

INFINITELY MANY SIGN-CHANGING SOLUTIONS FOR KIRCHHOFF TYPE PROBLEMS IN \mathbb{R}^3

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ABSTRACT. In this paper, we consider the following nonlinear Kirchhoff type problem:

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $a, b > 0$ are constants, the nonlinearity f is superlinear at infinity with subcritical growth and V is continuous and coercive. For the case when f is odd in u we obtain infinitely many sign-changing solutions for the above problem by using a combination of invariant sets method and the Ljusternik-Schirelman type minimax method. To the best of our knowledge, there are only few existence results for this problem. It is worth mentioning that the nonlinear term may not be 4-superlinear at infinity, in particular, it includes the power-type nonlinearity $|u|^{p-2}u$ with $p \in (2, 4]$.

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1. INTRODUCTION AND MAIN RESULTS

In this paper we are interested in establishing the multiplicity of sign-changing solutions to the following nonlinear Kirchhoff type problem

$$(1.1) \quad \begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = f(u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where $a, b > 0$ are constants, $V \in C(\mathbb{R}^3, \mathbb{R})$, and $f \in C(\mathbb{R}, \mathbb{R})$.

Problems like (1.1) have been widely investigated because they have a strong physical meaning. Indeed, (1.1) is related to the stationary analogue of the equation

$$(1.2) \quad u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u)$$

proposed by Kirchhoff in [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. In [17], Lions proposed an abstract framework for the problem and after that, problem (1.2) began to receive a lot of attention.

In (1.1), if we set $V(x) = 0$ and replace \mathbb{R}^3 and $f(u)$ by a bounded domain $\Omega \subset \mathbb{R}^N$ and $f(x, u)$, respectively, then we get the following Kirchhoff type equation

$$(1.3) \quad \begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2\right) \Delta u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

The above problem is a nonlocal one as the appearance of the term $\int_{\Omega} |\nabla u|^2 dx$ implies that (1.3) is not a pointwise identity. This phenomenon causes some mathematical difficulties, which make the study of (1.3) particularly interesting. In recent years, by using variational methods, the solvability of equation (1.3) with subcritical or critical growth nonlinearity has been paid much attention by various authors, see, e.g. [6, 24, 26, 27, 32, 33] and the references therein. For the results concerning the existence of sign-changing solutions for (1.3), we refer the reader to papers [25, 29, 38] which depend heavily on the nonlinearity term with 4-superlinear growth at infinity in the sense that

$$\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{t^4} = +\infty, \quad \text{uniformly in } x \in \Omega,$$

where $F(x, t) = \int_0^t f(x, s) ds$ and [23, 36] with the nonlinearity $f(x, u)$ may not be 4-superlinear at infinity.

If we replace $f(u)$ by $f(x, u)$ in (1.1), several authors have considered the following problem

$$(1.4) \quad -\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2\right) \Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N.$$

In recent years, there have been enormous results on existence, nonexistence and multiplicity of nontrivial solutions for such problem depending on the assumptions of the potential V and f . See, for example, [13, 16, 35] and the references therein. Recently, replacing a and b by $\varepsilon^2 a$ and εb in (1.4), respectively, many researches have studied a certain concentration phenomena for the following Kirchhoff type

equation

$$\begin{cases} \left(-\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(x, u), & \text{in } \mathbb{R}^3, \\ u > 0, u \in H^1(\mathbb{R}^3), \end{cases}$$

see e.g. [8, 9, 10, 22, 34]. We mention that there are only few works concerning the existence of sign-changing solutions for (1.4). We are only aware of the works [7, 11, 37]. In [7], Deng *et al.* studied the existence of radial sign-changing solutions with prescribed numbers of nodal domains for (1.4) in $H_r^1(\mathbb{R}^3)$, the subspace of radial functions of $H^1(\mathbb{R}^3)$ by using a Nehari manifold and gluing solution pieces together, when $V(x) = V(|x|)$ satisfies

(V₁) $V \in C([0, +\infty), \mathbb{R})$ is bounded below by a positive constant V_0 ;

and $f(x, u) = f(|x|, u)$ satisfies the following hypotheses:

- (f₁) $f(r, u) \in C^1([0, +\infty) \times \mathbb{R}, \mathbb{R})$ is odd in u for every $r \geq 0$;
- (f₂) $f(r, u) = o(|u|)$ as $u \rightarrow 0$ uniformly in $r \geq 0$;
- (f₃) for some constant $p \in (4, 6)$, $\lim_{u \rightarrow +\infty} \frac{f(r, u)}{u^{p-1}} = 0$ uniformly in $r \geq 0$;
- (f₄) $\lim_{u \rightarrow +\infty} \frac{F(r, u)}{u^4} = +\infty$, where $F(r, u) = \int_0^u f(r, t) dt$;
- (f₅) $\frac{f(r, u)}{|u|^3}$ is an increasing function of $u \in \mathbb{R} \setminus \{0\}$ for every $r \geq 0$.

In [11], Huang and Liu studied the existence of least energy sign-changing solutions with exactly two nodal domains for a variant of (1.4):

$$-\left(1 + \lambda \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\right) [\Delta u + V(x)u] = |u|^{p-2}u, \quad x \in \mathbb{R}^N.$$

where $\lambda > 0$, $p \in (4, 6)$ and V is assumed to guarantee the compactness. Ye [37] proved the existence of least energy sign-changing solutions for equation (1.4) with $f(x, u) = f(u)$ (i.e., (1.1)) by using constrained minimization of the sign-changing Nehari manifold and Brouwer degree theory under the conditions that V satisfies

- (V₁) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{\mathbb{R}^3} V(x) \geq V_0 > 0$ for some positive constant V_0 and is coercive, i.e., $\lim_{|x| \rightarrow \infty} V(x) = \infty$,

and the nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the following assumptions:

- (f₁) $\lim_{s \rightarrow 0} \frac{f(s)}{|s|^3} = 0$;
- (f₂) there exists $3 < q < 2^* - 1$ such that $\lim_{|s| \rightarrow +\infty} \frac{f(s)}{|s|^q} = 0$, where $2^* = +\infty$ if $N = 2$ and $2^* = 6$ if $N = 3$;
- (f₃) $\lim_{|s| \rightarrow +\infty} \frac{F(s)}{s^4} = +\infty$, where $F(s) = \int_0^s f(t) dt$;
- (f₄) the function $\frac{f(s)}{|s|^3}$ is nondecreasing on $\mathbb{R} \setminus \{0\}$.

To the best of our knowledge, there is no result in the literature on the existence of multiple sign-changing solutions for problems (1.1) and (1.4) without any symmetry. Motivated by the above works, in the present paper we study the existence of infinitely many sign-changing solutions for problem (1.1) with coercive potential V , that is, (V₁) holds and more general assumptions on f . More precisely, we assume that f satisfies the following assumptions:

- (f₁) $f \in C(\mathbb{R}, \mathbb{R})$ and $|f(u)| \leq C(1 + |u|^{p-1})$ for some $C > 0$ and $p \in (2, 6)$;
- (f₂) $f(u) = o(u)$ as $u \rightarrow 0$;
- (f₃) there exists $\mu > 2$ such that $\frac{1}{\mu} f(u)u \geq F(u) > 0$ for all $u \in \mathbb{R} \setminus \{0\}$, where $F(u) = \int_0^u f(s) ds$;
- (f₄) f is odd, i.e., $f(-u) = -f(u)$.

Now we state our first main result.

Theorem 1.1. *Suppose that (V_1) and (f_1) – (f_4) hold and $\mu > 4$. Then problem (1.1) admits infinitely many sign-changing solutions.*

Remark. The assumptions (V_1) plays a role only in guaranteeing the compactness of the (PS) sequence for the energy functional I associated with (1.1). We point out that Theorem (1.1) also holds when working in $H_r^1(\mathbb{R}^3)$ if V is a positive constant.

Recall that (f_3) is the so-called Ambrosetti-Rabinowitz condition ((AR) for short). It is easy to see that $\mu > 4$ guarantees the Palais-Smale ((PS) for short) sequence for I at any $c \in \mathbb{R}$ is bounded. But if $\mu < 4$, f may not be 4-superlinear at infinity, due to the effect of the nonlocal term, it is difficult to get a bounded (PS) sequence for I . Motivated by [13], to overcome this difficulty, in the case $\mu < 4$, we suppose that $V(x)$ satisfies the following additional condition

(V_2) V is weakly differentiable, $(DV(x), x) \in L^r(\mathbb{R}^3)$ for some $r \in [\frac{3}{2}, \infty]$ and

$$\frac{\mu - 2}{2}V(x) - (DV(x), x) \geq 0 \quad \text{for a.e. } x \in \mathbb{R}^3,$$

where μ is given by (f_3) .

It is worth mentioning that this assumption is different from that of [13]. Li and Ye [13] assumed

$$V(x) - (DV(x), x) \geq 0 \quad \text{for a.e. } x \in \mathbb{R}^3,$$

and then obtained a positive ground state solution to (1.1) with $f(u) = |u|^{p-2}u$ ($p \in (3, 6)$) by using the constrained minimization on a suitable Pohozaev-Nehari manifold. We remark that the case $2 < p \leq 3$ is not included in their result.

Then we have the following result.

Theorem 1.2. *Suppose that (V_1) – (V_2) and (f_1) – (f_4) hold. Then problem (1.1) admits infinitely many sign-changing solutions.*

Remark. (i) To the best of our knowledge, there is no existence result for sign-changing solutions to (1.1) in the literature even in the special case $f(u) = |u|^{p-2}u$ with $2 < p \leq 4$.

(ii) There exists function $V(x)$ satisfying the assumptions (V_1) – (V_2) . For example, let

$$V(x) = \ln(1 + |x|) + \frac{2}{\mu - 2}.$$

Clearly, (V_1) holds. Moreover, for $x \in \mathbb{R}^3 \setminus \{0\}$, $(DV(x), x) = \frac{|x|}{1+|x|}$. Therefore, $(DV(x), x) \in L^\infty(\mathbb{R}^3)$ and for a.e. $x \in \mathbb{R}^3$,

$$\frac{\mu - 2}{2}V(x) - (DV(x), x) = \frac{\mu - 2}{2} \ln(1 + |x|) + 1 - \frac{|x|}{1 + |x|} \geq \frac{\mu - 2}{2} \ln(1 + |x|) \geq 0,$$

and so condition (V_2) holds. Another example is $V(x) = \ln(1 + |x|^2) - \frac{|x|^2}{1+|x|^2} + \frac{\mu+2}{\mu-2}$. One can also check that $V(x)$ satisfies (V_1) – (V_2) .

Motivated by [15, 31], we will prove Theorems 1.1 and 1.2 by applying the usual Ljusternik-Schnirelman type minimax method in conjunction with invariant set method. More precisely, we will construct certain invariant sets of the gradient flow corresponding to the energy functional I such that all positive and negative solutions are contained in these invariant sets and then minimax arguments can be applied to construct sign-changing solutions outside these invariant sets. The

method of invariant sets of descending flow has been used widely in dealing with sign-changing solutions of elliptic problem, see [2, 3, 19] and the references therein. But for the nonlocal problem, there are serious technical difficulties to overcome. Here we would like to point out the difficulties we will encounter and our main ideas.

Due to the effect of the nonlocal term, the arguments of constructing invariant sets of descending flow in [2, 20, 31] cannot be directly applied to problem (1.1). To overcome this difficulty, we will adopt some ideas from [21] which studied the existence of infinitely many sign-changing solutions for a Schrödinger-Poisson system by using an abstract critical point developed by Liu et al. [18]. First, we will construct an auxiliary operator A (see Lemma 3.4 below) from which we can construct closed convex sets containing all the positive and negative solutions in their interior. However, A itself cannot be used to define the desired invariant sets of the flow, because the operator A is merely continuous under our assumptions. Inspired by [3], from A , we can get a locally Lipschitz continuous operator B (see Lemma 3.7 below) which inherits the main properties of A . Then, we can use B to define the flow (see Lemma 3.8 below). Finally, by using a suitable deformation lemma in the presence of invariant sets (see Lemma 3.9 below) and minimax procedures, we prove that problem (1.1) has infinitely many sign-changing solutions.

As mentioned above, if $\mu < 4$, the nonlinearity term f may not be 4-superlinear at infinity. It prevents us from obtaining a bounded (PS) sequence, let alone (PS) condition holds for I . Therefore, the above arguments cannot be applied directly to prove Theorem 1.2. To overcome the obstacle, inspired by [21], we consider the perturbed functionals $I_\lambda : E \rightarrow \mathbb{R}$ (see (4.1) below) defined by

$$I_\lambda(u) = I(u) - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r, \quad \lambda \in (0, 1],$$

where $r \in (\max\{4, p\}, 6)$, here p is from (f_1) . It will be shown that I_λ admits infinitely many sign-changing critical points $\{u_k^\lambda\}_{k \geq 2}$ by using the above framework. Then, by using a Pohozaev identity and (V_2) , we can prove that $u_k^\lambda \rightarrow u_k$ strongly in E as $\lambda \rightarrow 0^+$ and then the existence of infinitely many sign-changing solutions for (1.1) are obtained.

The remainder of this paper is organized as follows. In Section 2 we derive a variational setting for problem (1.1) and give some preliminary lemmas. We will prove Theorem 1.1 in Section 3. Section 4 is devoted to the proof of Theorem 1.2.

2. VARIATIONAL SETTING AND PRELIMINARY LEMMAS

Throughout this paper, we use the standard notations. We denote by $C, c_i, C_i, i = 1, 2, \dots$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem. $\|\cdot\|_q$ denotes the usual norm of $L^q(\mathbb{R}^3)$ for $q \in [2, \infty]$. For simplicity, we write $\int_{\mathbb{R}^3} h$ to mean the Lebesgue integral of $h(x)$ over \mathbb{R}^3 . For a functional $J : E \rightarrow \mathbb{R}$, we set $J^b := \{u \in E : J(u) \leq b\}$. We use “ \rightarrow ” and “ \rightharpoonup ” to denote the strong and weak convergence in the related function space respectively. We will write $o(1)$ to denote quantity that tends to 0 as $n \rightarrow \infty$.

Our argument is variational. In the paper, we work in the following Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < \infty \right\}$$

with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)|u|^2) \right)^{\frac{1}{2}}.$$

We denote its inner product by $(\cdot, \cdot)_E$. It is well known that $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2, 6]$. Moreover, as in [5], we have the following result which plays an important role in our proof.

Lemma 2.1. *Under (V_1) , the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6)$.*

Remark. Indeed, as in [4], (V_1) can be replaced by the more general condition.

(V'_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{\mathbb{R}^3} V(x) \geq V_0 > 0$ and there exists $r_0 > 0$ such that for any $M > 0$,

$$\lim_{|y| \rightarrow \infty} m(\{x \in \mathbb{R}^3 : V(x) \leq M\} \cap \{x \in \mathbb{R}^3 : |x - y| \leq r_0\}) = 0.$$

Since $E \hookrightarrow L^2(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ is a separable Hilbert space, E has a countable orthogonal basis $\{e_i\}$. In the following, for any $m \in \mathbb{N}$, we denote $E_m := \text{span}\{e_1, e_2, \dots, e_m\}$.

Under our assumptions, it is standard to show that the weak solutions to (1.1) correspond to the critical points of the energy functional $I \in C^1(E, \mathbb{R})$ defined by

$$(2.1) \quad I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)|u|^2) + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^3} F(u),$$

where $F(u) = \int_0^u f(s)ds$. Moreover, for any $u, v \in E$, we have

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + V(x)uv) + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla v - \int_{\mathbb{R}^3} f(u)v.$$

3. PROOF OF THEOREM 1.1

In this section, we devote to prove the existence of infinitely many sign-changing solutions to problem (1.1) with $\mu > 4$ by using a combination of invariant sets method and Ljusternik-Schnirelman type minimax results.

3.1. Some technical lemmas. In order to construct the minimax values for the functional I defined in (2.1), the following three technical lemmas are needed.

Lemma 3.1. *Under the assumptions (f_1) - (f_3) , the functional I satisfies the (PS) condition.*

Proof. By (f_3) , it is easy to check that any (PS) sequence for I at level $c \in \mathbb{R}$ is bounded. Thus, by Lemma 2.1, one can follow the same way as in the proof of Lemma 4 in [35] to complete the present proof. \square

Lemma 3.2. *Suppose (f_1) - (f_3) hold and $m \geq 1$. Then there exists $R_m = R(E_m) > 0$, such that*

$$\sup_{\mathcal{B}_{R_m}^c \cap E_m} I < 0,$$

where $\mathcal{B}_R^c := E \setminus \mathcal{B}_R$ and $\mathcal{B}_R := \{u \in E : \|u\|_E \leq R\}$.

Proof. By (f_1) - (f_3) , there exists $c_1, c_2 > 0$ such that $F(t) \geq c_1|t|^\mu - c_2|t|^2$ for all $t \in \mathbb{R}$. Thus,

$$\begin{aligned} I(u) &\leq \frac{1}{2}\|u\|_E^2 + \frac{b}{4a^2}\|u\|_E^4 - \int_{\mathbb{R}^3} F(u) \\ &\leq \frac{1}{2}\|u\|_E^2 + \frac{b}{4a^2}\|u\|_E^4 - c_1 \int_{\mathbb{R}^3} |u|^\mu + c_2 \int_{\mathbb{R}^3} |u|^2. \end{aligned}$$

Since $\mu > 4$ and any norm in finite dimensional space is equivalent, one concludes that

$$\lim_{\substack{u \in E_m \\ \|u\|_E \rightarrow \infty}} I(u) = -\infty$$

for any fixed $m \geq 1$. Therefore the result follows. \square

Lemma 3.3. *Assume (f_1) and (f_2) hold. Then there exist $\rho > 0$ and $\alpha > 0$ such that*

$$\inf_{\partial \mathcal{B}_\rho} I \geq \alpha.$$

Proof. By (f_1) and (f_2) , for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$(3.1) \quad |f(t)| \leq \delta|t| + C_\delta|t|^{p-1}$$

for $t \in \mathbb{R}$. Then, for $u \in E$, we have

$$(3.2) \quad \begin{aligned} I(u) &= \frac{1}{2}\|u\|_E^2 + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^3} F(u) \\ &\geq \frac{1}{2}\|u\|_E^2 - \frac{\delta}{2} \int_{\mathbb{R}^3} |u|^2 - \frac{C_\delta}{p} \int_{\mathbb{R}^3} |u|^p. \end{aligned}$$

By the Sobolev embedding theorem, for any $r \in [2, 6]$, there exists $C(r) > 0$ such that $\|u\|_r \leq C(r)\|u\|_E$. Choose δ satisfying that $C(2)\delta < \frac{1}{2}$. Then, it follows from (3.2) that

$$I(u) \geq \frac{1}{4}\|u\|_E^2 - \frac{C_\delta C(p)}{p}\|u\|_E^p.$$

Noting that $p > 2$, we conclude that there exist $\rho, \alpha > 0$ such that $\inf_{\partial \mathcal{B}_\rho} I \geq \alpha$, as required. \square

3.2. Invariant subsets of descending flow. In order to construct a descending flow guaranteeing existence of the desired invariant sets for the functional I , we introduce an auxiliary operator $A : E \rightarrow E$. Precisely, for any $u \in E$, we define $v = A(u)$ the unique solution to the following equation

$$(3.3) \quad - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta v + V(x)v = f(u), \quad v \in E.$$

Clearly, the set of fixed points of A is the same as the set of critical points of I .

Lemma 3.4. *The operator A is well defined and continuous.*

Proof. Let $u \in E$, and define

$$J(v) = \frac{1}{2} \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)v^2 - \int_{\mathbb{R}^3} f(u)v, \quad v \in E.$$

Obviously, $J \in C^1(E, \mathbb{R})$. And it is easy to check that J is weakly lower semicontinuous.

By (3.1) and the Sobolev embeddings, one has

$$(3.4) \quad \left| \int_{\mathbb{R}^3} f(u)v \right| \leq \delta \int_{\mathbb{R}^3} |u||v| + C_\delta \int_{\mathbb{R}^3} |u|^{p-1}|v| \leq \delta \|u\|_2 \|v\|_2 + C_\delta \|u\|_p^{p-1} \|v\|_p \\ \leq C \|v\|_E,$$

which implies

$$J(v) = \frac{1}{2} \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)v^2 - \int_{\mathbb{R}^3} f(u)v \\ \geq \frac{1}{2} \|v\|_E^2 - C \|v\|_E \rightarrow +\infty, \quad \text{as } \|v\|_E \rightarrow \infty,$$

where $C = C(\varepsilon, u)$ is a constant depending on ε and u . Therefore, J is coercive.

By (3.4), it is easy to see that J is bounded below and maps bounded sets into bounded sets. Now we prove J is also strictly convex. In fact,

$$\langle J'(v) - J'(w), v - w \rangle = \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} |\nabla(v-w)|^2 + \int_{\mathbb{R}^3} V(x)|v-w|^2 \\ \geq \|v-w\|_E^2 > 0, \quad \text{if } v \neq w,$$

where

$$\langle J'(v), \varphi \rangle = \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} \nabla v \cdot \nabla \varphi + \int_{\mathbb{R}^3} V(x)v\varphi - \int_{\mathbb{R}^3} f(u)\varphi, \quad \varphi \in E.$$

Thus, by Theorems 1.5.6 and 1.5.8 in [1], J admits a unique minimizer $v = A(u) \in E$, which is the unique solution to (3.3). Moreover, by (3.4), A maps bounded sets into bounded sets.

In the following, we prove that the operator A is continuous. Let $\{u_n\} \subset E$ with $u_n \rightarrow u$ in E strongly. Denote $v = A(u)$ and $v_n = A(u_n)$. Then we have

$$\|v_n - v\|_E^2 = b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v \nabla(v_n - v) - b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla v_n \cdot \nabla(v_n - v) \\ + \int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v) \\ \leq b \left(\int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} |\nabla u_n|^2 \right) \int_{\mathbb{R}^3} \nabla v \cdot \nabla(v_n - v) \\ + \int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v) \\ \triangleq I_1 + I_2.$$

By the Hölder inequality and Sobolev embedding theorem,

$$(3.5) \quad I_1 \leq b \left| \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} |\nabla u_n|^2 \right| \|v\|_2 \|v_n - v\|_2 \\ \leq C \left| \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} |\nabla u_n|^2 \right| \|v\|_E \|v_n - v\|_E.$$

Now, we estimate the second term I_2 . The proof is similar to that of [21]. Let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(t) \in [0, 1]$ for $t \in \mathbb{R}$, $\phi(t) = 1$ for $\|t\| \leq 1$ and $\phi(t) = 0$ for $\|t\| \geq 2$. Let $g_1(t) = \phi(t)f(t)$, $g_2(t) = f(t) - g_1(t)$. Then, by (f_1) and (f_2) , there

exists $C > 0$ such that $|g_1(s)| \leq C|s|$ and $|g_2(s)| \leq C|s|^{p-1}$ for $s \in \mathbb{R}$. Thus,

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^3} (g_1(u_n) - g_1(u))(v_n - v) + \int_{\mathbb{R}^3} (g_2(u_n) - g_2(u))(v_n - v) \\ &\leq \left(\int_{\mathbb{R}^3} |g_1(u_n) - g_1(u)|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |v_n - v|^2 \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\mathbb{R}^3} |g_2(u_n) - g_2(u)|^p \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^3} |v_n - v|^p \right)^{\frac{1}{p}} \\ &\leq C \|v_n - v\|_E \left[\left(\int_{\mathbb{R}^3} |g_1(u_n) - g_1(u)|^2 \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^3} |g_2(u_n) - g_2(u)|^p \right)^{\frac{p-1}{p}} \right], \end{aligned}$$

which, jointly with (3.5), implies

$$\begin{aligned} \|v_n - v\|_E &\leq C \left[\left| \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} |\nabla u_n|^2 \right| \|v\|_E + \left(\int_{\mathbb{R}^3} |g_1(u_n) - g_1(u)|^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \left(\int_{\mathbb{R}^3} |g_2(u_n) - g_2(u)|^p \right)^{\frac{p-1}{p}} \right]. \end{aligned}$$

Therefore, noting that $u_n \rightarrow u$ in E and by the dominated convergence theorem, one has $\|v_n - v\|_E \rightarrow 0$ as $n \rightarrow \infty$. This proof is completed. \square

Now we summarize some properties of the operator A which are useful to study our problem.

Lemma 3.5.

- (i) $\langle I'(u), u - A(u) \rangle \geq \|u - A(u)\|_E^2$ for all $u \in E$.
- (ii) $\|I'(u)\| \leq \|u - A(u)\|_E (1 + C\|u\|_E^2)$ for some $C > 0$ and all $u \in E$.
- (iii) For $M > 0$ and $\alpha > 0$, there exists $\beta > 0$ such that $\|u - A(u)\|_E \geq \beta$ for any $u \in E$ with $|I(u)| \leq M$ and $\|I'(u)\| \geq \alpha$.
- (iv) If f is odd, then so is A .

Proof. (i) Noting that $A(u)$ is the solution to (3.3), for $u \in E$, it is easy to see that

$$\begin{aligned} \langle I'(u), u - A(u) \rangle &= \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} |\nabla(u - A(u))|^2 \\ &\quad + \int_{\mathbb{R}^3} V(x)|u - A(u)|^2 \\ &\geq \|u - A(u)\|_E^2. \end{aligned} \tag{3.6}$$

(ii) For any $\varphi \in E$, By the Hölder inequality, one has

$$\begin{aligned} \langle I'(u), \varphi \rangle &= (u - A(u), \varphi)_E + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla(u - A(u)) \cdot \nabla \varphi \\ &\leq \|u - A(u)\|_E \|\varphi\|_E + C \|u\|_E^2 \|u - A(u)\|_E \|\varphi\|_E, \end{aligned}$$

which implies $\|I'(u)\| \leq \|u - A(u)\|_E (1 + C\|u\|_E^2)$ for all $u \in E$, here $C = b/a^2 > 0$ is a constant.

(iii) Since

$$\langle I'(u), u \rangle = (u - A(u), u)_E + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla(u - A(u)) \nabla u,$$

it follows from (f₃) that

$$\begin{aligned}
& I(u) - \frac{1}{\mu}(u - A(u), u)_E \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right) b \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\
&\quad + \frac{b}{\mu} \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla(u - A(u)) \nabla u + \int_{\mathbb{R}^3} \left(\frac{1}{\mu} f(u)u - F(u)\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{\mu}\right) b \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\
&\quad + \frac{b}{\mu} \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u - A(u)).
\end{aligned}$$

Consequently,

$$\begin{aligned}
(3.7) \quad & \|u\|_E^2 + \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\
&\leq C \left(|I(u)| + \|u\|_E \|u - A(u)\|_E + \int_{\mathbb{R}^3} |\nabla u|^2 \left| \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u - A(u)) \right| \right).
\end{aligned}$$

By the Hölder inequality and Young inequality, for any $\varepsilon > 0$,

$$\begin{aligned}
& \int_{\mathbb{R}^3} |\nabla u|^2 \left| \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u - A(u)) \right| \\
&\leq \int_{\mathbb{R}^3} |\nabla u|^2 \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla(u - A(u))|^2\right)^{\frac{1}{2}} \\
&\leq \varepsilon \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + C(\varepsilon) \|u\|_E^2 \|u - A(u)\|_E^2.
\end{aligned}$$

Then, for ε small enough, from (3.7), we have

$$(3.8) \quad \|u\|_E^2 \leq C_1 (|I(u)| + \|u\|_E \|u - A(u)\|_E + \|u\|_E^2 \|u - A(u)\|_E^2).$$

Arguing indirectly, suppose that there exists $\{u_n\} \subset E$ with $|I(u_n)| \leq M$ and $\|I'(u_n)\| \geq \alpha$ such that $\|u_n - A(u_n)\|_E \rightarrow 0$ as $n \rightarrow \infty$. Then it follows from (3.8) that $\{\|u_n\|_E\}$ is bounded. Thus, by (ii), one concludes that $\|I'(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction.

The conclusion (iv) is obviously, and the proof is completely. \square

Here and in the sequel, define the convex cones

$$P^+ := \{u \in E : u \geq 0\} \quad \text{and} \quad P^- := \{u \in E : u \leq 0\}.$$

For $\varepsilon > 0$, we denote

$$P_\varepsilon^+ := \{u \in E : \text{dist}(u, P^+) < \varepsilon\} \quad \text{and} \quad P_\varepsilon^- := \{u \in E : \text{dist}(u, P^-) < \varepsilon\},$$

where $\text{dist}(u, P^\pm) = \inf_{v \in P^\pm} \|u - v\|_E$. Obviously, $P_\varepsilon^- = -P_\varepsilon^+$. Let $W := P_\varepsilon^+ \cup P_\varepsilon^-$. It is easy to see that W is an open and symmetric subset of E and $Q := E \setminus W$ contains only sign-changing functions. The following result shows that for ε small, all sign-changing solutions to (1.1) are contained in Q .

Lemma 3.6. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there holds*

$$A(\overline{P}_\varepsilon^\pm) \subset P_\varepsilon^\pm.$$

Moreover, every nontrivial solutions $u \in P_\varepsilon^+$ and $u \in P_\varepsilon^-$ of (1.1) are positive and negative, respectively.

Proof. We only prove the case $A(\overline{P}_\varepsilon^+) \subset P_\varepsilon^+$, because the other case can be obtained similarly. We write $u \in E$ as $u = u^- + u^+$, where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. For $u \in E$, denote $v = A(u)$. By the Sobolev embedding theorem, for any $r \in [2, 6]$, there exists $C(r) > 0$ such that

$$\|u^-\|_r = \inf_{\phi \in P^+} \|u - \phi\|_r \leq C(r) \inf_{\phi \in P^+} \|u - \phi\|_E = C(r) \text{dist}(u, P^+).$$

Then, noting that $\text{dist}(v, P^+) \leq \|v^-\|_E$, by (3.1) and Hölder inequality, we have

$$\begin{aligned} \text{dist}(v, P^+) \|v^-\|_E &\leq \|v^-\|_E^2 = (v, v^-)_E \\ &= \int_{\mathbb{R}^3} f(u)v^- - b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v \cdot \nabla v^- \\ &\leq \int_{\mathbb{R}^3} f(u)v^- \leq \int_{\mathbb{R}^3} f(u^-)v^- \\ &\leq \int_{\mathbb{R}^3} (\delta|u^-| + C_\delta|u^-|^{p-1})|v^-| \\ &\leq \delta \|u^-\|_2 \|v^-\|_2 + C_\delta \|u^-\|_p^{p-1} \|v^-\|_p \\ &\leq C(\delta \text{dist}(u, P^+) + C_\delta \text{dist}(u, P^+)^{p-1}) \|v^-\|_E, \end{aligned}$$

which implies that

$$\text{dist}(v, P^+) \leq C(\delta \text{dist}(u, P^+) + C_\delta \text{dist}(u, P^+)^{p-1}).$$

Letting $\delta = \frac{1}{4C}$ and taking $\varepsilon_0 \in (0, \frac{1}{(4CC_\delta)^{1/(p-2)}})$, for any $\varepsilon \in (0, \varepsilon_0]$, one has

$$\text{dist}(v, P^+) \leq \frac{1}{2} \text{dist}(u, P^+) < \varepsilon, \quad \text{for any } u \in \overline{P}_\varepsilon^+.$$

This implies that $A(\overline{P}_\varepsilon^+) \subset P_\varepsilon^+$. If there exists $u \in P_\varepsilon^+$ such that $A(u) = u$, then $\text{dist}(u, P^+) = 0$, i.e. $u \in P^+$. Moreover, if $u \not\equiv 0$, by the maximum principle, $u > 0$ in \mathbb{R}^3 . \square

Note that the operator A is merely continuous. Denote the set of critical points of I by K . In order to construct a descending flow for I , we need to construct a locally Lipschitz continuous operator on $E_0 := E \setminus K$ which inherits the main properties of A . Together with Lemma 3.6 and following the same way as in the proof of Lemma 2.1 in [3], we have the following results.

Lemma 3.7. *There exists a locally Lipschitz continuous operator $B : E_0 \rightarrow E$ with the following properties:*

- (1) $B(\overline{P}_\varepsilon^\pm) \subset P_\varepsilon^\pm$ for all $\varepsilon \in (0, \varepsilon_0]$;
- (2) $\frac{1}{2} \|u - B(u)\|_E \leq \|u - A(u)\|_E \leq 2 \|u - B(u)\|_E$ for all $u \in E_0$;
- (3) $\langle I'(u), u - B(u) \rangle \geq \frac{1}{2} \|u - A(u)\|_E^2$ for all $u \in E_0$;
- (4) If f is odd, then so is B .

Set $H(u) = u - B(u)$. For $u \in E_0$, consider the following initial value problems:

$$(3.9) \quad \begin{cases} \frac{d}{dt}\sigma(t, u) = -H(\sigma(t, u)), \\ \sigma(0, u) = u. \end{cases}$$

Invoking the locally Lipschitz continuity of B , the existence and uniqueness theory of ODE implies that problem (3.9) has a unique solution, denoted by $\sigma(t, u)$ with maximal interval of existence $[0, T(u))$. By Lemma 3.7(3), it is easy to check that $I(\sigma(t, u))$ is strictly decreasing in $[0, T(u))$. Moreover, with the help of Lemma 3.7, using similar arguments to the proof of Lemma 3.2 in [3], we have the following result.

Lemma 3.8. *For any $\varepsilon \in (0, \varepsilon_0]$ and $u \in \overline{P}_\varepsilon^\pm \setminus K$, $\sigma(t, u) \in P_\varepsilon^\pm$ for all $t \in (0, T(u))$. Here ε_0 is given by Lemma 3.6.*

Remark. since $\partial P_\varepsilon^\pm \cap K = \emptyset$ by Lemma 3.6, the above lemma implies that $\sigma(t, u) \in P_\varepsilon^\pm$ for all $t \in (0, T(u))$ and $u \in \partial P_\varepsilon^\pm$. In the following, we may choose an $\varepsilon > 0$ sufficiently small such that $W = P_\varepsilon^+ \cup P_\varepsilon^-$ is an invariant set with respect to σ .

In order to construct nodal solutions by using the combination of invariant sets method and minimax method, we need a deformation lemma in the presence of invariant sets. In fact, we have the following result.

Lemma 3.9. *Let N be an open symmetric neighborhood of $K_c \setminus W$. Then there exists an $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$, there exists $\eta \in C([0, 1] \times E, E)$ satisfying:*

- (i) $\eta(t, u) = u$ for $t = 0$ or $u \notin I^{-1}([c - \varepsilon_0, c + \varepsilon_0])$;
- (ii) $\eta(1, I^{c+\varepsilon} \cup W \setminus N) \subset I^{c-\varepsilon} \cup W$ and $\eta(1, I^{c+\varepsilon} \cup W) \subset I^{c-\varepsilon} \cup W$ if $K_c \setminus W = \emptyset$;
- (iii) $\eta(t, -u) = -\eta(t, u)$ for all $(t, u) \in [0, 1] \times E$;
- (iv) $\eta(t, \overline{P}_\varepsilon^\pm) \subset \overline{P}_\varepsilon^\pm$ for any $t \in [0, 1]$.

Proof. The proof is similar to that of Lemma 5.1 in [20]. We state the proof here for the readers convenience. For $G \subset E$ and $a > 0$, we define $N_a(G) := \{u \in E : \text{dist}(u, G) < a\}$. Set $K_c^1 := K_c \cap W$ and $K_c^2 := K_c \setminus W$. Due to the (PS) condition, K_c^1 is compact and therefore $d := \text{dist}(K_c^1, Q) > 0$. Choose $\delta \in (0, \frac{d}{2})$ small enough such that $N_{3\delta}(K_c^2) \subset N$. Obviously, $N_\delta(K_c^1) \subset W$. Since I satisfies the (PS) condition, there exists $\varepsilon_0, \alpha > 0$ such that

$$\|I'(u)\| \geq \alpha, \quad \text{for } u \in I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus N_\delta(K_c).$$

Then, from Lemmas 3.5 and 3.7, there exists $\beta > 0$ such that

$$(3.10) \quad \|u - B(u)\|_E \geq \beta, \quad \text{for } u \in I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus N_\delta(K_c).$$

Without loss of generality, we may assume that $\varepsilon_0 \leq \frac{\beta\delta}{16}$. Define

$$E_1 := I^{-1}([c - \varepsilon_0, c + \varepsilon_0]) \setminus N_\delta(K_c^2),$$

and for any fixed $\varepsilon \in (0, \varepsilon_0)$, we set

$$E_2 := I^{-1}([c - \varepsilon, c + \varepsilon]) \setminus N_{2\delta}(K_c^2).$$

Let

$$\psi(u) = \frac{\text{dist}(u, E \setminus E_1)}{\text{dist}(u, E \setminus E_1) + \text{dist}(u, E_2)}.$$

Recall that $H(u) = u - B(u)$ for $u \in E_0 = E \setminus K$. By Lemma 3.7, $\psi(\cdot)H(\cdot)$ is locally Lipschitz continuous on E . Consider the following initial value problem

$$\frac{d}{dt}\xi(t, u) = -\frac{\psi(\xi(t, u))H(\xi(t, u))}{\|H(\xi(t, u))\|_E}, \quad \xi(0, u) = u.$$

Then $\xi(t, u)$ is well-defined and continuous on $\mathbb{R}^+ \times E$.

Define $\eta(t, u) := \xi(\delta t, u)$. It suffices to check (ii), because (i) is obviously and (iii)-(iv) are easily checked by Lemma 3.7(4) and 3.8. For (ii), arguing indirectly, we suppose that $\eta(1, u) \notin I^{c-\varepsilon} \cup W$ for some $u \in I^{c+\varepsilon} \cup W \setminus N$. Then $\eta(t, u) \notin I^{c-\varepsilon} \cup W$ for all $t \in [0, 1]$ since $I^{c-\varepsilon} \cup W$ is an invariant subset. As a result, $\eta(t, u) \notin N_\delta(K_c^1)$ for all $t \in [0, 1]$ since $N_\delta(K_c^1) \subset W$. Noting that

$$\|\eta(t, u) - u\|_E = \|\xi(\delta t, u) - u\|_E = \left\| \int_0^{\delta t} \xi'(s, u) ds \right\|_E \leq \delta, \quad \text{for any } t \in [0, 1],$$

we have $\eta(t, u) \notin N_{2\delta}(K_c^2)$ for all $t \in [0, 1]$ due to $u \notin N_{3\delta}(K_c^2)$. Therefore, for all $t \in [0, 1]$,

$$\eta(t, u) \in I^{-1}([c - \varepsilon, c + \varepsilon]) \setminus (N_{2\delta}(K_c^2) \cup N_\delta(K_c^1)).$$

As a consequence, for $t \in [0, 1]$,

$$\psi(\eta(t, u)) = 1 \quad \text{and} \quad \|u - B(u)\|_E \geq \beta.$$

Hence, by Lemmas 3.7(2)-(3) and (3.10), we have

$$\begin{aligned} I(\eta(t, u)) &= I(u) + \int_0^1 \frac{d}{ds} I(\eta(s, u)) ds \leq c + \varepsilon - \int_0^1 \frac{\delta}{8} \|\eta(s, u) - B(\eta(s, u))\|_E ds \\ &\leq c + \varepsilon - \frac{\beta\delta}{8} \leq c + \varepsilon - 2\varepsilon_0 < c - \varepsilon, \end{aligned}$$

a contradiction. The proof is completed. \square

3.3. The proof of Theorem 1.1. Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We adopt some techniques in the proof of Theorem 1.1 in [31] (see also [20, 28]). We divide our proof into three steps.

Step 1. We define a minimax value c_k for the functional $I(u)$ with $k = 2, 3, \dots$. Define

$$G_m := \{h \in C(\mathcal{B}_{R_m} \cap E_m, E) : h \text{ is odd and } h = \text{id on } \partial\mathcal{B}_{R_m} \cap E_m\},$$

where $R_m > 0$ is given by Lemma 3.2. It is easy to see that $G_m \neq \emptyset$, because $\text{id} \in G_m$ for all $m \in \mathbb{N}$. For $k \geq 2$, we set

$$\Gamma_k := \{h(\mathcal{B}_{R_m} \cap E_m \setminus Y) : h \in G_m, m \geq k, Y = -Y \text{ is open and } \gamma(Y) \leq m - k\},$$

where $\gamma(M)$ denote the Krasnoselskii's genus of the set M (cf. [30]). As Proposition 9.18 in [28], Γ_k possess the following properties:

- (1°) $\Gamma_k \neq \emptyset$ and $\Gamma_{k+1} \subset \Gamma_k$ for all $k \geq 2$.
- (2°) If $\phi \in C(E, E)$ is odd and $\phi = \text{id}$ on $\partial\mathcal{B}_{R_m} \cap E_m$, then $\phi(A) \in \Gamma_k$ if $A \in \Gamma_k$ for all $k \geq 2$.
- (3°) If $A \in \Gamma_k$, $Z = -Z$ is open and $\gamma(Z) \leq s < k$ and $k - s \geq 2$, then $A \setminus Z \in \Gamma_{k-s}$.

Now, for $k \geq 2$, we claim that for any $A \in \Gamma_k$, $A \cap Q \neq \emptyset$. Consider the attracting domain of 0 in E :

$$\mathcal{D} := \{u \in E : \sigma(t, u) \rightarrow 0, \text{ as } t \rightarrow \infty\}.$$

Since 0 is a local minimum of I and by the continuous dependence of ODE on initial data, \mathcal{D} is open. Moreover, $\partial\mathcal{D}$ is an invariant set and $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \subset \mathcal{D}$. Similar to Lemma 3.4 in [2], we have

$$I(u) > 0 \text{ for every } u \in \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \setminus \{0\}.$$

Given $k \geq 2$, let $A \in \Gamma_k$, that is

$$A = h(\mathcal{B}_{R_m} \cap E_m \setminus Y)$$

with $\gamma(Y) \leq m - k$. Define

$$\mathcal{O} := \{u \in \mathcal{B}_{R_m} \cap E_m : h(u) \in \mathcal{D}\}.$$

Obviously, \mathcal{O} is a bounded open symmetric set with $0 \in \mathcal{O}$ and $\overline{\mathcal{O}} \subset \mathcal{B}_{R_m} \cap E_m$. Therefore, by the Borsuk-Ulam theorem, $\gamma(\partial\mathcal{O}) = m$. Moreover, by the continuity of h , $h(\partial\mathcal{O}) \subset \partial\mathcal{D}$. Consequently,

$$h(\partial\mathcal{O} \setminus Y) \subset A \cap \partial\mathcal{D}.$$

Thus, by using the ‘‘monotone, sub-additive and supervariant’’ property of the genus (cf. Proposition 5.4 in [30]), we have

$$\gamma(A \cap \partial\mathcal{D}) \geq \gamma(h(\partial\mathcal{O} \setminus Y)) \geq \gamma(\partial\mathcal{O} \setminus Y) \geq \gamma(\partial\mathcal{O}) - \gamma(Y) \geq k.$$

Noting that $P_\varepsilon^+ \cap P_\varepsilon^- \cap \partial\mathcal{D} = \emptyset$, one has $\gamma(W \cap \partial\mathcal{D}) \leq 1$. Hence, for $k \geq 2$, we conclude that

$$\gamma(A \cap Q \cap \partial\mathcal{D}) \geq \gamma(A \cap \partial\mathcal{D}) - \gamma(W \cap \partial\mathcal{D}) \geq k - 1 \geq 1,$$

which implies

$$(3.11) \quad A \cap Q \cap \partial\mathcal{D} \neq \emptyset.$$

Then, it follows that $A \cap Q \neq \emptyset$ for any $A \in \Gamma_k$ with $k \geq 2$. Thus, we finish the proof of the claim. Hence, for $k = 2, 3, \dots$, we can define a minimax value c_k by

$$c_k := \inf_{A \in \Gamma_k} \sup_{A \cap Q} I.$$

Choosing ρ given in Lemma 3.3 small enough if necessary, we have $\partial\mathcal{B}_\rho \subset \mathcal{D}$. Then by (3.11), for any $A \in \Gamma_k$, one has

$$\sup_{A \cap Q} I \geq \inf_{\partial\mathcal{D}} I \geq \inf_{\partial\mathcal{B}_\rho} I \geq \alpha > 0$$

by Lemma 3.3. As a consequence, $c_k \geq \alpha > 0$. Moreover, by (1^o), $c_{k+1} \geq c_k$ for any $k \geq 2$.

Step 2. We show that for all $k \geq 2$, there exists a sign-changing critical point u_k such that $I(u_k) = c_k$ i.e.,

$$(3.12) \quad K_{c_k} \cap Q \neq \emptyset.$$

To prove (3.12), arguing by contradiction, we suppose $K_{c_k} \cap Q = \emptyset$. By Lemma 3.9, there exist $\varepsilon > 0$ and a map $\eta \in C([0, 1] \times E, E)$ such that $\eta(1, \cdot)$ is odd, $\eta(1, u) = u$ for $u \in I^{c_k - 2\varepsilon}$ and

$$(3.13) \quad \eta(1, I^{c_k + \varepsilon} \cup W) \subset I^{c_k - \varepsilon} \cup W.$$

It follows from the definition of c_k that there exists $A \in \Gamma_k$ such that

$$\sup_{A \cap Q} I \leq c_k + \varepsilon.$$

Set $B = \eta(1, A)$. Then, by (3.13), we have

$$\sup_{B \cap Q} I \leq c_k - \varepsilon.$$

Noting that $B \in \Gamma_k$ by Lemma 3.2 and (2°) above, one concludes that $c_k \leq c_k - \varepsilon$, a contradiction.

Step 3. Finally, we shall prove that $c_k \rightarrow \infty$, as $k \rightarrow \infty$. This implies that $I(u)$ has infinitely many sign-changing critical points.

Arguing indirectly, we assume $c_k \rightarrow c < \infty$, as $k \rightarrow \infty$. Owing to the (PS) condition, it follows that K_c is nonempty and compact. Moreover, we have

$$K_c^2 := K_c \cap Q \neq \emptyset.$$

In fact, suppose $\{u_k\}$ is a sequence of sign-changing solutions to (1.1) with $I(u_k) = c_k$. Then, $\langle I'(u_k), u_k^\pm \rangle = 0$ and therefore

$$\|u_k^\pm\|_E^2 + b \int_{\mathbb{R}^3} |\nabla u_k|^2 \int_{\mathbb{R}^3} |\nabla u_k^\pm|^2 = \int_{\mathbb{R}^3} f(u_k^\pm) u_k^\pm.$$

Thus by (3.1) and Sobolev embedding theorem, we have $\|u_k^\pm\|_E \geq \delta_0 > 0$, where δ_0 is a constant independent of k . Noting that I satisfies the (PS) condition, passing to a subsequence if necessary, there exists $u \in K_c$ such that $u_k \rightarrow u$. Then, the above inequality implies that u is still sign-changing and hence $K_c^2 \neq \emptyset$.

Suppose $\gamma(K_c^2) = \tau$. Since $0 \notin K_c^2$ and K_c^2 is compact, by the ‘‘continuous’’ property of the genus, there exists a open neighborhood N in E with $K_c^2 \subset N$ such that $\gamma(N) = \tau$. From Lemma 3.9, there exist $\varepsilon > 0$ and a map $\eta \in C([0, 1] \times E, E)$ such that $\eta(1, \cdot)$ is odd, $\eta(1, u) = u$ for $u \in I^{c-2\varepsilon}$ and

$$(3.14) \quad \eta(1, I^{c+\varepsilon} \cup W \setminus N) \subset I^{c-\varepsilon} \cup W.$$

Since $c_k \rightarrow c$ as $k \rightarrow \infty$, we can choose k sufficiently large, such that

$$(3.15) \quad c_k \geq c - \frac{1}{2}\varepsilon.$$

Note that $c_{k+\tau} \geq c_k$. By the definition of $c_{k+\tau}$, there exists $A \in \Gamma_{k+\tau}$, i.e.,

$$A = h(\mathcal{B}_R \cap E_m \setminus Y),$$

where $h \in G_m$, $m \geq k + \tau$, $\gamma(Y) \leq m - (k + \tau)$, such that

$$I(u) \leq c_{k+\tau} + \frac{1}{4}\varepsilon < c + \varepsilon, \quad \text{for any } u \in A \cap Q.$$

Therefore $A \subset I^{c+\varepsilon} \cup W$. Then, from (3.14), we have

$$(3.16) \quad \eta(1, A \setminus N) \subset I^{c-\varepsilon} \cup W.$$

Set $Y_1 = Y \cup h^{-1}(N)$. Clearly, Y_1 is symmetric and open, and

$$\gamma(Y_1) \leq \gamma(Y) + \gamma(h^{-1}(N)) \leq m - (k + \tau) + \tau = m - k.$$

Then, by (2°) and (3°) above, one concludes that

$$\tilde{A} := \eta(1, h(\mathcal{B}_R \cap E_m \setminus Y_1)) \in \Gamma_k.$$

As a result, by (3.16)

$$c_k \leq \sup_{\tilde{A} \cap Q} I \leq \sup_{\eta(1, A \setminus N) \cap Q} I \leq c - \varepsilon.$$

This is a contradiction to (3.15) and hence the proof is completed. \square

4. PROOF OF THEOREM 1.2

Fix a number $r \in (\max\{4, p\}, 6)$. As in [21], we introduce a family of functional defined by

$$(4.1) \quad I_\lambda(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u|^2 - \int_{\mathbb{R}^3} F(u) - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r$$

for $\lambda \in (0, 1]$. It is standard to show that $I_\lambda \in C^1(E, \mathbb{R})$.

In order to study our problem, we give some preliminary results. Firstly, as Lemma 2.1 in [13], we have the following Pohozaev type identity.

Lemma 4.1. *Assume (V_0) – (V_1) and (f_1) – (f_3) hold. Let u be a critical point of I_λ in E , then*

$$\begin{aligned} \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(x)|u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} (DV(x), x)|u|^2 + \frac{b}{2} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 \\ - 3 \int_{\mathbb{R}^3} F(u) - \frac{3\lambda}{r} \int_{\mathbb{R}^3} |u|^r = 0. \end{aligned}$$

Lemma 4.2. *For $\lambda \in (0, 1]$, I_λ satisfies the (PS) condition.*

Proof. Here we adopt a technique in the proof of Lemma 4.2 in [21]. Assume that there exist $\{u_n\} \subset E$ and $c \in \mathbb{R}$ such that $I_\lambda(u_n) \rightarrow c$ and $I'_\lambda(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Choose a number $\gamma \in (4, r)$. For $u \in E$, we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\gamma} \langle I'_\lambda(u_n), u_n \rangle &= \left(\frac{1}{2} - \frac{1}{\gamma} \right) \|u_n\|_E^2 + \left(\frac{1}{4} - \frac{1}{\gamma} \right) b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 \\ &\quad + \int_{\mathbb{R}^3} \left(\frac{1}{\gamma} f(u_n)u_n - F(u_n) \right) + \left(\frac{1}{\gamma} - \frac{1}{r} \right) \lambda \|u_n\|_r^r. \end{aligned}$$

Then, by (f_1) and (f_2) , one sees that

$$\|u_n\|_E^2 + \lambda \|u_n\|_r^r \leq C_1 (|I_\lambda(u_n)| + \|u_n\|_E \|I'_\lambda(u_n)\| + \|u_n\|_p^p).$$

Thus, by the Young inequality, for large n ,

$$(4.2) \quad \|u_n\|_E^2 + \lambda \|u_n\|_r^r \leq C_2 (1 + \|u_n\|_p^p).$$

Now we show that $\{u_n\}$ is bounded in E . Arguing indirectly, we assume that $\|u_n\|_E \rightarrow \infty$ as $n \rightarrow \infty$. Let $w_n = u_n / \|u_n\|_E$. Then $\|w_n\| = 1$ and thus up to a subsequence if necessary, there exists $w \in E$ such that $w_n \rightharpoonup w$ in E and $w_n \rightarrow w$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 6)$. Since $2 < p < r$, by (4.2), it is easy to see that

$$\|w\|_r^r \leq 0,$$

which implies $w = 0$. On the other hand, from (4.2), there exists $C(\lambda) > 0$ such that for large n ,

$$\|u_n\|_2^2 + \|u_n\|_r^r \leq C(\lambda) \|u_n\|_p^p \leq C(\lambda) \|u_n\|_2^{tp} \|u_n\|_r^{(1-t)p},$$

here we have used the interpolation inequality and $t \in (0, 1)$ satisfying $\frac{1}{p} = \frac{t}{2} + \frac{1-t}{r}$. As a consequence, there exist $C_1(\lambda), C_2(\lambda) > 0$ such that, for large n ,

$$C_1(\lambda) \|u_n\|_2^{\frac{2}{t}} \leq \|u_n\|_r \leq C_2(\lambda) \|u_n\|_2^{\frac{2}{t}}.$$

Therefore $\|u_n\|_p^p \leq C_3(\lambda)\|u_n\|_2^2$ and hence, by (4.2), for large n , we have

$$\|u_n\|_E^2 \leq C_4(\lambda)\|u_n\|_2^2,$$

which implies that $\|w_n\|_2^2 \geq (C_4(\lambda))^{-1}$. Noting that $w_n \rightarrow w$ in $L^2(\mathbb{R}^3)$, we see that $\|w\|_2^2 \geq (C_4(\lambda))^{-1}$. This contradicts to $w = 0$ and therefore $\{u_n\}$ is bounded in E .

Then, up to a subsequence, we can assume that $u_n \rightharpoonup u$ in E as $n \rightarrow \infty$. By Lemma 2.1, we see that

$$(4.3) \quad u_n \rightarrow u \quad \text{in } L^s(\mathbb{R}^3), \quad \text{for any } s \in [2, 6).$$

Note that

$$\begin{aligned} \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle &= \|u_n - u\|_E^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla(u_n - u) \\ &\quad + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla(u_n - u) \\ &\quad - \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) \\ &\quad - \lambda \int_{\mathbb{R}^3} (|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u) \\ &\geq \|u_n - u\|_E^2 \\ &\quad + b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) \\ &\quad - \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) \\ &\quad - \lambda \int_{\mathbb{R}^3} (|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u). \end{aligned}$$

By (f_1) - (f_2) and (4.3), it is standard to show that

$$\int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) + \lambda \int_{\mathbb{R}^3} (|u_n|^{r-2}u_n - |u|^{r-2}u)(u_n - u) = o(1),$$

as $n \rightarrow \infty$. Moreover, by the boundedness of $\{u_n\}$ in E , (4.3) and the fact that $u_n \rightharpoonup u$ in E , one has

$$b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_n - u) = o(1), \quad \text{as } n \rightarrow \infty.$$

Thus, from $I'_\lambda(u_n) \rightarrow 0$, we conclude that $\|u_n - u\|_E^2 \rightarrow 0$ as $n \rightarrow \infty$ and hence the proof is completed. \square

Note that for any $\lambda \in (0, 1]$,

$$I_\lambda(u) = I(u) - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r \leq I(u).$$

Thus, one can follow the same line of the proof of Lemma 3.2 to obtain the following result.

Lemma 4.3. *Suppose (f_1) - (f_3) hold and $m \geq 1$. Then there exists $R_m > 0$ independent of λ , such that*

$$\sup_{\mathcal{B}_{R_m}^c \cap E_m} I_\lambda < 0.$$

Lemma 4.4. *Assume (f_1) and (f_2) hold. Then for any $\lambda \in (0, 1]$, there exist $\rho_\lambda > 0$ and $\alpha_\lambda > 0$ such that*

$$\inf_{\partial \mathcal{B}_{\rho_\lambda}} I_\lambda \geq \alpha_\lambda.$$

Proof. By (3.1), for $u \in E$, we have

$$(4.4) \quad \begin{aligned} I_\lambda(u) &= \frac{1}{2} \|u\|_E^2 + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^3} F(u) - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r \\ &\geq \frac{1}{2} \|u\|_E^2 - \frac{\delta}{2} \int_{\mathbb{R}^3} |u|^2 - \frac{C_\delta}{p} \int_{\mathbb{R}^3} |u|^p - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r. \end{aligned}$$

Notice that for any $s \in [2, 6]$, there exists $C(s) > 0$ such that $\|u\|_s \leq C(s)\|u\|_E$. Choose δ satisfying that $C(2)\delta < \frac{1}{2}$. Then, from (4.4) we have

$$I_\lambda(u) \geq \frac{1}{4} \|u\|_E^2 - \frac{C_\delta C(p)}{p} \|u\|_E^p - \frac{\lambda C(r)}{r} \|u\|_E^r.$$

Noting that $r > p > 2$, one obtains that there exist $\rho_\lambda, \alpha_\lambda > 0$ such that $\inf_{\partial \mathcal{B}_{\rho_\lambda}} I_\lambda \geq \alpha_\lambda$, as required. \square

Let $\lambda \in (0, 1]$, for any $u \in E$, we consider the following equation

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta v + V(x)v = f(u) + \lambda |u|^{r-2}u, \quad v \in E.$$

Similar to Lemma 3.4, one can prove that the above equation has a unique solution, denoted by $v = A_\lambda(u) \in E$ and the operator $A_\lambda : E \rightarrow E$ is continuous. As in Section 3, we shall study some properties of the operator A_λ .

Lemma 4.5.

- (i) $\langle I'_\lambda(u), u - A_\lambda(u) \rangle \geq \|u - A_\lambda(u)\|_E^2$ for all $u \in E$.
- (ii) There exists $C > 0$ independent of λ such that $\|I'_\lambda(u)\| \leq \|u - A_\lambda(u)\|_E(1 + C\|u\|_E^2)$ for all $u \in E$.
- (iii) For $M > 0$ and $\alpha > 0$, there exists $\beta_\lambda > 0$ such that $\|u - A_\lambda(u)\|_E \geq \beta_\lambda$ for any $u \in E$ with $|I_\lambda(u)| \leq M$ and $\|I'_\lambda(u)\| \geq \alpha$.
- (iv) If f is odd, then so is A_λ .

Proof. We only prove (iii), because the proofs of (i)-(ii) and (iv) are similar to that of Lemma 3.5. Fix a number $\gamma \in (4, r)$. Notice that

$$\langle I'_\lambda(u), u \rangle = (u - A_\lambda(u), u)_E + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla(u - A_\lambda(u)) \cdot \nabla u,$$

which implies that

$$\begin{aligned} I_\lambda(u) - \frac{1}{\gamma} (u - A_\lambda(u), u)_E &= \left(\frac{1}{2} - \frac{1}{\gamma}\right) \|u\|_E^2 + \left(\frac{1}{4} - \frac{1}{\gamma}\right) b \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 \\ &\quad + \frac{b}{\gamma} \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla(u - A_\lambda(u)) \cdot \nabla u \\ &\quad + \int_{\mathbb{R}^3} \left(\frac{1}{\gamma} f(u)u - F(u)\right) + \left(\frac{1}{\gamma} - \frac{1}{r}\right) \lambda \|u\|_E^r. \end{aligned}$$

From (f_1) and (f_2) , we have

$$\begin{aligned} & \|u\|_E^2 + \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \lambda \|u\|_r^r \\ & \leq C \left(|I_\lambda(u)| + \|u\|_E \|u - A_\lambda(u)\|_E + \int_{\mathbb{R}^3} |\nabla u|^2 \left| \int_{\mathbb{R}^3} \nabla u \cdot \nabla (u - A_\lambda(u)) \right| + \|u\|_p^p \right). \end{aligned}$$

Then, by the Hölder inequality and Young inequality, one sees that

$$(4.5) \quad \|u\|_E^2 + \lambda \|u\|_r^r \leq C_1 (|I(u)| + \|u\|_E \|u - A_\lambda(u)\|_E + \|u\|_E^2 \|u - A_\lambda(u)\|_E^2 + \|u\|_p^p).$$

Arguing indirectly, suppose that there exists $\{u_n\} \subset E$ with $|I_\lambda(u_n)| \leq M$ and $\|I'_\lambda(u_n)\| \geq \alpha$ such that $\|u_n - A_\lambda(u_n)\|_E \rightarrow 0$ as $n \rightarrow \infty$. Then, it follows from (4.5) that, for large n ,

$$\|u\|_E^2 + \lambda \|u\|_r^r \leq C_2 (1 + \|u\|_p^p).$$

Similar to the proof of Lemma 4.2, one can show that $\{u_n\}$ is bounded in E . Hence, jointly with (ii), this implies $\|I'_\lambda(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$, a contradiction. This ends the proof. \square

Lemma 4.6. *There exists $\varepsilon_0 > 0$ independent of λ such that for any $\varepsilon \in (0, \varepsilon_0]$, there holds*

$$A_\lambda(\overline{P}_\varepsilon^\pm) \subset P_\varepsilon^\pm.$$

Moreover, every nontrivial solutions $u \in P_\varepsilon^+$ and $u \in P_\varepsilon^-$ of (1.1) are positive and negative, respectively.

Proof. As in the proof of Lemma 3.6, we have

$$\begin{aligned} \text{dist}(v, P^+) \|v^-\|_E & \leq \int_{\mathbb{R}^3} f(u)v^- + \lambda \int_{\mathbb{R}^3} |u|^{r-2} u v^- - b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v \cdot \nabla v^- \\ & \leq \int_{\mathbb{R}^3} (\delta |u^-| + C_\delta |u^-|^{p-1} + |u^-|^{r-1}) |v^-| \\ & \leq C(\delta \text{dist}(u, P^+) + C_\delta \text{dist}(u, P^+)^{p-1} + \text{dist}(u, P^+)^{r-1}) \|v^-\|_E, \end{aligned}$$

here C is a constant independent of λ . Thus, choosing $\delta = \frac{1}{2C}$ and $\varepsilon_0 > 0$ satisfying with $CC_\delta \varepsilon_0^{p-2} + C\varepsilon_0^{r-2} \leq \frac{1}{4}$, for any $\varepsilon \in (0, \varepsilon_0]$, one concludes that

$$\text{dist}(v, P^+) \leq \frac{1}{2} \text{dist}(u, P^+) < \varepsilon, \quad \text{for any } u \in \overline{P}_\varepsilon^+.$$

This implies that $A_\lambda(\overline{P}_\varepsilon^+) \subset P_\varepsilon^+$. Moreover, we can show that any nontrivial solutions $u \in P_\varepsilon^+$ is positive. The other case can be proved similarly. \square

Denote the set of critical points of I_λ by K_λ . As in Lemma 3.7, we have the following results.

Lemma 4.7. *There exists a locally Lipschitz continuous operator $B_\lambda : E \setminus K_\lambda \rightarrow E$ with the following properties:*

- (1) $B_\lambda(\overline{P}_\varepsilon^\pm) \subset P_\varepsilon^\pm$ for all $\varepsilon \in (0, \varepsilon_0]$;
- (2) $\frac{1}{2} \|u - B_\lambda(u)\|_E \leq \|u - A_\lambda(u)\|_E \leq 2 \|u - B_\lambda(u)\|_E$ for all $u \in E \setminus K_\lambda$;
- (3) $\langle I'_\lambda(u), u - B_\lambda(u) \rangle \geq \frac{1}{2} \|u - A_\lambda(u)\|_E^2$ for all $u \in E \setminus K_\lambda$;
- (4) If f is odd, then so is B_λ .

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Set

$$G_m := \{h \in C(\mathcal{B}_{R_m} \cap E_m, E) : h \text{ is odd and } h = \text{id on } \partial\mathcal{B}_{R_m} \cap E_m\},$$

where $R_m > 0$ is given by Lemma 4.3. For $k \geq 2$, we denote Γ_k by

$$\Gamma_k := \{h(\mathcal{B}_{R_m} \cap E_m \setminus Y) : h \in G_m, m \geq k, Y = -Y \text{ is open and } \gamma(Y) \leq m - k\}.$$

With the help of Lemmas 4.2–4.7, as in Section 3, for any fixed $\lambda \in (0, 1]$, we can define a minimax value c_k^λ for the functional $I_\lambda(u)$ as

$$c_k^\lambda := \inf_{A \in \Gamma_k} \sup_{A \cap Q} I_\lambda, \quad k = 2, 3, \dots.$$

Moreover, one can show that for all $k \geq 2$ there exists $u_k^\lambda \in Q \cap K_{c_k^\lambda}$ and

$$0 < \alpha_\lambda \leq c_k^\lambda \rightarrow \infty, \quad \text{as } k \rightarrow \infty,$$

where α_λ is defined in Lemma 4.4. Then, we have the following

Claim: For any fixed $k \geq 2$, the sequence $\{u_k^\lambda\}_{\lambda \in (0, 1]}$ obtained above is bounded in E .

Indeed, notice that for any $\lambda \in (0, 1]$, $I_\lambda(u) \leq I(u)$ for $u \in E$. Then, for any given $k \geq 2$, by the definition of c_k^λ , we can obtain that

$$(4.6) \quad c_k^\lambda \leq c(m) := \sup_{\mathcal{B}_{R_m} \cap E_m} I < +\infty, \quad \text{for all } \lambda \in (0, 1],$$

where $m \geq k$ is fixed and $R_m > 0$ is given by Lemma 4.3. Moreover, we have

$$(4.7) \quad \begin{aligned} & \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u_k^\lambda|^2 \\ & - \int_{\mathbb{R}^3} F(u_k^\lambda) - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u_k^\lambda|^r = c_k^\lambda \end{aligned}$$

$$(4.8) \quad \begin{aligned} & a \int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 + b \left(\int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 \right)^2 + \int_{\mathbb{R}^3} V(x) |u_k^\lambda|^2 \\ & - \int_{\mathbb{R}^3} f(u_k^\lambda) u_k^\lambda - \lambda \int_{\mathbb{R}^3} |u_k^\lambda|^r = 0 \end{aligned}$$

and the Pohozaev type identity

$$(4.9) \quad \begin{aligned} & \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(x) |u_k^\lambda|^2 + \frac{1}{2} \int_{\mathbb{R}^3} (DV(x), x) |u_k^\lambda|^2 + \frac{b}{2} \left(\int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 \right)^2 \\ & - 3 \int_{\mathbb{R}^3} F(u_k^\lambda) - \frac{3\lambda}{r} \int_{\mathbb{R}^3} |u_k^\lambda|^r = 0. \end{aligned}$$

Multiplying (4.7) by $\mu + 6$, (4.8) by -1 and (4.9) by -2 and adding them up, one concludes that

$$\begin{aligned} & \frac{\mu + 2}{2} a \int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 + \frac{\mu - 2}{4} b \left(\int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 \right)^2 \\ & + \int_{\mathbb{R}^3} \left(\frac{\mu - 2}{2} V(x) - (DV(x), x) \right) |u_k^\lambda|^2 + \int_{\mathbb{R}^3} (f(u_k^\lambda) u_k^\lambda - \mu F(u_k^\lambda)) \\ & + \frac{r - \mu}{r} \lambda \int_{\mathbb{R}^3} |u_k^\lambda|^r = (\mu + 6) c_k^\lambda. \end{aligned}$$

Thus, noting that $2 < \mu \leq p < r$, by (V_2) , (f_3) and (4.6), we see that $\int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2$ and $\lambda \int_{\mathbb{R}^3} |u_k^\lambda|^r$ are bounded uniformly in $\lambda \in (0, 1]$. Using this fact, from (f_3) , (4.7)

and (4.8), we deduce that $\{u_k^\lambda\}_{\lambda \in (0,1]}$ is bounded in E and hence the proof of the claim is finished.

Then, up to a subsequence, we can assume that $u_k^\lambda \rightharpoonup u_k$ in E as $\lambda \rightarrow 0^+$ and $u_k^\lambda \rightarrow u_k$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 6)$. Note that

$$\begin{aligned} \langle I'_\lambda(u_k^\lambda) - I'(u_k), u_k^\lambda - u_k \rangle &= \|u_k^\lambda - u_k\|_E^2 + b \int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 \int_{\mathbb{R}^3} \nabla u_k^\lambda \cdot \nabla (u_k^\lambda - u_k) \\ &\quad + b \int_{\mathbb{R}^3} |\nabla u_k|^2 \int_{\mathbb{R}^3} \nabla u_k \cdot \nabla (u_k^\lambda - u_k) \\ &\quad - \int_{\mathbb{R}^3} (f(u_k^\lambda) - f(u_k))(u_k^\lambda - u_k) - \lambda \int_{\mathbb{R}^3} |u_k^\lambda|^{r-2} u_k^\lambda (u_k^\lambda - u_k) \\ &\geq \|u_k^\lambda - u_k\|_E^2 + b \int_{\mathbb{R}^3} (|\nabla u_k^\lambda|^2 - |\nabla u_k|^2) \int_{\mathbb{R}^3} \nabla u_k \cdot \nabla (u_k^\lambda - u_k) \\ &\quad - \int_{\mathbb{R}^3} (f(u_k^\lambda) - f(u_k))(u_k^\lambda - u_k) - \lambda \int_{\mathbb{R}^3} |u_k^\lambda|^{r-2} u_k^\lambda (u_k^\lambda - u_k). \end{aligned}$$

Then, by (f_1) - (f_2) and $I'_\lambda(u_k^\lambda) = 0$, as in the proof of Lemma 4.2, we see that $\|u_k^\lambda - u_k\|_E^2 \rightarrow 0$ as $\lambda \rightarrow 0^+$. Consequently, $I'(u_k) = 0$ and $I(u_k) = c_k := \lim_{\lambda \rightarrow 0^+} c_\lambda^k$.

We claim that u_k is still sign-changing. Indeed, using the fact that $\langle I'_\lambda(u_k^\lambda), u_k^{\lambda^\pm} \rangle = 0$, we have

$$\begin{aligned} \|u_k^{\lambda^\pm}\|_E^2 &\leq \|u_k^{\lambda^\pm}\|_E^2 + b \int_{\mathbb{R}^3} |\nabla u_k^\lambda|^2 \int_{\mathbb{R}^3} |\nabla u_k^{\lambda^\pm}|^2 = \int_{\mathbb{R}^3} f(u_k^{\lambda^\pm}) u_k^{\lambda^\pm} + \lambda \int_{\mathbb{R}^3} |u_k^{\lambda^\pm}|^r \\ &\leq \int_{\mathbb{R}^3} f(u_k^{\lambda^\pm}) u_k^{\lambda^\pm} + \int_{\mathbb{R}^3} |u_k^{\lambda^\pm}|^r, \end{aligned}$$

which, jointly with (3.1) and Sobolev embedding theorem, implies that $\|u_k^{\lambda^\pm}\|_E \geq \alpha_0 > 0$, where α_0 is a constant independent of λ . From this fact and $u_k^\lambda \rightarrow u_k$ in E , it is easy to see that $\|u_k^\pm\|_E \geq \alpha_0$. Therefore, u is a sign-changing solution of (1.1).

Noting that c_λ^k is nonincreasing in λ , we see that $c_k \geq c_\lambda^k$ for any $\lambda \in (0, 1]$. Since $c_k^\lambda \rightarrow \infty$ as $k \rightarrow \infty$, one deduces that

$$\lim_{k \rightarrow \infty} c_k = \infty.$$

This implies that equation (1.1) admits infinitely many sign-changing solutions and therefore the proof is completed. \square

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