vartheta(v) + \hat{k}_\lambda(\overline{u}^+)] h dz \\ &\text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{18}$$

In (18) we choose $h = -\overline{u}^- \in W_0^{1,p}(\Omega)$. Using Lemma 2.2($c$) and since $\beta > ||\xi||_\infty$ we obtain

$$\begin{aligned} &c_{14}||\overline{u}^-||^p \leqslant 0 \text{ for some } c_{14} > 0, \\ &\Rightarrow \overline{u} \geqslant 0,\ \overline{u} \neq 0. \end{aligned}$$

Then from (18) we have

$$\begin{aligned} &-\operatorname{div} a(D\overline{u}(z)) + \xi(z)\overline{u}(z)^{p-1} = \vartheta(v(z)) + k_\lambda(\overline{u}(z)) \text{ for almost all } z \in \Omega, \\ &\Rightarrow \overline{u} \in W_0^{1,p}(\Omega) \text{ is a positive of problem } (10)_\lambda \text{ for } \lambda \in (0,\lambda_0]. \end{aligned} \tag{19}$$

From (19) and Theorem 7.1 of Ladyzhenskaya & Uraltseva [8, p. 286], we have $\overline{u} \in L^\infty(\Omega)$. Hence $k_\lambda(\overline{u}(\cdot)) \in L^\infty(\Omega)$. Recall that $\vartheta(v(\cdot)) \in L^s(\Omega)$ with $s > N$. From Theorem 9.15 of Gilbarg & Trudinger [6, p. 241], we know that there exists a unique solution $y_0 \in W^{2,s}(\Omega)$ to the following linear Dirichlet problem

$$-\Delta y(z) = \vartheta(v(z)) \text{ in } \Omega,\ y|_{\partial\Omega} = 0.$$

By the Sobolev embedding theorem, we have

$$W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{s} > 0.$$

Let $\eta_0(z) = Dy_0(z)$. Then $\eta_0 \in C^\alpha(\overline{\Omega}, \mathbb{R}^N)$ and we have

$$-\operatorname{div}\left(a(D\overline{u}(z)) - \eta_0(z)\right) + \xi(z)\overline{u}(z)^{p-1} = k_\lambda(\overline{u}(z)) \text{ for almost all } z \in \Omega.$$

The regularity theory of Lieberman [11] implies that $\overline{u} \in C_+ \backslash \{0\}$. Moreover, from (19) we have

$$\begin{aligned} & \operatorname{div} a(D\overline{u}(z)) \leqslant ||\xi||_\infty \overline{u}(z)^{p-1} \text{ for almost all } z \in \Omega, \\ \Rightarrow \quad & \overline{u} \in \operatorname{int} C_+ \end{aligned}$$

(from the nonlinear maximum principle, see Pucci & Serrin [21, pp. 111,120]).

Let $\hat{S_\lambda}$ denote the set of positive solutions of problem $(10)_\lambda$. We have just seen that $\emptyset \neq \hat{S_\lambda} \subseteq \operatorname{int} C_+$ for $\lambda \in (0, \lambda_0]$. Moreover, from Papageorgiou, Rădulescu & Repovš [16, Proposition 18], we know that $\hat{S_\lambda}$ is downward directed (that is, if $u_1, u_2, \in \hat{S_\lambda}$, then we can find $u \in \hat{S_\lambda}$ such that $u \leqslant u_1, u \leqslant u_2$). So, by Lemma 3.10 of Hu & Papageorgiou [7, p. 178], we can find a decreasing sequence $\{\overline{u}_n\}_{n\geqslant 1} \subseteq \hat{S_\lambda}$ such that

$$\inf \hat{S_\lambda} = \inf_{n\geqslant 1} \overline{u}_n.$$

For every $n \in \mathbb{N}$ we have

$$\langle A(\overline{u}_n), h\rangle + \int_\Omega \xi(z)\overline{u}_n^{p-1} h dz = \int_\Omega [\vartheta(v) + k_\lambda(\overline{u}_n)] h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{20}$$

Choosing $h = \overline{u}_n \in W_0^{1,p}$ and since $0 \leqslant \overline{u}_n \leqslant \overline{u}_1$ for all $n \in \mathbb{N}$, using Lemma 2(c), we see that $\{\overline{u}_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we have

$$\overline{u}_n \xrightarrow{w} \overline{u}_\lambda \text{ in } W_0^{1,p}(\Omega). \tag{21}$$

Next, in (20) we choose $h = \overline{u}_n - \overline{u} \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (21). Then

$$\begin{aligned} & \lim_{n\to\infty} \langle A(\overline{u}_n), \overline{u}_n - \overline{u}_\lambda\rangle = 0, \\ \Rightarrow \quad & \overline{u}_n \to \overline{u}_\lambda \text{ in } W_0^{1,p}(\Omega), \ \overline{u}_\lambda \geqslant 0 \\ & \text{(recall that } A(\cdot) \text{ is of type } (S)_+, \text{ see Section 2).} \end{aligned} \tag{22}$$

We pass to the limit as $n \to \infty$ in (20) and use (22). Then

$$\begin{aligned} & \langle A(\overline{u}_\lambda), h\rangle + \int_\Omega \xi(z)\overline{u}_\lambda^{p-1} h dz = \int_\Omega [\vartheta(v) + k_\lambda(\overline{u}_\lambda)] h dz \\ & \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow \quad & \overline{u}_\lambda \text{ is a nonnegative solution of } (10)_\lambda. \end{aligned}$$

Note that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} -\operatorname{div} a(D\overline{u}_n(z)) + \xi^+(z)\overline{u}_n(z)^{p-1} & \geqslant \vartheta(v(z)) + k_\lambda(\overline{u}_n(z)) \\ & \geqslant \vartheta(v(z)) = -\operatorname{div} a(Dv(z)) + \xi^+(z)v(z)^{p-1} \\ & \quad \text{for almost all } z \in \Omega, \\ \Rightarrow v \leqslant \overline{u}_n \text{ for all } n \in \mathbb{N} & \end{aligned}$$

(by the weak comparison principle, see Damascelli [3, Theorem 1.2])

$$\Rightarrow v \leqslant \overline{u}_\lambda \text{ (see (22)), hence } \overline{u}_\lambda \neq 0. \tag{23}$$

Therefore $\overline{u}_\lambda \in \hat{S_\lambda} \subseteq \operatorname{int} C_+$ and $\overline{u}_\lambda = \inf \hat{S_\lambda}$. □

We will use $\bar{u}_\lambda \in \operatorname{int} C_+$ from Proposition 3.2 to show the nonemptiness of $\mathcal{L}$.

Proposition 3.3. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold, then $\mathcal{L} \neq \emptyset$ and $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. From (9) we have

$$\lambda x^{q-1} + f(z,x) \leqslant k_\lambda(x) \text{ for almost all } z \in \Omega, \text{ and all } x \geqslant 0,\ \lambda > 0. \tag{24}$$

For $\lambda \in (0, \lambda_0]$ we have

$$\begin{aligned} -\mathrm{div}\, a(D\bar{u}_\lambda(z)) + \xi(z)\bar{u}_\lambda(z)^{p-1} &= \vartheta(v(z)) + k_\lambda(\bar{u}_\lambda(z)) \\ &\quad \text{(see Proposition 3.2)} \\ &\geqslant \vartheta(\bar{u}_\lambda(z)) + k_\lambda(\bar{u}(z)) \\ &\quad \text{(see (23) and hypothesis } H(\vartheta)(ii)) \\ &\geqslant \vartheta(\bar{u}_\lambda(z)) + \lambda\bar{u}_\lambda(z)^{q-1} + f(z,\bar{u}_\lambda(z)) \qquad (25) \\ &\quad \text{for almost all } z \in \Omega \text{ (see (24))}. \end{aligned}$$

With $\beta > ||\xi||_\infty$ and $\lambda \in (0, \lambda_0]$, we consider the following truncation-perturbation of the reaction in problem (P_λ):

$$\gamma_\lambda(z,x) = \begin{cases} \vartheta(v(z)) + \lambda v(z)^{q-1} + f(z,v(z)) + \beta v(z)^{p-1} & \text{if } x < v(z) \\ \vartheta(x) + \lambda x^{q-1} + f(z,x) + \beta x^{p-1} & \text{if } v(z) \leqslant x \leqslant \bar{u}_\lambda(z) \\ \vartheta(\bar{u}_\lambda(z)) + \lambda\bar{u}_\lambda(z)^{q-1} + f(z,\bar{u}_\lambda(z)) + \beta\bar{u}_\lambda(z)^{p-1} & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \tag{26}$$

This is a Carathéodory function. We set $\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)ds$ and consider the functional $\hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \int_\Omega \Gamma_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Using Proposition 3 of Papageorgiou & Smyrlis [17], we see that $\hat{\sigma}_\lambda \in C^1(W_0^{1,p}(\Omega))$. Also, from (26), Corollary 2.3 and since $\beta > ||\xi||_\infty$, we see that $\hat{\sigma}_\lambda(\cdot)$ is coercive. In addition, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\sigma}_\lambda(u_\lambda) = \inf\{\hat{\sigma}(u) : u \in W_0^{1,p}(\Omega)\}, \\ \Rightarrow\ & \hat{\sigma}'_\lambda(u_\lambda) = 0, \\ \Rightarrow\ & \langle A(u_\lambda, h)\rangle + \int_\Omega [\xi(z)+\beta]|u_\lambda|^{p-2}u_\lambda h dz = \int_\Omega \gamma_\lambda(z,u_\lambda)hdz \qquad (27) \\ & \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned}$$

In (27) first we choose $h = (u_\lambda - \bar{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned} & \left\langle A(u_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \right\rangle + \int_\Omega [\xi(z)+\beta]u_\lambda^{p-1}(u_\lambda - \bar{u}_\lambda)^+ dz \\ =\ & \int_\Omega [\vartheta(\bar{u}_\lambda) + \lambda\bar{u}_\lambda^{q-1} + f(z,\bar{u}_\lambda) + \beta\bar{u}_\lambda^{p-1}](u_\lambda - \bar{u}_\lambda)^+ dz \text{ (see (26))} \\ \leqslant\ & \left\langle A(\bar{u}_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \right\rangle + \int_\Omega [\xi(z)+\beta]\bar{u}_\lambda^{p-1}(u_\lambda - \bar{u}_\lambda)^+ dz \text{ (see (25))}, \\ \Rightarrow\ & u_\lambda \leqslant \bar{u}_\lambda \text{ (since } \beta > ||\xi||_\infty). \end{aligned}$$

Next, in (27) we choose $h = (v - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned} & \left\langle A(u_\lambda), (v-u_\lambda)^+ \right\rangle + \int_\Omega [\xi(z)+\beta]|u_\lambda|^{p-2}u_\lambda (v-u_\lambda)^+ dz \\ = & \int_\Omega [\vartheta(v) + \lambda v^{q-1} + f(z,v) + \beta v^{p-1}](v-u_\lambda)^+ dz \text{ (see (26))} \\ \geqslant & \int_\Omega [\vartheta(v) + \beta v^{p-1}](v-u_\lambda)^+ dz \text{ (since } f \geqslant 0) \\ = & \left\langle A(v), (v-u_\lambda)^+ \right\rangle + \int_\Omega [\xi(z)+\beta] v^{p-1}(v-u_\lambda)^+ dz \text{ (see Proposition 3.1),} \\ \Rightarrow & \ v \leqslant u_\lambda. \end{aligned}$$

So, we have proved that

$$u_\lambda \in [v, \bar{u}_\lambda] \quad (\lambda \in (0, \lambda_0]). \tag{28}$$

It follows from (26), (27) and (28) that

$$-\mathrm{div}\, a(Du_\lambda(z)) + \xi(z)u_\lambda(z)^{p-1} = \vartheta(u_\lambda(z)) + \lambda u_\lambda(z)^{q-1} + f(z, u_\lambda(z))$$
for almost all $z \in \Omega$.

Note that $\vartheta_\lambda(u_\lambda) \leqslant \vartheta(v)$ (see (28) and hypothesis $H(\vartheta)(ii)$) and $\vartheta(v) \in L^s(\Omega)$. So, as before (see the proof of Proposition 3.2), we infer that

$$u_\lambda \in \mathrm{int}\, C_+ .$$

Therefore we have seen that

$$\begin{aligned} & (0, \lambda_0] \subseteq \mathcal{L}, \text{ hence } \mathcal{L} \neq \emptyset \\ \text{and} \quad & S_\lambda \subseteq \mathrm{int}\, C_+. \end{aligned}$$

The proof is now complete. □

For $\eta > 0$, let $\tilde{u}_\eta \in \mathrm{int}\, C_+$ be the unique solution of the following Dirichlet problem

$$-\mathrm{div}\, a(Du(z)) + \xi^+(z)u(z)^{p-1} = \eta \text{ in } \Omega, \ u|_{\partial\Omega} = 0.$$

By Proposition 9 of Papageorgiou, Rădulescu & Repovš [16], we see that given $u \in S_\lambda \subseteq \mathrm{int}\, C_+$ (that is, $\lambda \in \mathcal{L}$), we can find small $\eta > 0$ such that

$$\tilde{u}_\eta \leqslant u \text{ and } \eta \leqslant \vartheta(\tilde{u}_\eta). \tag{29}$$

We will use this to obtain a lower bound for the elements of S_λ.

Proposition 3.4. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold and $\lambda \in \mathcal{L}$, then $v \leqslant u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda \subseteq \mathrm{int}\, C_+$. Then on account of (29) we can define the following Carathéodory function

$$e(z,x) = \begin{cases} \vartheta(\tilde{u}_\eta(z)) & \text{if } x < \tilde{u}_\eta(z) \\ \vartheta(x) & \text{if } \tilde{u}_\eta(z) \leqslant x \leqslant u(z) \\ \vartheta(u(z)) & \text{if } u(z) < x. \end{cases} \tag{30}$$

We set $E(z,x) = \int_0^x e(z,s)ds$ and consider the functional $\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega \xi^+(z)|u|^p dz - \int_\Omega E(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

As before, Proposition 3 of Papageorgiou & Smyrlis [17] implies that $\mu \in C^1(W_0^{1,p}(\Omega))$. The coercivity of $\mu(\cdot)$ (see (30)) and the sequential weak lower semicontinuity guarantee the existence of $\tilde{v} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \mu(\tilde{v}) = \inf\{\mu(u) : u \in W_0^{1,p}(\Omega)\}, \\ \Rightarrow\ & \mu'(\tilde{v}) = 0, \\ \Rightarrow\ & \langle A(\tilde{v}), h\rangle + \int_\Omega \xi^+(z)|\tilde{v}|^{p-2}\tilde{v}hdz = \int_\Omega e(z,\tilde{v})hdz \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{31}$$

In (31) we choose $h = (\tilde{v} - u)^+ \in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned} & \left\langle A(\tilde{v}), (\tilde{v}-u)^+\right\rangle + \int_\Omega \xi^+(z)\tilde{v}^{p-1}(\tilde{v}-u)^+dz \\ = & \int_\Omega \vartheta(u)(\tilde{v}-u)^+dz \text{ (see (30))} \\ \leqslant & \int_\Omega [\vartheta(u) + \lambda u^{q-1} + f(z,u)](\tilde{v}-u)^+dz \text{ (since } u \in \operatorname{int} C_+,\ f \geqslant 0) \\ \leqslant & \left\langle A(u), (\tilde{v}-u)^+\right\rangle + \int_\Omega \xi^+(z)u^{p-1}(\tilde{v}-u)^+dz \text{ (since } u \in S_\lambda), \\ \Rightarrow\ & \tilde{v} \leqslant u. \end{aligned}$$

Similarly, if in (31) we choose $h = (\tilde{u}_\eta - \tilde{v})^+ \in W_0^{1,p}(\Omega)$, then we have

$$\begin{aligned} & \left\langle A(\tilde{v}), (\tilde{u}_\eta-\tilde{v})^+\right\rangle + \int_\Omega \xi^+(z)|\tilde{v}|^{p-2}\tilde{v}(\tilde{u}_\eta-\tilde{v})^+dz \\ = & \int_\Omega \vartheta(\tilde{u}_\eta)(\tilde{u}_\eta-\tilde{v})^+dz \text{ (see (30))} \\ \geqslant & \int_\Omega \eta(\tilde{u}_\eta-\tilde{v})^+dz \text{ (see (29))} \\ = & \left\langle A(\tilde{u}_\eta), (\tilde{u}_\eta-v)^+\right\rangle + \int_\Omega \xi^+(z)\tilde{u}_\eta^{p-1}(\tilde{u}_\eta-v)^+dz, \\ \Rightarrow\ & \tilde{u}_\eta \leqslant \tilde{v}. \end{aligned}$$

So, we have proved that

$$\tilde{v} \in [\tilde{u}_\eta, u]. \tag{32}$$

It follows from (30), (31), (32) that $\tilde{v}$ is a positive solution of (18). Then on account of Proposition 3.1, we have

$$\begin{aligned} & \tilde{v} = v \in \operatorname{int} C_+, \\ \Rightarrow\ & v \leqslant u \text{ for all } u \in S_\lambda \text{ (see (32))}. \end{aligned}$$

The proof is now complete. □

Next, we show a structural property of the set $\mathcal{L}$, namely that $\mathcal{L}$ is an interval. Moreover, we establish a kind of strong monotonicity property for the solution set S_λ.

Proposition 3.5. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold, $\lambda \in \mathcal{L}, 0 < \mu < \lambda$ and $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. From Proposition 3.4 we know that $v \leqslant u_\lambda$. Then with $\beta > ||\xi||_\infty$ we can define the following truncation-perturbation of the reaction in problem (P_μ):

(33)
$$e_\mu(z,x) = \begin{cases} \vartheta(v(z)) + \mu v(z)^{q-1} + f(z,v(z)) + \beta v(z)^{p-1} & \text{if } x < v(z) \\ \vartheta(x) + \mu x^{q-1} + f(z,x) + \beta x^{p-1} & \text{if } v(z) \leqslant x \leqslant u_\lambda(z) \\ \vartheta(u_\lambda(z)) + \mu u_\lambda(z)^{q-1} + f(z,u_\lambda(z)) + \beta u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x. \end{cases}$$

Evidently, $e_\mu(z,x)$ is a Carathéodory function. We set $E_\mu(z,x) = \int_0^x e_\mu(z,s)ds$ and consider the C^1-functional $\hat{\psi}_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\mu(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \int_\Omega E_\mu(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Clearly, $\hat{\psi}_\mu(\cdot)$ is coercive (see (33) and recall that $\beta > ||\xi||_\infty$). It is also sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{\psi}_\mu(u_\mu) = \inf\{\hat{\psi}_\mu(u) : u \in W_0^{1,p}(\Omega)\},$$
$$\Rightarrow \quad \hat{\psi}'_\mu(u_\mu) = 0,$$
$$(34)\quad \langle A(u_\mu), h\rangle + \int_\Omega [\xi(z)+\beta]|u_\mu|^{p-2}u_\mu h dz = \int_\Omega e_\mu(z,u_\mu)hdz \text{ for all } h \in W_0^{1,p}(\Omega).$$

In (34) we first use $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \left\langle A(u_\mu), (u_\mu - u_\lambda)^+\right\rangle + \int_\Omega [\xi(z)+\beta]u_\mu^{p-1}(u_\mu - u_\lambda)^+ dz \\ = & \int_\Omega [\vartheta(u_\lambda) + \mu u_\lambda^{q-1} + f(z,u_\lambda) + \beta u_\lambda^{p-1}](u_\mu - u_\lambda)^+ dz \text{ (see (33))} \\ \leqslant & \int_\Omega [\vartheta(u_\lambda) + \lambda u_\lambda^{q-1} + f(z,u_\lambda) + \beta u_\lambda^{p-1}](u_\mu - u_\lambda)^+ dz \text{ (since } \lambda > \mu) \\ = & \left\langle A(u_\lambda), (u_\mu - u_\lambda)^+\right\rangle + \int_\Omega [\xi(z)+\beta]u_\lambda^{p-1}(u_\mu - u_\lambda)^+ dz \text{ (since } u_\lambda \in S_\lambda), \\ \Rightarrow \quad & u_\mu \leqslant u_\lambda \text{ (recall that } \beta > ||\xi||_\infty). \end{aligned}$$

Next, in (34) we use $h = (v - u_\mu)^+ \in W_0^{1,p}(\Omega)$. Then from Proposition 3.1 and since $f \geqslant 0$, we obtain

$$v \leqslant u_\mu\,.$$

We have proved that

$$u_\mu \in [v, u_\lambda]. \tag{35}$$

It follows from (33), (34), (35) that $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ and so $\mu \in \mathcal{L}$.

Let $\rho = ||u_\lambda||_\infty$ and let $\hat{\xi}_\rho > 0$ as postulated by hypothesis $H(f)(v)$. We have

$$\begin{aligned} & -\operatorname{div} a(Du_\mu) + [\xi(z)+\hat{\xi}_\rho]u_\mu^{p-1} - \vartheta(u_\mu) \\ = \quad & \mu u_\mu^{q-1} + f(z,u_\mu) + \hat{\xi}_\rho u_\mu^{p-1} \\ \leqslant \quad & \lambda u_\lambda^{q-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \text{ (see hypothesis } H(f)(vi),\ (35) \text{ and recall that } \mu < \lambda) \\ = (36) & -\operatorname{div} a(Du_\lambda) + [\xi(z)+\hat{\xi}_\rho]u_\lambda^{p-1} - \vartheta(u_\lambda). \end{aligned}$$

From (36) and Proposition 4 of Papageorgiou & Smyrlis [17], we obtain

$$u_\lambda - u_\mu \in \operatorname{int} C_+.$$

The proof is now complete. ∎

Proposition 3.6. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold, then $\lambda^* < +\infty$.*

Proof. Recall that by hypotheses $H(f)(ii),(iii)$, we have

$$\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for almost all } z\in\Omega.$$

So, we can find $M>0$ such that

$$f(z,x)\geqslant x^{p-1} \text{ for almost all } z\in\Omega, \text{ and all } x\geqslant M. \tag{37}$$

Hypotheses $H(\vartheta)$ imply that we can find small $\delta\in(0,1]$ such that

$$\vartheta(x)\geqslant\vartheta(\delta)\geqslant 1\geqslant\delta^{p-1}\geqslant x^{p-1} \text{ for all } x\in(0,\delta]\,. \tag{38}$$

Finally, hypotheses $H(f)(i),(v)$ imply that we can find big $\lambda_0>0$ such that

$$\lambda_0 x^{q-1}+f(z,x)\geqslant x^{p-1} \text{ for almost all } z\in\Omega \text{ and all } \delta\leqslant x\leqslant M. \tag{39}$$

Combining (37), (38), (39) we have

$$\vartheta(x)+\lambda_0 x^{q-1}+f(z,x)\geqslant x^{p-1} \text{ for almost all } z\in\Omega \text{ and all } x\geqslant 0. \tag{40}$$

Let $\lambda>\lambda_0$ and assume that $\lambda\in\mathcal{L}$. Then according to Proposition 3.3 we can find $u_\lambda\in S_\lambda\subseteq \operatorname{int} C_+$. Let $\Omega_0\subseteq\Omega$ be an open set with $\overline{\Omega}_0\subseteq\Omega$ and C^2-boundary $\partial\Omega_0$. We have

$$0<m_0=\min_{\overline{\Omega}_0} u_\lambda.$$

For $\epsilon>0$, let $m_0^\epsilon=m_0+\epsilon$ and with $\rho=||u_\lambda||_\infty$, let $\hat{\xi}_\rho>0$ be as postulated by hypothesis $H(f)(v)$. We can always take $\hat{\xi}_\rho>||\xi||_\infty$. We have

$$\begin{aligned}
& -\operatorname{div} a(Dm_0^\epsilon)+[\xi(z)+\hat{\xi}_\rho](m_0^\epsilon)^{p-1}-\vartheta(m_0^\epsilon)\\
\leqslant\ & [\xi(z)+\hat{\xi}_\rho]m_0^{p-1}+\chi(\epsilon)-\vartheta(m_0)\\
& \text{with } \chi(\epsilon)\to 0^+ \text{ as } \epsilon\to 0^+ \text{ (see hypotheses } H(\vartheta))\\
<\ & [\xi(z)+\hat{\xi}_\rho]u_\lambda^{p-1}+u_\lambda^{p-1}-\vartheta(u_\lambda)+\chi(\epsilon)\\
<\ & [\xi(z)+\hat{\xi}_\rho]u_\lambda^{p-1}+\lambda_0 u_\lambda^{p-1}+f(z,u_\lambda)+\chi(\epsilon) \text{ (see (40))}\\
=\ & [\xi(z)+\hat{\xi}_\rho]u_\lambda^{p-1}+\lambda u_\lambda^{q-1}+f(z,u_\lambda)-(\lambda-\lambda_0)u_\lambda^{q-1}+\chi(\epsilon)\\
<\ & [\xi(z)+\hat{\xi}_\rho]u_\lambda^{p-1}+\lambda u_\lambda^{q-1}+f(z,u_\lambda) \text{ for } \epsilon>0 \text{ small enough}\\
=\ & -\operatorname{div} a(Du_\lambda)+[\xi(z)+\hat{\xi}_p]u_\lambda^{p-1}-\vartheta(u_\lambda) \text{ for almost all } z\in\Omega_0 \text{ (recall that } u_\lambda\in S_\lambda).
\end{aligned} \tag{41}$$

Then from (40) and Proposition 2.4, we see that for small enough $\epsilon>0$ we have

$$u_\lambda-m_0^\epsilon\in\operatorname{int}\hat{C}_+(\overline{\Omega}_0),$$

which contradicts the definition of m_0. Hence $\lambda\notin\mathcal{L}$ and so $\lambda^*\leqslant\lambda_0<+\infty$. ∎

By Propositions 3.5 and 3.6 it follows that

$$(0,\lambda^*)\subseteq\mathcal{L}\subseteq(0,\lambda^*]\,. \tag{42}$$

Proposition 3.7. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold and $\lambda\in(0,\lambda^*)$, then problem (P_λ) admits at least two positive solutions*

$$u_0,\hat{u}\in\operatorname{int}C_+,\ u_0\neq\hat{u}.$$

Proof. Let $0 < \mu < \lambda < \eta < \lambda^*$. We have $\mu, \eta \in \mathcal{L}$ (see (42)). On account of Proposition 3.5 we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+,\ u_0 \in S_\lambda \subseteq \operatorname{int} C_+,\ u_\eta \in S_\eta \subseteq \operatorname{int} C_+$ such that

$$\begin{aligned} & u_0 - u_\eta \in \operatorname{int} C_+ \text{ and } u_\eta - u_0 \in \operatorname{int} C_+, \\ \Rightarrow \quad & u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[u_\mu, u_\eta]. \end{aligned} \tag{43}$$

With $\beta > ||\xi||_\infty$, we introduce the Carathéodory function $d_\lambda(z,x)$ defined by

$$d_\lambda(z,x) = \begin{cases} \vartheta(u_\mu(z)) + \lambda u_\mu(z)^{q-1} + f(z, u_\mu(z)) + \beta u_\mu(z)^{p-1} & \text{if } x \leqslant u_\mu(z) \\ \vartheta(x) + \lambda x^{q-1} + f(z,x) + \beta x^{p-1} & \text{if } u_\mu(z) < x. \end{cases} \tag{44}$$

We set $D_\lambda(z,x) = \displaystyle\int_0^x d_\lambda(z,s)ds$ and consider the functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \int_\Omega D_\lambda(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We know that $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$ (see Papageorgiou & Smyrlis [17, Proposition 3]). Also, let

$$\hat{d}_\lambda(z,x) = \begin{cases} d_\lambda(z,x) & \text{if } x \leqslant u_\eta(z) \\ d_\lambda(z, u_\eta(z)) & \text{if } u_\eta(z) < x. \end{cases} \tag{45}$$

This is a Carathéodory function. We set $\hat{D}_\lambda(z,x) = \displaystyle\int_0^x \hat{d}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]|u|^p dz - \int_\Omega \hat{D}_\lambda(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Using (44) and (45) and the nonlinear regularity theory (see the proof of Proposition 3.3), we show that

$$K_{\varphi_\lambda} \subseteq [u_\mu) \cap \operatorname{int} C_+, \tag{46}$$

$$K_{\hat{\varphi}_\lambda} \subseteq [u_\mu, u_\eta] \cap \operatorname{int} C_+. \tag{47}$$

From (47) we see that we can assume that

$$K_{\hat{\varphi}_\lambda} = \{u_0\} \tag{48}$$

or otherwise we already have a second positive solution for (P_λ) (see (45)) and so we are done.

Clearly, $\hat{\varphi}_\lambda(\cdot)$ is coercive (see (45)) and sequentially weakly lower semicontinuous. So, we can find $\hat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{\varphi}_\lambda(\hat{u_0}) = \inf\{\hat{\varphi}_\lambda(u) : u \in W_0^{1,p}(\Omega)\}, \\ \Rightarrow \quad & \hat{u}_0 \in K_{\hat{\varphi}_\lambda}, \\ \Rightarrow \quad & \hat{u}_0 = u_0 \text{ (see (48))}. \end{aligned} \tag{49}$$

But from (44) and (45) we see that

$$\hat{\varphi}_\lambda|_{[u_\mu,u_\eta]} = \varphi_\lambda|_{[u_\mu,u_\eta]}. \tag{50}$$

It follows from (43), (49), (50) that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda,$$

$$\Rightarrow \quad u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see [5]).} \tag{51}$$

On account of (44) and (46), we may assume that

$$K_{\varphi_\lambda} \text{ is finite.} \tag{52}$$

Otherwise we already have an infinity of positive smooth solutions. From (51), (52) and Theorem 5.7.6 of Papageorgiou, Rădulescu & Repovš [15], we see that we can find small $\rho \in (0,1)$ such that

$$\varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : ||u - u_0|| = \rho\} = m_\rho. \tag{53}$$

Hypothesis $H(f)(ii)$ and Corollary 2.3 imply that if $u \in \operatorname{int} C_+$, then

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{54}$$

Claim 3.1. *φ_λ satisfies the C-condition.*

Consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi_\lambda(u_n)| \leqslant c_{15} \text{ for some } c_{15} > 0, \text{ and all } n \in \mathbb{N}, \tag{55}$$

$$(1 + ||u_n||)\varphi_\lambda'(u_n) \to \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{56}$$

From (56) we have

$$\left|\langle A(u_n), h\rangle + \int_\Omega [\xi(z) + \beta]|u_n|^{p-2}u_n h dz - \int_\Omega d_\lambda(z, u_n) h dz\right| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \tag{57}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\epsilon_n \to 0^+$.

In (57) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. From (44) and Lemma 2.2, we have

$$\frac{c_1}{p-1}||Du_n^-||_p^p + \int_\Omega [\xi(z) + \beta](u_n^-)^p dz \leqslant \epsilon_n + c_{16}||u_n^-||$$

for some $c_{16} > 0$, and all $n \in \mathbb{N}$,

$$\Rightarrow \quad \{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (recall that } \beta > ||\xi||_\infty). \tag{58}$$

Next, in (57) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$-\int_\Omega (a(Du_n^+), Du_n^+)_{\mathbb{R}^N} dz - \int_\Omega [\xi(z) + \beta](u_n^+)^p dz + \int_\Omega [\lambda(u_n^+)^q + f(z, u_n^+)u_n^+] dz \leqslant c_{17} \tag{59}$$

for some $c_{17} > 0$ and all $n \in \mathbb{N}$ (see (44) and hypothesis $H(\vartheta)(ii)$).

From (55) and (58) we obtain

$$\int_\Omega pG(Du_n^+)dz + \int_\Omega [\xi(z) + \beta](u_n^+)^p dz - \int_\Omega \left[\frac{\lambda p}{q}(u_n^+)^q + pF(z, u_n^+)\right] dz \leqslant c_{18} \tag{60}$$

for some $c_{18} > 0$ and all $n \in \mathbb{N}$.

Adding (59) and (60) and using hypothesis $H(a)(iv)$, we obtain

$$\int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)] dz \leqslant c_{19} + \lambda \left[\frac{p}{q} - 1\right] ||u_n^+||_q^q \tag{61}$$

for some $c_{19} > 0$, all $n \in \mathbb{N}$.

From hypotheses $H(f)(i),(iii)$ we see that we can find $\hat{\beta}_1 \in (0, \hat{\beta}_0)$ and $c_{20} > 0$ such that

$$\hat{\beta}_1 x^\sigma - c_{20} \leqslant f(z,x)x - pF(z,x) \text{ for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{62}$$

Using (62) in (61) and recalling that $q < \sigma$ (see hypothesis $H(f)(iii)$) we obtain that

$$\{u_n^+\}_{n\geqslant 1} \subseteq L^\sigma(\Omega) \text{ is bounded.} \tag{63}$$

First, suppose that $N \neq p$. It is clear from hypothesis $H(f)(iii)$ that we may assume that $\sigma < r < p^*$ (recall that $p^* = +\infty$ if $N \leqslant p$). Let $t \in (0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\sigma} + \frac{t}{p^*}.$$

From the interpolation inequality (see, for example, Papageorgiou & Winkert [18, Proposition 2.3.17, p.116]), we have

$$\begin{aligned} & ||u_n^+||_r \leqslant ||u_n^+||_\sigma^{1-t} ||u_n^+||_{p^*}^t, \\ \Rightarrow \quad & ||u_n^+||_r^r \leqslant c_{21} ||u_n^+||^{tr} \\ & \text{for some } c_{21} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (63)).} \end{aligned} \tag{64}$$

From hypothesis $H(f)(i)$, we have

$$f(z,x)x \leqslant c_{22}[1 + x^r] \text{ for almost all } z \in \Omega, \text{ all } x \geqslant 0 \text{ and some } c_{22} > 0. \tag{65}$$

In (57) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and use Lemma 2.2. Then

$$\begin{aligned} & \frac{c_1}{p-1} ||Du_n^+||_p^p + \int_\Omega [\xi(z) + \beta](u_n^+)^p dz \leqslant \epsilon_n + \int_\Omega d_\lambda(z, u_n) u_n^+ dz, \\ \Rightarrow \quad & \frac{c_1}{p-1} ||Du_n^+||_p^p \leqslant c_{23} + \int_\Omega [\lambda (u_n^+)^q + f(z, u_n^+) u_n^+] dz \\ & \text{for some } c_{23} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (44))} \\ \leqslant \quad & c_{24}[1 + \lambda ||u_n^+||^q + ||u_n^+||^{tr}] \qquad (66) \\ & \text{for some } c_{24} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (64) and (65)).} \end{aligned}$$

The hypothesis on σ (see $H(f)(iii)$) implies that $tr < p$. Also we have $q < p$. Therefore it follows from (66) that

$$\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{67}$$

If $p = N$, then $p^* = +\infty$ and by the Sobolev embedding theorem, we have that $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leqslant s < \infty$. So, we need to replace in the previous argument p^* by $s > r > \sigma$ big enough. More precisely, as before, let $t \in (0,1)$ be such that

$$\begin{aligned} & \frac{1}{r} = \frac{1-t}{\sigma} + \frac{t}{s}, \\ \Rightarrow \quad & tr = \frac{s(r-\sigma)}{s-\sigma} \to r - \sigma \text{ as } s \to +\infty. \end{aligned}$$

Recall that $r - \sigma < p$ (see hypothesis $H(f)(iii)$). Hence for large enough $s > r$

$$tr = \frac{s(r-\sigma)}{s-\sigma} < p.$$

Then for such large $s > r$, the previous argument is valid and we again obtain (67).

From (58) and (67) we have that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega). \tag{68}$$

In (57) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$, and use (68). Then

$$\begin{aligned} & \lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\ \Rightarrow\ & u_n \to u \text{ in } W_0^{1,p}(\Omega) \\ & \text{(using the } (S)_+ \text{ property of } A(\cdot), \text{ see Section 2)}, \\ \Rightarrow\ & \varphi_\lambda(\cdot) \text{ satisfies the } C-\text{condition}. \end{aligned}$$

This proves Claim 1.

From (53), (54) and Claim 1, we see that we can apply the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\varphi_\lambda} \subseteq [u_\mu) \cap \operatorname{int} C_+ \text{ (see (46)) and } m_\rho \leqslant \varphi_\lambda(\hat{u}) \text{ (see (53))}. \tag{69}$$

It follows rom (44) and (69) that

$$\hat{u} \in S_\lambda \subseteq \operatorname{int} C_+ \text{ and } u_0 \neq \hat{u}.$$

The proof is now complete. ☐

Proposition 3.8. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\geqslant 1} \subseteq (0, \lambda^*)$ be such that $\lambda_n \uparrow \lambda^*$. We know that $\lambda_n \in \mathcal{L}$ for all $n \in \mathbb{N}$ and so we can find $u_n = u_{\lambda_n} \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ ($n \in \mathbb{N}$) increasing (see Proposition 3.5).

Let $\hat{\varphi}_{\lambda_n}(\cdot)$ be the functional from the proof of Proposition 3.7, with $u_\mu = u_{n-1},\ u_\mu = u_{n+1} (n \geqslant 2)$. Then we have

$$\begin{aligned} \hat{\varphi}_{\lambda_n}(u_n) \ &\leqslant\ \hat{\varphi}_{\lambda_n}(u_{n-1}) \\ &=\ \int_\Omega G(Du_{n-1})dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]u_{n-1}^p dz - \int_\Omega [\vartheta(u_{n-1}) + \lambda_n u_{n-1}^{q-1} + \\ &\qquad f(z, u_{n-1} + \beta u_{n-1}^{p-1})]u_{n-1}dz \\ &\leqslant\ \int_\Omega G(Du_{n-1})dz + \frac{1}{p}\int_\Omega [\xi(z)+\beta]u_{n-1}^p dz - \int_\Omega [\vartheta(u_{n-1}) + \lambda_{n-1} u_{n-1}^{q-1} + \\ &\qquad f(z, u_{n-1}) + \beta u_{n-1}^{p-1}]u_{n-1}dz \\ &\leqslant\ \int_\Omega (a(Du_{n-1}), Du_{n+1})dz + \int_\Omega \xi(z)u_{n-1}^p dz - \int_\Omega [\vartheta(u_{n-1}) + \lambda_{n-1} u_{n-1}^{q-1} + \\ &\qquad f(z, u_{n-1})]u_{n-1}dz \text{ (see (3) and recall that } \beta > ||\xi||_\infty) \\ &=\ 0 \text{ (since } u_{n-1} \in S_{\lambda_{n-1}}). \end{aligned} \tag{70}$$

Also, we have

$$\langle A(u_n), h\rangle + \int_\Omega [\xi(z)+\beta]u_n^{p-1} h dz = \int_\Omega d_{\lambda_n}(z, u_n) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ and all } n \in \mathbb{N}. \tag{71}$$

Using (70), (71) and reasoning as in the proof of Proposition 3.7 (see Claim 1), we obtain that

$$\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

From this, as in the proof of Proposition 3.7, exploiting the $(S)_+$ property of $A(\cdot)$, we obtain

$$u_n \to u_* \text{ in } W_0^{1,p}(\Omega). \tag{72}$$

Passing to the limit as $n \to \infty$ in (71) and using (72), we have

$$u_* \in S_{\lambda_*} \subseteq \operatorname{int} C_+ \text{ and so } \lambda^* \in \mathcal{L}.$$

The proof is now complete. □

This proposition implies that

$$\mathcal{L} = (0, \lambda^*] .$$

Summarizing the situation for problem (P_λ), we can state the following bifurcation-type result.

Theorem 3.9. *If hypotheses $H(a), H(\xi), H(\vartheta), H(f)$ hold, then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$ problem P_λ has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \ u_0 \neq \hat{u};$$

(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution $u_* \in \operatorname{int} C_+$;*
(c) *for all $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.*

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(N.S. Papageorgiou) Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

(V.D. Rădulescu) Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

(D.D. Repovš) Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
Email address: dusan.repovs@guest.arnes.si