A NEW CLASS OF HOMOLOGY AND COHOMOLOGY 3-MANIFOLDS

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ABSTRACT. We show that for any set of primes $\mathcal{P}$ there exists a space $M_{\mathcal{P}}$ which is a homology and cohomology 3-manifold with coefficients in $\mathbb{Z}_p$ for $p \in \mathcal{P}$ and is not a homology or cohomology 3-manifold with coefficients in $\mathbb{Z}_q$ for $q \notin \mathcal{P}$. Moreover, $M_{\mathcal{P}}$ is neither a homology nor cohomology 3-manifold with coefficients in $\mathbb{Z}$ or $\mathbb{Q}$.

1. INTRODUCTION

In 1908 Tietze [7] constructed his famous 3-manifolds $L(p, q)$ called *lens spaces*. These spaces have many interesting properties. For example, lens spaces $L(5, 1)$ and $L(5, 2)$ have isomorphic fundamental groups and the same homology groups, but they do not have the same homotopy type (this was first proved in 1919 by Alexander [1]).

It is also well known that for every prime q , the lens space $M^3 = L(q, 1)$ has the following properties:

- M^3 is a $\mathbb{Z}_p$ -homology 3-sphere for every prime $p \neq q$;
- M^3 is not a $\mathbb{Z}_q$ -homology 3-sphere; and
- M^3 is not a $\mathbb{Z}$ -homology 3-sphere.

We shall generalize this classical result as follows:

Theorem 1.1. *Given any set of primes $\mathcal{P}$, there exists a 3-dimensional compact metric space $M_{\mathcal{P}}$ which has the following properties:*

- $M_{\mathcal{P}}$ is a $\mathbb{Z}_p$ -homology 3-manifold for every $p \in \mathcal{P}$;
- $M_{\mathcal{P}}$ is a $\mathbb{Z}_p$ -cohomology 3-manifold for every $p \in \mathcal{P}$;
- $M_{\mathcal{P}}$ is not a $\mathbb{Z}_q$ -homology 3-manifold for any $q \notin \mathcal{P}$;
- $M_{\mathcal{P}}$ is not a $\mathbb{Z}_q$ -cohomology 3-manifold for any $q \notin \mathcal{P}$;
- $M_{\mathcal{P}}$ is not a $\mathbb{Z}$ -homology 3-manifold;
- $M_{\mathcal{P}}$ is not a $\mathbb{Z}$ -cohomology 3-manifold;
- $M_{\mathcal{P}}$ is not a $\mathbb{Q}$ -homology 3-manifold; and
- $M_{\mathcal{P}}$ is not a $\mathbb{Q}$ -cohomology 3-manifold.

2. PRELIMINARIES

First, we fix the terminology, notation, and recall some well known facts. We let L be the ring of integers $\mathbb{Z}$ or a field.

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Definition 2.1. (cf. [2, Corollary V.16.9]) *A locally compact space X is called a cohomology manifold of dimension n over L (denoted as $n - cm_L$) if*

- (1) *X has finite cohomological dimension over L ($\dim_L X < \infty$);*
- (2) *X is cohomologically locally connected over L (clc_L^∞); and*
- (3) *for all $x \in X$,*

$$\check{H}^p(X, X - \{x\}; L) \cong \begin{cases} L & \text{for } p = n \\ 0 & \text{for } p \neq n \end{cases}$$

where $\check{H}^*$ are Čech cohomology groups with coefficients in L .

Definition 2.2. *A locally compact space X is called a homology manifold of dimension n over L (denoted as $n - hm_L$) if*

- (1) *X has finite cohomological dimension over L ($\dim_L X < \infty$);*
- (2) *for all $x \in X$,*

$$H_p(X, X - \{x\}; L) \cong \begin{cases} L & \text{for } p = n \\ 0 & \text{for } p \neq n \end{cases}$$

where H_* are Borel-Moore homology groups with coefficients in L .

Homology manifolds will stand for homology $\mathbb{Z}$ -manifolds.

Any n -dimensional cohomology manifold ($n - cm_L$) is also an n -dimensional homology manifold ($n - hm_L$) by [2, Theorem V.16.8]. Therefore we will construct only cohomology manifolds which will be automatically homology manifolds by this theorem.

Example 2.3. For the construction and some simple properties of lens spaces see [5, 6]. In particular, the homology groups of the lens space $M^3 = L(q, 1)$ are

$$H_n(M^3; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z}_q & n = 1 \\ 0 & n = 2 \text{ or } n \geq 4 \end{cases}$$

By the Universal Coefficients Theorem, we have for any abelian group G ,

$$H_n(M^3; G) \cong H_n(M^3; \mathbb{Z}) \otimes G \oplus H_{n-1}(M^3; \mathbb{Z}) * G.$$

Therefore, if p and q are prime and $p \neq q$ then

$$H_n(M^3; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } n = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

whereas

$$H_n(M^3; \mathbb{Z}_q) \cong \begin{cases} \mathbb{Z}_q & n = 0, 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

Consider now the suspension $N^4 = \Sigma M^3$ of M^3 . Local conditions of Definitions 2.1 and 2.2 are clearly satisfied for N^4 since M^3 is a 3-manifold. Therefore N^4 is a $4 - hm_{\mathbb{Z}_p}$ and a $4 - cm_{\mathbb{Z}_p}$. However, N^4 is neither a $4 - hm_{\mathbb{Z}_q}$ nor a $4 - cm_{\mathbb{Z}_q}$ if p and q are prime and $p \neq q$ (cf. [2]).

3. PROOF OF THEOREM 1.1

Let $\mathcal{P} = \{p_i\}_{i \in K}$, for $K = \mathbb{N}$ or $K = \{1, \dots, k\}$, be a set of some prime numbers. If the set K is infinite then we define the numbers n_i as $n_i = p_1 \cdot p_2 \cdot p_3 \cdots p_i$. If the set K is finite and consists of exactly k elements, then define n_i as $n_i = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ for all i .

Let X be a solenoid in the 3-dimensional sphere S^3 , i.e. the inverse limit of solid tori corresponding to the following inverse system:

$$\mathbb{Z} \xleftarrow{n_1} \mathbb{Z} \xleftarrow{n_2} \mathbb{Z} \xleftarrow{n_3} \dots$$

naturally embedded in S^3 , see e.g. [3, Chapter IX, Exercise E.4].

Let us prove that the quotient space S^3/X is a cohomology 3-manifold $cm_{\mathbb{Z}_p}$. It is obvious that S^3/X is 3-dimensional, compact and metrizable. So the space S^3/X satisfies the condition (1) of Definition 2.1.

To prove that the space S^3/X satisfies the conditions (2) and (3) of Definition 2.1, let's calculate first the groups $\check{H}^n(S^3/X, \{x\}; G)$ with respect to the one-point subset $\{x\} = X/X$ for $G \cong \mathbb{Z}_p, p \in \mathcal{P}$; $G \cong \mathbb{Z}_q, q \notin \mathcal{P}$; $G \cong \mathbb{Z}$; $G \cong \mathbb{Q}$. Since S^3/X is connected and 3-dimensional it follows that

$$(1) \quad \check{H}^0(S^3/X, \{x\}; G) \cong 0 \quad \text{and} \quad \check{H}^n(S^3/X, \{x\}; G) \cong 0 \quad \text{for } n > 3.$$

Let U_i be the open i -th solid torus neighborhood of X in S^3 (see [3]). Then $\{U_i/X\}_{i \in \mathbb{N}}$ is a neighborhood base of x in S^3/X . By continuity of the Čech cohomology and by the Excision Axiom it follows that:

$$\begin{aligned} \check{H}^n(S^3/X, \{x\}; G) &\cong \check{H}^n(\varprojlim S^3/\overline{U}_i, \overline{U}_i/\overline{U}_i; G) \cong \\ &\cong \varprojlim \check{H}^n(S^3/\overline{U}_i, \overline{U}_i/\overline{U}_i; G) \cong \varprojlim \check{H}^n(S^3, \overline{U}_i; G). \end{aligned}$$

For $n = 1$ we have the exact sequence of the pair $(S^3, \overline{U}_i)$:

$$\check{H}^0(S^3; G) \longrightarrow \check{H}^0(\overline{U}_i; G) \longrightarrow \check{H}^1(S^3, \overline{U}_i; G) \longrightarrow \check{H}^1(S^3, G) \longrightarrow \check{H}^1(\overline{U}_i; G).$$

Since the 1-dimensional cohomology of the 3-sphere is trivial for any group of coefficients G and $\overline{U}_i$ is connected space for every i , it follows that $\check{H}^1(S^3, \overline{U}_i; G) \cong 0$, therefore $\check{H}^n(S^3/X, \{x\}; G) \cong 0$ and, in particular,

$$(2) \quad \check{H}^1(S^3/X, \{x\}; \mathbb{Z}_p) \cong 0$$

and

$$(3) \quad \check{H}^1(S^3/X, \{x\}; \mathbb{Z}_q) \cong 0, \check{H}^1(S^3/X, \{x\}; \mathbb{Z}) \cong 0, \check{H}^1(S^3/X, \{x\}; \mathbb{Q}) \cong 0.$$

For $n = 2$ we have the following exact sequence of the pair $(S^3, \overline{U}_i)$:

$$\check{H}^1(S^3; G) \longrightarrow \check{H}^1(\overline{U}_i; G) \longrightarrow \check{H}^2(S^3, \overline{U}_i; G) \longrightarrow \check{H}^2(S^3, G) \longrightarrow \check{H}^2(\overline{U}_i; G).$$

The cohomology groups $\check{H}^1(S^3; G)$ and $\check{H}^2(S^3, G)$ are trivial, and the groups $\check{H}^1(\overline{U}_i; G)$ are isomorphic to G since U_i has the homotopy type of a circle. The homomorphisms $\check{H}^1(\overline{U}_i; G) \rightarrow \check{H}^1(\overline{U}_{i+1}; G)$ are multiplications by n_i that take the group G into itself.

Therefore for the group of coefficients $G \cong \mathbb{Z}_p$ it follows that

$$(4) \quad \check{H}^2(S^3/X, \{x\}; \mathbb{Z}_p) \cong 0.$$

However,

$$(5) \quad \check{H}^2(S^3/X, \{x\}; \mathbb{Z}_q) \not\cong 0, \check{H}^2(S^3/X, \{x\}; \mathbb{Z}) \not\cong 0, \check{H}^2(S^3/X, \{x\}; \mathbb{Q}) \not\cong 0.$$

For $n = 3$ consider the following cohomology exact sequence for the pair $(S^3, \overline{U}_i)$:

$$\check{H}^2(\overline{U}_i; G) \longrightarrow \check{H}^3(S^3, \overline{U}_i; G) \longrightarrow \check{H}^3(S^3, G) \longrightarrow \check{H}^3(\overline{U}_i; G).$$

Since $\overline{U}_i \simeq S^1$, it follows that:

$$(6) \quad \check{H}^3(S^3/X, x; G) \cong G.$$

Let us calculate the groups $\check{H}^n(S^3/X - \{x\}; \mathbb{Z}_p)$. The space $S^3/X - \{x\}$ is the union $\bigcup_{i=1}^{\infty} (S^3 - U_i)$ of an increasing sequence of “complementary” solid tori.

For $n = 1$ we have the following exact sequence of Milnor-Harlap [4, Theorem 1]:

$$0 \rightarrow \varprojlim^{(1)} \check{H}^0(S^3 - U_i; \mathbb{Z}_p) \rightarrow \check{H}^1(S^3 - X; \mathbb{Z}_p) \rightarrow \varprojlim \check{H}^1(S^3 - U_i; \mathbb{Z}_p) \rightarrow 0,$$

where $\varprojlim^{(1)}$ is the first derived functor of the functor of the inverse limit. Since $p \in \mathcal{P}$ it follows that the inverse limit $\varprojlim \check{H}^1(S^3 - U_i; \mathbb{Z}_p)$ is trivial. The group $\varprojlim^{(1)} \check{H}^0(S^3 - U_i; \mathbb{Z}_p)$ is trivial since the corresponding inverse sequence satisfies the Mittag-Leffler (ML) condition, so we have

$$(7) \quad \check{H}^1(S^3 - X; \mathbb{Z}_p) \cong 0.$$

Analogously, it is easy to see that

$$(8) \quad \check{H}^1(S^3 - X; \mathbb{Z}_q) \not\cong 0 \text{ for } q \notin \mathcal{P}, \quad \check{H}^1(S^3 - X; \mathbb{Z}) \cong 0, \quad \check{H}^1(S^3 - X; \mathbb{Q}) \not\cong 0.$$

Let $n = 2$, then we have the Milnor-Harlap exact sequence for the presentation $S^3/X - \{x\} = \bigcup_{i=1}^{\infty} (S^3 - U_i)$:

$$0 \rightarrow \varprojlim^{(1)} \check{H}^1(S^3 - U_i; \mathbb{Z}_p) \rightarrow \check{H}^2(S^3 - X; \mathbb{Z}_p) \rightarrow \varprojlim \check{H}^2(S^3 - U_i; \mathbb{Z}_p) \rightarrow 0.$$

The groups $\varprojlim^{(1)} \check{H}^1(S^3 - U_i; \mathbb{Z}_p)$ are trivial since the groups $\check{H}^1(S^3 - U_i; \mathbb{Z}_p)$ are isomorphic to the finite group $\mathbb{Z}_p$ and the corresponding inverse sequence satisfies the ML condition. The groups $\check{H}^2(S^3 - U_i; \mathbb{Z}_p)$ are also trivial since the “complementary” solid tori have the homotopy type of the circle. Therefore

$$(9) \quad \check{H}^2(S^3 - X; \mathbb{Z}_p) \cong 0.$$

For $n = 3$ we have the exact sequence of Milnor-Harlap for the same presentation of $S^3/X - \{x\}$ as before:

$$0 \rightarrow \varprojlim^{(1)} \check{H}^2(S^3 - U_i; \mathbb{Z}_p) \rightarrow \check{H}^3(S^3 - X; \mathbb{Z}_p) \rightarrow \varprojlim \check{H}^3(S^3 - U_i; \mathbb{Z}_p) \rightarrow 0.$$

The groups $\check{H}^3(S^3 - U_i; \mathbb{Z}_p)$ and $\check{H}^2(S^3 - U_i; \mathbb{Z}_p)$ are trivial since the spaces $S^3 - U_i$ have the homotopy type of a circle. Therefore:

$$(10) \quad \check{H}^3(S^3 - X; \mathbb{Z}_p) \cong 0.$$

Next, let us calculate the groups $\check{H}^n(S^3/X, S^3/X - X/X; G)$ for certain groups G .

Since the space S^3/X is connected and $\dim S^3/X = 3$ it follows that these groups are trivial groups for $n = 0, n > 3$.

Since the space $S^3 - X$ is connected and $\check{H}^1(S^3/X; \mathbb{Z}_p) \cong 0$ by (2), it follows by the exact cohomology sequence of the pair $(S^3/X, S^3/X - X/X)$ (or the pair $(S^3/X, S^3 - X)$ since $S^3/X - X/X \cong S^3 - X$) that

$$(11) \quad \check{H}^1(S^3/X, S^3 - X; \mathbb{Z}_p) \cong 0.$$

By the exact sequence:

$$\check{H}^1(S^3 - X; \mathbb{Z}_p) \xrightarrow{\delta} \check{H}^2(S^3/X, S^3 - X; \mathbb{Z}_p) \longrightarrow \check{H}^2(S^3/X; \mathbb{Z}_p) \longrightarrow \check{H}^2(S^3 - X; \mathbb{Z}_p)$$

and since the groups $\check{H}^1(S^3 - X; \mathbb{Z}_p)$ and $\check{H}^2(S^3/X; \mathbb{Z}_p)$ are trivial by (7) and (4) it follows that

$$(12) \quad \check{H}^2(S^3/X, S^3 - X; \mathbb{Z}_p) \cong 0.$$

For any group of coefficients the corresponding homomorphism δ is a monomorphism by (3). Since the groups $\check{H}^1(S^3 - X; \mathbb{Z}_q)$ for $q \notin \mathcal{P}$, $\check{H}^1(S^3 - X; \mathbb{Q})$ are nontrivial by (8), and the groups $\check{H}^1(S^3/X; \mathbb{Z}_q)$, $\check{H}^1(S^3/X; \mathbb{Q})$ are trivial, it follows that

$$(13) \quad \check{H}^2(S^3/X, S^3 - X; \mathbb{Z}_q) \not\cong 0, \check{H}^2(S^3/X, S^3 - X; \mathbb{Q}) \not\cong 0.$$

Consider the following exact sequence of the pair $(S^3/X, S^3 - X; \mathbb{Z}_p)$:

$$\check{H}^2(S^3 - X; \mathbb{Z}_p) \xrightarrow{\delta} \check{H}^3(S^3/X, S^3 - X; \mathbb{Z}_p) \longrightarrow \check{H}^3(S^3/X; \mathbb{Z}_p) \longrightarrow \check{H}^3(S^3 - X; \mathbb{Z}_p).$$

The groups $\check{H}^2(S^3 - X; \mathbb{Z}_p)$ and $\check{H}^3(S^3 - X; \mathbb{Z}_p)$ are trivial by (9) and (10), respectively. Next, observe that $\check{H}^3(S^3/X; \mathbb{Z}_p) \cong \mathbb{Z}_p$. Therefore

$$(14) \quad \check{H}^3(S^3/X, S^3 - X; \mathbb{Z}_p) \cong \mathbb{Z}_p.$$

Let us show that S^3/X is $clc_{\mathbb{Z}_p}^\infty$ at all points. Obviously, the space S^3/X is a $clc_{\mathbb{Z}_p}^\infty$ space for all points except at the point $x = X/X$ since $S^3 - X$ is an open manifold. As mentioned before, the sets $\{U_i/X\}_{i \in \mathbb{N}}$ form a neighborhood base of the point x . Consider the groups $\check{H}^n(U_i/X, X/X; \mathbb{Z}_p)$. By the Excision Axiom it follows that $\check{H}^n(U_i/X, X/X; \mathbb{Z}_p) \cong \check{H}^n(U_i, X; \mathbb{Z}_p)$.

From the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \cong \check{H}^0(X; \mathbb{Z}_p) & \longrightarrow & \check{H}^1(U_i, X; \mathbb{Z}_p) & \longrightarrow & \check{H}^1(U_i; \mathbb{Z}_p) & \cong & \mathbb{Z}_p \\ \downarrow & & \downarrow \pi^i & & \downarrow \times n_i & & \\ 0 \cong \check{H}^0(X; \mathbb{Z}_p) & \longrightarrow & \check{H}^1(U_{i+1}, X; \mathbb{Z}_p) & \longrightarrow & \check{H}^1(U_{i+1}; \mathbb{Z}_p) & \cong & \mathbb{Z}_p \end{array}$$

it follows that for a large enough i , the homomorphism

$$\check{H}^1(U_i, X; \mathbb{Z}_p) \xrightarrow{\pi^i} \check{H}^1(U_{i+1}, X; \mathbb{Z}_p)$$

is trivial. Therefore

$$(15) \quad S^3/X \text{ is a } 1 - clc_{\mathbb{Z}_p} \text{ space.}$$

By the analogous diagram for the group of coefficients $\mathbb{Z}$ it is easy to see that the homomorphism $\check{H}^1(U_i, X; \mathbb{Z}) \xrightarrow{\pi^i} \check{H}^1(U_{i+1}, X; \mathbb{Z})$ is a monomorphism of the group $\mathbb{Z}$. Therefore

$$(16) \quad S^3/X \text{ is not } 1 - clc_{\mathbb{Z}}.$$

By the exact sequence

$$\check{H}^1(X; \mathbb{Z}_p) \longrightarrow \check{H}^2(U_i, X; \mathbb{Z}_p) \longrightarrow \check{H}^2(U_i; \mathbb{Z}_p)$$

since $\check{H}^2(U_i, \mathbb{Z}_p) \cong 0$ and the Čech cohomology group $\check{H}^1(X; \mathbb{Z}_p)$ is obviously isomorphic to the direct limit of the sequence

$$\mathbb{Z}_p \xrightarrow{\times n_1} \mathbb{Z}_p \xrightarrow{\times n_2} \mathbb{Z}_p \xrightarrow{\times n_3} \dots$$

it follows that $\check{H}^2(U_i, X; \mathbb{Z}_p) \cong 0$. By the Excision Axiom it follows that

$$\check{H}^2(U_i/X, X/X; \mathbb{Z}_p) \cong 0$$

and

$$(17) \quad S^3/X \text{ is a } 2 - clc_{\mathbb{Z}_p} \text{ space.}$$

By the following exact sequence of the pair (U_i, X)

$$\check{H}^2(X; \mathbb{Z}_p) \longrightarrow \check{H}^3(U_i, X; \mathbb{Z}_p) \longrightarrow \check{H}^3(U_i, \mathbb{Z}_p)$$

and since the space X is 1-dimensional and U_i is homotopy equivalent to the circle it follows that $\check{H}^3(U_i, X; \mathbb{Z}_p) \cong 0$ therefore $\check{H}^3(U_i/X, X/X; \mathbb{Z}_p) \cong 0$ and

$$(18) \quad S^3/X \text{ is a } 3 - clc_{\mathbb{Z}_p} \text{ space.}$$

By the local connectedness of the space S^3/X , by (15), (17), (18) and since $\dim S^3/X = 3$ it follows that S^3/X is a $clc_{\mathbb{Z}_p}^\infty$ space and S^3/X satisfies the condition (2) of Definition 2.1.

It follows by (11), (12), and (14) that S^3/X satisfies the condition (3) of Definition 2.1, therefore S^3/X is a $3 - cm_{\mathbb{Z}_p}$ and a $3 - hm_{\mathbb{Z}_p}$.

However, the space S^3/X is neither a $3 - cm_{\mathbb{Z}_q}$ nor a $3 - cm_{\mathbb{Q}}$ since $\check{H}^2(S^3/X, S^3 - X; \mathbb{Z}_q) \not\cong 0$ and $\check{H}^2(S^3/X, S^3 - X; \mathbb{Q}) \not\cong 0$ by (16), and is not a $3 - cm_{\mathbb{Z}}$ since it is not a $1 - clc_{\mathbb{Z}}$. This completes the proof. $\square$

4. EPILOGUE

The spaces which we have constructed are not ANR's, so the following remains an open question:

Question 4.1. *Let $\mathcal{P}$ be any set of prime numbers. Does there exist a 3-dimensional ANR $X_{\mathcal{P}}$ with the following properties:*

- (1) $X_{\mathcal{P}}$ is a $\mathbb{Z}_p$ -homology 3-manifold for every $p \in \mathcal{P}$; and
- (2) $X_{\mathcal{P}}$ is not a $\mathbb{Z}_q$ -homology 3-manifold for any $q \notin \mathcal{P}$?

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REFERENCES

- [1] J. W. Alexander, *Note on two three-dimensional manifolds with the same group*, Trans Amer. Math. Soc. 20 (1919), 339–342.
- [2] G. E. Bredon, *Sheaf Theory*, 2nd Ed., Graduate Texts in Math. No. 170, Springer, Berlin, 1997.
- [3] S. Eilenberg, N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, N. J., 1952.

- [4] A. È. Harlap, *Local homology and cohomology, homology dimension and generalized manifolds*, Math. Sb. (N. S.) 96 (138) (1975), 347–373. (in Russian); English transl.: Math. USSR-Sb. 25:3 (1975), 323–349.
- [5] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, Cambridge, 2002.
- [6] H. Seifert, W. Threlfall, *A Textbook of Topology*, Academic Press, New York, 1980.
- [7] H. Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Für Math. Und Phys. 19 (1908), 1–118.

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