On Gehring–Martin–Tan groups with an elliptic generator

Dušan D. Repovš and Andrei Yu. Vesnin

Abstract. The Gehring–Martin–Tan inequality for 2-generator subgroups of $\mathrm{PSL}(2, \mathbb{C})$ is one of the best known discreteness conditions. A Kleinian group G is called a Gehring–Martin–Tan group if the equality holds for the group G . We give a method for constructing Gehring–Martin–Tan groups with a generator of order four and present some examples. These groups arise as groups of finite volume hyperbolic 3-orbifolds.

Mathematics Subject Classification (2010). Primary 57M50; Secondary 20H15.

Keywords. Hyperbolic orbifold, discreteness condition, two-generated group.

1. Introduction

In this paper we are interested in discreteness conditions for groups of isometries of a hyperbolic 3-space $\mathbb{H}^3$. It was shown by Jørgensen that it suffices to understand the discreteness problem for a class of two-generated groups. The most famous necessary discreteness conditions are the Shimizu lemma and the Jørgensen inequality [2]. There are some generalizations of these conditions for complex and quaternionic hyperbolic spaces, see for example [5, 9, 15].

A description of the set of all two-generated discrete non-elementary groups of isometries $\mathbb{H}^3$ for which the equality holds in the Jørgensen inequality is an open problem of special interest. For many elegant results concerning this problem see [4, 8, 17] and references therein.

We shall consider a different discreteness condition for two-generated groups which was independently proved by Gehring and Martin [8] and Tan [18]. Like the Jørgensen inequality, this condition has a form of an inequality involving the trace of one of the generators and the trace of the commutator of generators. We shall say that a group is a *GMT-group* if it

can be generated by two elements for which the equality holds. The following problem arises naturally.

Problem. *Find all GMT-groups.*

The problem is still open, and in this paper we shall present a method to construct new examples of GMT-groups from known examples.

The most interesting example of a GMT-group is a group, related to the well-known figure-eight knot. Denote by $\mathcal{F}(n)$ the orbifold with the underlying space S^3 and the singular set the figure-eight knot $\mathcal{F}$ with singularity index n , $n \geq 4$. According to [1], orbifolds $\mathcal{F}(n)$ are extreme in the following sense: Let L_n denote the set of all orientable hyperbolic 3-orbifolds with non-empty singular set and with all torsion orders bounded below by n . Therefore

$$L_2 \supset L_3 \supset L_4 \supset \dots \quad \text{and} \quad \bigcap L_n = \emptyset.$$

Then for all $n \geq 4$, the unique lowest-volume element of L_n is the orbifold $\mathcal{F}(n)$. A formula for $\text{vol } \mathcal{F}(n)$ was given in [21]. It was shown in [19] that $\mathcal{F}(4)$ is extreme in the sense of discreteness conditions: the orbifold group of $\mathcal{F}(4)$ is a GMT-group. Below we shall use the orbifold group of $\mathcal{F}(4)$ as a starting point for constructions of new examples of GMT-groups.

In Section 2 we shall give basic definitions and describe some properties of GMT-groups. In particular, we shall prove Lemma 2.4 which gives a method for constructing new GMT-groups. Next, we shall apply this method. In Section 3 we shall prove that some 3-orbifold hyperbolic groups related to the figure-eight knot are GMT-groups. In Section 4 we shall give examples of GMT-groups which are subgroups of the Picard group.

2. Gehring–Martin–Tan discreteness condition

Let $\mathbb{H}^3$ be the three-dimensional hyperbolic space presented by the Poincaré model in the upper halfspace. Then the boundary $\partial\mathbb{H}^3$ can be identified with $\mathbb{C}$. It is well-known that the group $\text{Iso}(\mathbb{H}^3)$ of all orientation preserving isometries of $\mathbb{H}^3$ is isomorphic to

$$\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \{\pm I\},$$

where I denotes the unit matrix. In the sequel we shall not distinguish between a matrix $M \in \text{SL}(2, \mathbb{C})$ and its equivalence class $\{\pm M\} \in \text{PSL}(2, \mathbb{C})$. An action of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$$

on

$$\mathbb{H}^3 = \{(z, t) \mid z \in \mathbb{C}, t \in \mathbb{R}_+\}$$

is defined by the following rule:

$$g(z, t) = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2t^2}, \frac{t}{|cz + d|^2 + |c|^2t^2} \right).$$

Recall that a matrix $M \in \text{SL}(2, \mathbb{C}) \setminus \{\pm I\}$ is said to be:

- *elliptic* if $\text{tr}^2(M) \in [0, 4)$;

- *parabolic* if $\text{tr}^2(M) = 4$; and
- *loxodromic* if $\text{tr}^2(M) \in \mathbb{C} \setminus [0, 4]$.

In particular, a loxodromic element is said to be:

- *hyperbolic* if $\text{tr}(M) \in (-\infty, 0) \cup (2, +\infty)$.

We shall say that an element of group $\text{PSL}(2, \mathbb{C})$ is *elliptic*, *parabolic*, or *loxodromic* if its representative in $\text{SL}(2, \mathbb{C})$ is of such type.

A group $G < \text{PSL}(2, \mathbb{C})$ is said to be *discrete* if it is a discrete set in the matrix quotient-topology. A group $G < \text{PSL}(2, \mathbb{C})$ is said to be *elementary* if there exists a finite G -orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$, and *non-elementary* otherwise.

In 1977 Jørgensen proved [7] that a non-elementary group $G < \text{PSL}(2, \mathbb{C})$ is discrete if and only if any two elements $f, g \in G$ generate a discrete group. His result motivated many other investigations of discreteness conditions for two-generated groups. In the present paper we shall discuss the necessary discreteness condition obtained in 1989 by Gehring and Martin [8] and independently by Tan [18]. We shall formulate their result as follows:

Theorem 2.1 (See [8, 18]). *Suppose that $f, g \in \text{PSL}(2, \mathbb{C})$ generate a discrete group. If $\text{tr}[f, g] \neq 1$ then the following inequality holds:*

$$|\text{tr}^2(f) - 2| + |\text{tr}[f, g] - 1| \geq 1. \quad (2.1)$$

This result makes following definitions natural. For $f, g \in \text{PSL}(2, \mathbb{C})$ such that $\text{tr}[f, g] \neq 1$ define

$$\mathcal{G}(f, g) = |\text{tr}^2(f) - 2| + |\text{tr}[f, g] - 1|.$$

Let $G < \text{PSL}(2, \mathbb{C})$ be a two-generated group. The value

$$\mathcal{G}(G) = \inf_{\langle f, g \rangle = G} \mathcal{G}(f, g)$$

is referred to as the *Gehring–Martin–Tan number* (or, shortly *GMT-number*) of G . A two-generated discrete group $G < \text{PSL}(2, \mathbb{C})$ is said to be a *GMT-group* if it can be generated by f and g such that $\mathcal{G}(f, g) = 1$.

The following statement shows (see also [20]) that property $\mathcal{G}(f, g) = 1$ implies many restrictions for f .

Lemma 2.2. *Suppose that $f, g \in \text{PSL}(2, \mathbb{C})$ generate a discrete group and $\text{tr}[f, g] \neq 1$. Assume that f is one of the following transformations:*

- (i) *parabolic*;
- (ii) *hyperbolic*;
- (iii) *elliptic of order 2 or 3; or*
- (iv) *elliptic with trace $\text{tr}^2(f) = 4 \cos^2(\pi k/n)$, where $(n, k) = 1$, $n/k \geq 6$.*

Then for any g we have $\mathcal{G}(f, g) > 1$.

Proof. The result follows immediately from the classification of elements of $\text{PSL}(2, \mathbb{C})$ and from the fact that $\mathcal{G}(f, g)$ is defined for pairs f, g such that $\text{tr}[f, g] \neq 1$.

- (i) If f is parabolic then $\text{tr}^2(f) = 4$ and therefore $|\text{tr}^2(f) - 2| = 2 > 1$.
- (ii) If f is hyperbolic then $\text{tr}(f) \in (-\infty, 0) \cup (2, \infty)$, so $|\text{tr}^2(f) - 2| > 2$.

- (iii) If f is elliptic of order 2 then $\text{tr}^2(f) = 0$, so $|\text{tr}^2(f) - 2| = 2$. If f is elliptic of order 3 then $\text{tr}^2(f) = 1$, so $|\text{tr}^2(f) - 2| = 1$. Since $\text{tr}[f, g] \neq 1$ we get $\mathcal{G}(f, g) > 1$.
- (iv) If f is elliptic with trace $\text{tr}^2(f) = 4\cos^2(\pi k/n)$, where $(n, k) = 1$ and $n/k \geq 6$, then $\text{tr}^2(f) \geq 4\cos^2(\pi/6) = 3$, so $|\text{tr}^2(f) - 2| \geq 1$. Since $\text{tr}[f, g] \neq 1$ we get $\mathcal{G}(f, g) > 1$.

□

The following statement gives a way for finding GMT-subgroups of GMT-groups with a generator of order four.

Lemma 2.3 (See [20]). *Let $\langle f, g \rangle$ be a GMT-group with $\mathcal{G}(f, g) = 1$, where f is elliptic of order four. Then a group generated by f and $h = gfg^{-1}$ is a GMT-group with $\mathcal{G}(f, h) = 1$.*

The following statement gives a method for constructing GMT-groups as extensions of GMT-groups with a generator of order four.

Lemma 2.4. *Let $\langle f, g \rangle$ be a GMT-group with $\mathcal{G}(f, g) = 1$, where f is elliptic of order four. Assume that $h \in \text{PSL}(2, \mathbb{C})$ is an involution of $\langle f, g \rangle$ with one of the following conjugation actions:*

- (i) $hfh^{-1} = g$;
- (ii) $hfh^{-1} = f^{-1}$; or
- (iii) $hfh^{-1} = fg^{-1}f^{-1}$.

Then $\langle f, h \rangle$ is a GMT-group.

Proof. Since f is elliptic of order four, we have $\text{tr}^2(f) = 2$, hence $\mathcal{G}(f, g) = |\text{tr}[f, g] - 1|$ and $\mathcal{G}(f, h) = |\text{tr}[f, h] - 1|$. Since h is an involution, it follows that either $\langle f, g \rangle$ is a subgroup of index two in $\langle f, h \rangle$, or $\langle f, g \rangle$ and $\langle f, h \rangle$ coincide. Therefore $\langle f, h \rangle$ is discrete. Recall that by [2] the identify

$$\text{tr}[f, hfh^{-1}] = (\text{tr}[f, h] - 2)(\text{tr}[f, h] - \text{tr}^2(f) + 2) + 2 \quad (2.2)$$

holds for any $f, h \in \text{PSL}(2, \mathbb{C})$. Then using $\text{tr}^2(f) = 2$, we get from (2.2) that

$$\text{tr}[f, hfh^{-1}] = (\text{tr}[f, h] - 2)\text{tr}[f, h] + 2.$$

Hence

$$|\text{tr}[f, hfh^{-1}] - 1| = |\text{tr}[f, h] - 1|^2. \quad (2.3)$$

Consider the case (i) with $hfh^{-1} = g$. Hence

$$|\text{tr}[f, hfh^{-1}] - 1| = |\text{tr}[f, g] - 1| = \mathcal{G}(f, g) = 1.$$

Therefore, $\mathcal{G}(f, h) = 1$.

Consider the case (ii) with $hfh^{-1} = f^{-1}$. Hence

$$|\text{tr}[f, hfh^{-1}] - 1| = |\text{tr}[f, f^{-1}] - 1| = 1.$$

Therefore, $\mathcal{G}(f, h) = 1$.

Consider the case (iii) with $hfh^{-1} = fg^{-1}f^{-1}$. Then

$$\begin{aligned} |\text{tr}[f, hfh^{-1}] - 1| &= |\text{tr}[f, fg^{-1}f^{-1}] - 1| = |\text{tr}(fg^{-1}f^{-1}g) - 1| \\ &= |\text{tr}([f, g]^{-1}) - 1| = |\text{tr}[f, g] - 1| = 1. \end{aligned}$$

Here we used that for $\alpha \in \mathrm{SL}(2, \mathbb{C})$ the relation $\mathrm{tr}(\alpha^{-1}) = \mathrm{tr} \alpha$ holds. Therefore $\mathcal{G}(f, h) = 1$. $\square$

In the next section we shall realize a method given by Lemma 2.4.

3. The figure-eight knot and related orbifolds

Let us denote by $\mathcal{F}$ the figure-eight knot in the 3-sphere S^3 presented by its diagram in Fig. 1. The knot group $\pi_1(S^3 \setminus \mathcal{F})$ can be easily found by the

FIGURE 1. Generators of the group $\pi_1(S^3 \setminus \mathcal{F})$.

Wirtinger algorithm. Taking generators b and ρ , the corresponding loops are marked in Fig. 1, we get

$$\pi_1(S^3 \setminus \mathcal{F}) = \langle \rho, b \mid \rho^{-1} [b, \rho] = [b, \rho] b \rangle,$$

where $[b, \rho] = b\rho b^{-1}\rho^{-1}$. It is well-known [16] that the group $\pi_1(S^3 \setminus \mathcal{F})$ has a faithful representation in $\mathrm{PSL}(2, \mathbb{C})$.

Denote by $\mathcal{F}(n)$ the orbifold with the underlying space S^3 and singular set $\mathcal{F}$ with singularity index n , where $n \geq 4$. Cyclic n -fold coverings of $\mathcal{F}(n)$ are known as *Fibonacci manifolds*, see [13, 22] for their interesting properties. We call $\mathcal{F}(n)$ the *figure-eight orbifold*. Denote its orbifold group by $\Gamma_n = \pi^{\mathrm{orb}} \mathcal{F}(n)$. The group Γ_n has the following presentation:

$$\Gamma_n = \langle \rho_n, b_n \mid \rho_n^n = b_n^n = 1, \quad \rho_n^{-1} [b_n, \rho_n] = [b_n, \rho_n] b_n \rangle,$$

where generators ρ_n and b_n correspond to loops $\rho, b \in \pi_1(S^3 \setminus \mathcal{F})$. It is well known that for $n \geq 4$ the group Γ_n has a faithful representation in $\mathrm{PSL}(2, \mathbb{C})$. According to [14], this representation is defined on generators by

$$\rho_n = \begin{pmatrix} \cos \frac{\pi}{n} & ie^{d_n/2} \sin \frac{\pi}{n} \\ ie^{-d_n/2} \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}, b_n = \begin{pmatrix} \cos \frac{\pi}{n} & ie^{-d_n/2} \sin \frac{\pi}{n} \\ ie^{d_n/2} \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}. \quad (3.1)$$

The quantity d_n , defined as the complex distance between the axis of f_n and the axis of g_n , is given by

$$\cosh d_n = \frac{1}{4} \left(1 + \cot^2(\pi/n) - i\sqrt{3 \cot^4(\pi/n) + 14 \cot^2(\pi/n) - 5} \right). \quad (3.2)$$

The image of Γ_n under this representation is a non-elementary discrete group. In what follows, we shall not distinguish between the group Γ_n and its image under the faithful representation.

GMT-numbers of the figure-eight knot group and of the figure-eight orbifold groups were studied in [19]. It was shown that $\mathcal{G}(\pi_1(S^3 \setminus \mathcal{F})) = 3$ and the following result was obtained.

Theorem 3.1 (See [19]). *Let $n \geq 4$. Then the following inequalities hold for the figure-eight orbifold groups:*

$$1 \leq \mathcal{G}(\Gamma_n) \leq 3 - 4 \sin^2 \frac{\pi}{n}.$$

By writing the above inequalities for $n = 4$ we immediately get the following:

Corollary 3.2 (See [19]). *The figure-eight orbifold group Γ_4 is a GMT-group.*

This result can be checked directly. Indeed, by (3.1) we have $\text{tr}^2(\rho_n) = 4 \cos^2(\pi/n)$, so $|\text{tr}^2(\rho_4) - 2| = 0$. Also, by [19, Lemma 1] for any $\lambda \in \mathbb{R}$ we have

$$|\text{tr}[\rho_n, b_n] - \lambda| = \sqrt{(\lambda^2 - 3\lambda + 3) + 4(\lambda - 1) \sin^2(\pi/n)},$$

and this gives $|\text{tr}[\rho_n, b_n] - 1| = 1$ for any n . Hence $\mathcal{G}(\rho_n, b_n) = 1$.

4. Quotient orbifolds of the figure-eight orbifold

Lemma 4.1. *The figure-eight orbifold group Γ_n , $n \geq 4$, has involutions of types (i), (ii), and (iii) from Lemma 2.4.*

Proof. Being one of the simplest knots, the figure-eight knot has been intensively studied. In 1914 Dehn demonstrated [6] that the figure-eight group Γ has eight outer automorphisms forming the dihedral group

$$\langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1 \rangle,$$

where

$$\begin{cases} \sigma(\rho) = b, \\ \sigma(b) = \rho, \end{cases} \quad \text{and} \quad \begin{cases} \tau(\rho) = \rho b \rho^{-1}, \\ \tau(b) = b^{-1} \rho b. \end{cases}$$

Later, in 1931, Magnus proved [11] that Γ has no other outer automorphisms. It is obvious that the actions

$$\begin{cases} \sigma(\rho_n) = b_n, \\ \sigma(b_n) = \rho_n, \end{cases} \quad \text{and} \quad \begin{cases} \tau(\rho_n) = \rho_n b_n \rho_n^{-1}, \\ \tau(b_n) = b_n^{-1} \rho_n b_n. \end{cases}$$

are outer automorphisms of Γ_n for any n . The action of the group $\langle \sigma, \tau \rangle$ by automorphisms on Γ_n is presented in Table 1 (see also [23]).

Since $\mathcal{F}(n)$, $n \geq 4$, is a hyperbolic 3-orbifold of finite volume (see for example [3]). Note that there are two involutions acting as described in the statement of Lemma 2.4. For any n there exists $h_1 \in \text{Iso}(\mathbb{H}^3)$ such that

$$\sigma(\rho_n) = h_1 \rho_n h_1^{-1} = b_n$$

TABLE 1. Automorphism of Γ_n .

	σ	τ	τ^2	τ^3	$\sigma\tau$	$\sigma\tau^2$	$\sigma\tau^3$
ρ_n	b_n	$\rho_n b_n \rho_n^{-1}$	b_n^{-1}	$b_n^{-1} \rho_n^{-1} b_n$	$b_n^{-1} \rho_n b_n$	ρ_n^{-1}	$\rho_n b_n^{-1} \rho_n^{-1}$
b_n	ρ_n	$b_n^{-1} \rho_n b_n$	ρ_n^{-1}	$\rho_n b_n^{-1} \rho_n^{-1}$	$\rho_n b_n \rho_n^{-1}$	b_n^{-1}	$b_n^{-1} \rho_n^{-1} b_n$

and σ is of type (i) from Lemma 2.4. For any n there exists $h_2 \in \text{Iso}(\mathbb{H}^3)$ such that

$$\sigma\tau^2(\rho_n) = h_2 \rho_n h_2^{-1} = \rho_n^{-1}$$

and involution $\sigma\tau^2$ is of type (ii). For any n there exists $h_3 \in \text{Iso}(\mathbb{H}^3)$ which realizes τ^2 with conjugation by b_n :

$$b_n(\tau^2(b_n))b_n^{-1} = h_3 b_n h_3^{-1} = b_n \rho_n^{-1} b_n^{-1}$$

and involution h_3 is of type (iii). $\square$

Let us define two orbifolds with a 3-sphere S^3 as the underlying space. Denote by $\mathcal{O}_1(n)$ the orbifold with singular set the spatial theta-graph presented in Fig. 2 with singularities 2, 2 and n at its edges, as indicated in the figure. Denote by $\mathcal{O}_2(n)$ the orbifold with singular set the 2-component link 6_2^2 with singularities 2 and n at its components, as presented in Fig. 2.

FIGURE 2. Singular sets of $\mathcal{O}_1(n)$ and $\mathcal{O}_2(n)$.

Theorem 4.2. *The orbifold group $\pi^{\text{orb}}\mathcal{O}_1(4)$ is a GMT-group.*

Proof. For a fixed n consider an involution $h_1 \in \text{Iso}(\mathbb{H}^3)$ from the proof of Lemma 4.1 such that $h_1 \rho_n h_1^{-1} = b_n$ and $h_1 b_n h_1^{-1} = \rho_n$. For ρ_n and b_n given by (3.1) we have $h_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. An extension of Γ_n by h_1 has the following presentation:

$$\Delta_n = \langle \rho_n, b_n, h_1 \mid \rho_n^n = b_n^n = h_1^2 = 1, \rho_n^{-1} [b_n, \rho_n] = [b_n, \rho_n] b_n, h_1 \rho_n h_1^{-1} = b_n \rangle. \quad (4.1)$$

It is easy to see that the conjugation by h_1 is induced by an involution of S^3 whose axis corresponds to the dotted line in Fig. 1 and intersects the figure-eight knot $\mathcal{F}$ in two points. This symmetry induces an isometry, also denoted by h_1 , of the orbifold $\mathcal{F}(n)$.

FIGURE 3. Singular set of $\mathcal{F}(n)/h_1$.

The quotient space $\mathcal{F}(n)/h_1$ has S^3 as its underlying space. Its singular set is a spatial graph with two vertices presented by a diagram in Fig. 3. This graph can be described as the torus knot 5_1 with a tunnel AB . Points A and B are the images of intersection points of the singular set of $\mathcal{F}(n)$ with the axis of the involution h_1 . Two edges of this graph, which are images of the axis of h_1 , have singularity index 2, and the third edge, which is the image of the singular set of $\mathcal{F}(n)$, has singularity index n .

It can be checked directly (see for example, [23]) that the orbifold group of $\mathcal{F}(n)/h_1$ is isomorphic to Δ_n with generators ρ , b and h_1 , as pictured in Figure 3. Indeed, the relations (4.1) hold by the Wirtinger algorithm, in particular, the relation $\rho_n^{-1} [b_n, \rho_n] = [b_n, \rho_n] b_n$ is a consequence of the fact that the loop around its unknotting tunnel AB (see [12] about unknotting tunnels) is an element of order two in the orbifold group. Obviously, spatial theta-graphs presented in diagrams in Figure 2 (left) and Figure 3 are equivalent, so $\pi^{\text{orb}} \mathcal{O}_1(n) = \Delta(n)$. Eliminating b_n from (4.1) we get that Δ_n is a two-generated group with generators ρ_n and h_1 .

Suppose that $n = 4$. By Corollary 3.2, Γ_4 is a GMT-group and the pair Γ_4 and h_1 satisfies the case (i) of Lemma 2.4. Hence $\pi^{\text{orb}} \mathcal{O}_1(4)$ is a GMT-group. $\square$

Theorem 4.3. *The orbifold group $\pi^{\text{orb}} \mathcal{O}_2(4)$ is a GMT-group.*

Proof. To see the symmetry h_3 we shall redraw the singular set of the orbifold $\mathcal{F}(n)$ as in Figure 5. Denote $\lambda = b\rho b^{-1}$, see Figure 5. It is easy to see that h_3 corresponds to such rotational symmetry of order two that b goes to λ^{-1} and λ goes to b^{-1} . Therefore, $h_3 b h_3^{-1} = b\rho^{-1} b^{-1}$ that corresponds to the case (iii) of Lemma 2.4. Using $\rho = b^{-1} h_3 b^{-1} h_3 b$ from the defining relation $\rho^{-1} [b, \rho] = [b, \rho] b$ we get the relation

$$b h_3 b h_3 b^{-1} h_3 b^{-1} h_3 b h_3 = h_3 b h_3 b^{-1} h_3 b^{-1} h_3 b h_3 b$$

that corresponds to the canonical defining relation of the two-generated fundamental group of the two-bridge link $10/3$ pictured in Figure 2 on the right (see also [23]). Thus, the group generated by ρ_n , b_n and h_3 is the orbifold

FIGURE 4. Singular set of the orbifold $\mathcal{F}(n)$.

group $\pi^{\text{orb}}\mathcal{O}_2(4)$. Suppose that $n = 4$. By Corollary 3.2, Γ_4 is a GMT-group and the pair Γ_4 and h_3 satisfies the case (iii) of Lemma 2.4. Hence $\pi^{\text{orb}}\mathcal{O}_2(n)$ is a GMT-group. $\square$

5. More examples of GMT-groups

In this section we shall give two more examples of GMT-groups. Let us denote the orbifolds with the singular set presented in Fig. 5 by $\mathcal{O}_3$ and $\mathcal{O}_4$. The singular set of $\mathcal{O}_3$ is a spatial graph with two vertices that can be described as a Hopf link with an unknotting tunnel. The singular set of $\mathcal{O}_4$ can be described as a double link with an unknotting tunnel. Singularity indices are presented in Fig. 5. Both singular vertices of $\mathcal{O}_3$ belong to $\partial\mathbb{H}^3$. One of

FIGURE 5. Singular sets of orbifolds $\mathcal{O}_3$ and $\mathcal{O}_4$.

singular vertices of $\mathcal{O}_4$ belongs to $\partial\mathbb{H}^3$, whereas the other one lies in $\mathbb{H}^3$.

Theorem 5.1. *Orbifold groups $\pi^{\text{orb}}\mathcal{O}_3$ and $\pi^{\text{orb}}\mathcal{O}_4$ are GMT-groups.*

Proof. The orbifold group of $\pi^{\text{orb}}\mathcal{O}_3$ has the following presentation:

$$\pi^{\text{orb}}\mathcal{O}_3 = \langle a, b \mid a^4 = b^4 = 1, [a, b]^2 = 1 \rangle,$$

where a relation $[a, b]^2 = 1$ corresponds to a loop around a tunnel. Concerning the hyperbolicity of this orbifold see for example, [10]. Let us use letters a and b also for images of generators in the group $\text{Iso}(\mathbb{H}^3)$, corresponding to a

faithful representation. Then $\text{tr}^2(a) = 2$ and $\text{tr}[a, b] = 0$. Hence $\pi^{\text{orb}}\mathcal{O}_3$ is a GMT-group.

It can be seen from Fig. 5 that the singular set of $\mathcal{O}_3$ has a symmetry of order two that exchanges a and b . This symmetry induces an involution τ of $\pi^{\text{orb}}\mathcal{O}_3$ defined by

$$\tau(a) = pap^{-1} = b \quad \text{and} \quad \tau(b) = pbp^{-1} = a$$

for some $p \in \text{Iso}(\mathbb{H}^3)$. It is easy to verify that the quotient orbifold $\mathcal{O}_3/p$ is isometric to $\mathcal{O}_4$. Moreover, $\pi^{\text{orb}}\mathcal{O}_4$ is two-generated with generators a , p , and p satisfies the case (i) of Lemma 2.4. Hence, $\pi^{\text{orb}}\mathcal{O}_4$ is a GMT-group. It has the following presentation:

$$\pi^{\text{orb}}\mathcal{O}_4 = \langle a, p \mid a^4 = p^2 = 1, \quad (apapa^{-1}pa^{-1}p)^2 = 1 \rangle. \quad \square$$

Acknowledgment

This research was supported by the Slovenian Research Agency grant BI-RU/14-15-001, P1-0292, J1-6721 and J1-7025, the Ministry of Education and Science of the Russian Federation grant 16-01-00414, and the Laboratory of Quantum Topology, Chelyabinsk State University, grant 14.Z50.31.0020.

References

- [1] C.K. Atkinson, D. Futer, *The lowest volume 3-orbifolds with high torsion*. Trans. Amer. Math. Soc., DOI: 10.1090/tran/6920, to appear. arXiv:1507.07894 [math.GT].
- [2] A.F. Beardon, *The Geometry of Discrete Groups*. New York – Heidelberg – Berlin, Springer-Verlag, 1983.
- [3] M. Boileau, S. Maillot, J. Porti, *Three-dimensional orbifolds and their geometric structures*. Panoramas et Synthèses, **15**, Société Mathématique de France, Paris, 2003.
- [4] J. Callahan, *Jørgensen number and arithmeticity*. Conform. Geom. Dyn. **13** (2009), 160–186.
- [5] W. Cao, H. Tan, *Jørgensen inequality for quaternionic hyperbolic space with elliptic elements*. Bull. Austral. Math. Soc. **81** (2010), 121–131.
- [6] M. Dehn, *Dei beiden Kleeblattschlingen*. Math. Ann. **75** (1914), 402–413.
- [7] T. Jørgensen, *A note on subgroups of $\text{SL}(2, \mathbb{C})$* . Quart. J. Math. Oxford Ser. (2) **28** (1977), no. 110, 209–211.
- [8] F.W. Gehring, G.J. Martin, *Stability and extremality in Jørgensen's inequality*. Complex Variables Theory Appl. **12** (1989), 277–282.
- [9] K. Gongopadhyay, A. Mukherjee, *Extremality of quaternionic Jørgensen inequality*. arXiv:1503.08802 [math.CV].
- [10] E. Klimenko, N. Kopteva, *Two-generated Kleinian orbifolds*. arXiv:math/0606066.
- [11] W. Magnus, *Untersuchungen fiber einige unendliche diskontinuierliche Gruppen*. Math. Ann. **105** (1931) 52–74
- [12] K. Marimoto, M. Sakuma, Y. Yokota, *Identifying tunnel number one knots*. J. Math. Soc. Japan **48** (1996), no. 4, 667–688.

- [13] S. Matveev, C. Petronio, A. Vesnin, *Two-sided asymptotic bounds for the complexity of some closed hyperbolic three-manifolds*. J. Austral. Math. Soc. **86** (2009), 205–219.
- [14] A. Mednykh, A. Rasskazov, *On the structure of the canonical fundamental set for the 2-bridge link orbifolds*. Univ. Bielefeld, 1998. (Preprint 98-062, available online www.mathematik.uni-bielefeld.de/sfb343)
- [15] J.R. Parker, *Shimizu’s lemma for complex hyperbolic space*. Int. J. Math. **3** (1992), 291–308.
- [16] R. Riley, *A quadratic parabolic group*. Math. Proc. Camb. Phil. Soc. **77** (1975), 281–288.
- [17] H. Sato, *The Jørgensen number of the Whitehead link group*. Bol. Soc. Mat. Mexicana **10** (2004), 495–502.
- [18] D. Tan, *On two-generator discrete groups of Möbius transformations*. Proc. Amer. Math. Soc. **106** (1989), 763–770.
- [19] A. Vesnin, A. Masley, *On Jørgensen numbers and their analogues for groups of figure-eight orbifolds*. Siberian Math. J. **55** (2014), 807–816.
- [20] A. Vesnin, A. Masley, *Two-generated subgroups of $\mathrm{PSL}(2, \mathbb{C})$ which are extreme for Jørgensen inequality and its analogues*. In: Proc. Seminar on vector and tensor analysis with their applications to geometry, mechanics and physics. Moscow State University, **30** (2015), 1–54.
- [21] A. Vesnin, A. Mednykh, *Hyperbolic volumes of the Fibonacci manifolds*. Siberian Math. J. **36** (1995), 235–245.
- [22] A. Vesnin, A. Mednykh, *Fibonacci manifolds as two-fold coverings over the three-dimensional sphere and the Meyerhoff–Neumann conjecture*. Siberian Math. J. **37** (1996), 461–467.
- [23] A. Vesnin, A. Rasskazov, *Isometries of hyperbolic Fibonacci manifolds*. Siberian Math. J. **40** (1999), 9–22.

Dušan D. Repovš

Faculty of Education and Faculty of Mathematics and Physics

University of Ljubljana, Slovenia

& Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

e-mail: dušan.repovs@guest.arnes.si

Andrei Yu. Vesnin

Laboratory of Quantum Topology, Chelyabinsk State University, Russia

& Sobolev Institute of Mathematics, Novosibirsk, Russia

e-mail: vesnin@math.nsc.ru