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ASSOCIATE SPACE WITH RESPECT TO A SEMI-FINITE MEASURE

Master's Thesis

Adviser: Assistant Professor Marko Kandić

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Contents

Program dela

Natančno predstavite članek

C.Avalos-Ramos, F.Galaz-Fontes, *Associate space with respect to a semi-finite measure*, Indag. Math. **28** (2017), 261-267,

pri čemer s pomočjo dodatne literature razvijete relevantno teorijo Banachovih mrež in funkcijskih prostorov.

doc. dr. Marko Kandić

Work plan

Precisely present the paper

C.Avalos-Ramos, F.Galaz-Fontes, *Associate space with respect to a semi-finite measure*, Indag. Math. **28** (2017), 261-267,

whereby by using additional literature you develop relevant theory of Banach lattices and function spaces.

doc. dr. Marko Kandić

Osnovna literatura (Basic references)

[1] C. Avalos-Ramos and F. Galaz-Fontes, *Associate space with respect to a semi-finite measure*, Indag. Math. **28**(2) (2017), 261–267

Podpis mentorja (adviser):

Pridruženi prostor glede na semi-končno mero

Povzetek

Uvedemo osnovno teorijo vektorskih mrež in urejenostno omejenih operatorjev. Preučujemo urejenostno zvezne operatorje, urejenostni dualni prostor normiranih mrež in več različic Fatoujeve lastnosti normiranih mrež. Nato uvedemo normirane funkcijske prostore, nasičene funkcijske polnorme in pridruženi prostor normiranemu funkcijskemu prostoru, na koncu pa tudi semi-končne in lokalizabilne mere. Dokažemo, da je pridruženi prostor E' poljubnega nasičenega Banachovega funkcijskega prostora E glede na semi-končno mero μ enak urejenostno zveznemu dualnemu prostoru $E_n^\sim$ natanko tedaj, ko ima E' krepko Fatoujevo lastnost. Če je E nadalje σ-urejenostno zvezen, dokažemo, da so $E_n^\sim$, σ-urejenostno zvezni dualni prostor $E_c^\sim$ in normirani dualni prostor E^* enaki, kar pomeni, da lahko v prej omenjenem rezultatu enakost $E' = E_n^\sim$ nadomestimo z $E' = E^*$. Če je μ lokalizabilna, dokažemo, da ima E' krepko Fatoujevo lastnost, kar pomeni, da je $E' = E_n^\sim$. Na koncu navedemo primer, ko pravkar omenjena enakost ne drži.

Associate space with respect to a semi-finite measure

Abstract

We introduce the basic theory of vector lattices and order bounded operators. Order continuous operators, the order dual space, normed lattices and several variants of the Fatou property of normed lattices are studied. We then introduce normed function spaces, saturated function seminorms and the associate space to a normed function space, as well as semi-finite and localizable measures. We prove that the associate space E' of an arbitrary saturated Banach function space E with respect to a semi-finite measure μ equals the order continuous dual space $E_n^\sim$ if and only if E' has the strong Fatou property. If E is furthermore σ-order continuous we prove that $E_n^\sim$, the σ-order continuous dual space $E_c^\sim$ and the norm dual E^* are equal which implies that in the previously mentioned result the equality $E' = E_n^\sim$ can be replaced with $E' = E^*$. Also, if μ is localizable, we prove that E' has the strong Fatou property which implies that $E' = E_n^\sim$. Finally, we give an example where the aforementioned equality fails.

Math. Subj. Class. (2010): 46E30, 47B65, 46B42

Ključne besede: Semi-končne in lokalizabilne mere; Banachovi funkcijski prostori; Pridruženi prostor; urejenostna in σ-urejenostna zveznost; Fatoujeva lastnost

Keywords: Semi-finite and localizable measures; Banach function spaces; Associate space; Order and σ-order continuity; Fatou property

1 Introduction

Almost every classical Banach space is equipped with a natural order that is compatible with the algebraic and topological structures of the space. In this thesis we study vector lattices which are partially ordered vector spaces where the order and algebraic structure are compatible. An address of F. Riesz in 1928 on the decomposition of linear functionals (into their positive and negative parts), at the International Congress of Mathematicians in Bologna, Italy, marked the beginnings of the study of vector lattices and positive operators.

Vector lattices, also called Riesz spaces or K-lineals, were first considered by F. Riesz, L. Kantorovič, and H. Freudenthal. Subsequently other important contributions came from the Soviet Union (L.V. Kantorovič, A.J. Judin, A.G. Pinsker, and B.Z. Vulikh), Japan (H. Nakano, T. Ogasawara, and K. Yosida), and the United States (G. Birkhoff, H.F. Bohnenblust, S. Kakutani, and M.M. Stone). In 1950 the book Functional Analysis in Partially Ordered Spaces by L. V. Kantorovič, B. Z. Vulikh, and A. G. Pinsker appeared in the Soviet literature. This book (that has not been translated into English) contained an excellent treatment, up to that date, of positive operators and their applications.

In the late 1960s, 1970s and 1980s the theory rapidly increased. Important contributions came from the Dutch school (W.A.J. Luxemburg, A.C. Zaanen) and the Tübingen school (H.H. Schaefer). In the mid-seventies the research on this subject was essentially influenced by the books of H.H. Schaefer (1974) and W.A.J. Luxemburg and A.C. Zaanen (1971). Afterwards other important books concerning this subject appeared, A.C. Zaanen (1983), H.U. Schwarz (1984), and C.D. Aliprantis and O. Burkinshaw (1985).

It was the merit of H.H. Schaefer to present the theory of Banach lattices and positive operators as an inseparable part of the general Banach space and operator theory. In particular, deep results of the general theory and classical analysis were used to prove related properties in the case of general Banach lattices.

Since 1978 many developments and changes have occured in the field of Riesz spaces the most important of which was the application of locally solid Riesz spaces to economics. As it turned out, the lattice and topological structures of ordered spaces were the main ingredients needed for an economic framework on which a fruitful and credible economic analysis can be founded.

In the last forty-five years the applications of the theory of vector lattices have grown remarkably. This development has been fostered in many branches of mathematics (such as optimization, numerical methods, positive solutions of equations, positive systems, positive semigroups, measure theory etc.) by the manifold aspects summarized under the heading of *positivity*. It is impossible to provide only a rough survey of the applications spread over many fields of the present-day mathematical research, so we shall only refer to some of them: game theory, nuclear reactor theory, statistical decision theory, structured population dynamics, economics, equilibrium theory, convex operators, extremal problems, Choquet theory, variational methods, positive solutions of operator equations, integral operators, fixed point equations, maximum principles, positive systems, semigroups of positive operators, measure theory, stochastic processes, martingale theory.

On the other hand, special problems e.g. cones in Banach spaces, dominated operators, integral operators, order continuous norms and miscellaneous others complete the general theory by many new and particular aspects.

Particular examples of vector lattices are those whose elements are measurable functions, such as L^p spaces, Lorentz spaces and Orlicz spaces. During almost a century of their existence, Lebesgue spaces have constantly played a primary role in analysis. However, it has been known almost from the very beginning that the Lebesgue scale is not sufficiently general to provide a satisfactory description of fine properties of functions required by practical tasks. This was noted during the early 1920s by Kolgomorov, Zygmund, Titchmarsh and others, mostly in connection with research of properties of operators on function spaces. Thus, naturally, during the first half of the twentieth century, new fine scales of function spaces have been introduced. The efforts of Young, Orlicz, Hardy, Littlewood, Zygmund, Halperin, Köthe, Marcinkiewicz, Lorentz, Luxemburg, Morrey, Campananto and many others resulted in the development of a powerful and qualitatively new mathematical discipline of function spaces.

The basic theory of Banach function norms and Banach function spaces was in some sense a culmination of efforts to cover Orlicz spaces with other types of spaces under a common theme, performed from the 1930s to the 1950s by Orlicz, Lorentz, Luxemburg, Zaanen, Köthe, Halperin and others. This material gradually appeared mostly in works of the mentioned authors. The systematic treatment of this topic can be found in Luxemburg and Zaanen [7] and Bennett and Sharpley [2].

The thesis is divided into 4 chapters. The first chapter is a short introduction to the thesis. In the second chapter we introduce the basic theory of vector lattices and operators between them. We state the Archimedean property and Dedekind completeness for vector lattices which are analogous to the same properties for the real numbers. Then we study ideals and bands which are subspaces with special properties. The basic theory of linear operators is then discussed, starting with Kantorovič's extension theorem for additive maps. We define the notion of an order bounded operator, and show that the set of all order bounded operators from a vector lattice to a Dedekind complete vector lattice is again a Dedekind complete vector lattice. We study order continuous and σ-order continuous operators and define the order, order continuous and σ-order continuous dual space. We move on to discussing normed vector lattices and prove that the norm dual and order dual of a Banach lattice coincide. The Fatou properties are then introduced which play an important role in the main results. We end this chapter with a short section on order and σ-order continuous norms.

The third chapter is an introduction to the theory of function spaces. We state the Riesz-Fischer property of a function norm ρ, which is equivalent to the norm completeness of the function space L_ρ determined by ρ. Then we discuss saturated function norms and revisit the Fatou properties. The notions of semi-finite and localizable measure are then stated and several results concerning function spaces with respect to a localizable measure are proved. Then the associate space is introduced which is one of the central objects appearing in the results in Chapter 4. Finally, we prove that if the underlying measure is σ-finite then the associate space and the order continuous dual of a normed function space coincide.

The fourth chapter contains the main results of the thesis. If μ is a semi-finite measure and E is a Banach function space with respect to μ we prove that the associate space E' is isometrically isomorphic to the order continuous dual space $E_n^\sim$ if and only if E' has the strong Fatou property. Our second result is that under the stronger assumption that μ is localizable the associate space E' always has the strong Fatou property which implies that $E' = E_n^\sim$. Then we provide an example of a Banach function space E such that $E' \neq E_n^\sim$.

Throughout the text $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. The symbol $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^+$ denotes the set of all real numbers greater than or equal to 0. All vector spaces are assumed to be over $\mathbb{R}$. The operations sup and inf take precedence over $+$ and $-$. We make the convention that $0 \cdot \infty = \infty \cdot 0 = 0$. If f is a (equivalence class of) measurable function on the measure space (Ω, Σ, μ) then we define $\operatorname{supp}(f) = \{x \in \Omega : f(x) \neq 0\}$ as a measurable set defined up to unions and differences with sets of measure 0. We denote the essential supremum of f by $\operatorname{ess\,sup} f = \sup\{a \in \mathbb{R}^+ : \mu(\{x \in \Omega : |f(x)| \geq a\}) > 0\}$. Also, for measurable functions f and g and measurable set B we say that:

(a) $f = g$ on B if $f(x) = g(x)$ holds for almost all $x \in B$;

(b) $f \leq g$ on B if $f(x) \leq g(x)$ holds for almost all $x \in B$;

(c) $f < g$ on B if $f(x) < g(x)$ holds for almost all $x \in B$.

2 Vector Lattices

As was stated in the introduction, almost every classical Banach space is naturally equipped with a partial order that is compatible with its algebraic and topological structures. In this chapter we develop the basic theory of vector lattices and linear operators. We follow the treatment in [3, Chapter 1, Chapter 4] by Aliprantis and Burkinshaw.

2.1 Basic Definitions and Examples

Before we define vector lattices, we define the more general notion of an ordered vector space.

Definition 2.1. An *ordered vector space* is a vector space V that is also partially ordered by a relation $\leq$ such that the following two conditions hold:

(a) if $x \leq y$ then $x + z \leq y + z$ for any $x, y, z \in V$;

(b) if $x \leq y$ then $cx \leq cy$ for any $x, y \in V$ and $c \in \mathbb{R}^+$.

Let V be an ordered vector space. A vector $x \in V$ that satisfies $x \geq 0$ is called *positive*. The set $\{x \in V : x \geq 0\}$ is called the *positive cone* of V and is denoted by V^+. We write $x < y$ to mean $x \leq y$ and $x \neq y$. For a subset $A \subseteq V$ we say that A is *order bounded from above* (or just bounded from above) if there exists a vector $y \in V$ such that $x \leq y$ for all $x \in A$. We then call y an *upper bound* of A. The smallest upper bound of A, if it exists, is called the *supremum* of A and is denoted by $\sup A$. We analogously define the *lower bound* and the *infimum* of A. A subset of V that is bounded from above and below is called *order bounded.* For $x, y \in V$ such that $x \leq y$ the set $\{z \in V : x \leq z \leq y\}$ is called an *interval* and is denoted by $[x, y]$.

Definition 2.2. A *vector lattice* is an ordered vector space E such that for all $x, y \in E$ the set $\{x, y\}$ has the supremum and the infimum.

For two elements x and y of a vector lattice we denote $\sup\{x, y\}$ and $\inf\{x, y\}$ by $x \vee y$ and $x \wedge y$, respectively.

Example 2.3.

1. Let S be any nonempty set and $\mathbb{R}^S$ be the vector space of all functions $f : S \to \mathbb{R}$. We define a partial order $\leq$ on $\mathbb{R}^S$ by

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \text{ for all } x \in S.$$

   We easily see that with this ordering $\mathbb{R}^S$ becomes an ordered vector space. We will show that $\mathbb{R}^S$ is a vector lattice. For $f, g \in \mathbb{R}^S$ we claim that their supremum in $\mathbb{R}^S$ is the function $h : S \to \mathbb{R}$ defined by

$$h(x) = \max\{f(x), g(x)\}.$$

Clearly $h \geq f$ because for any $x \in S$ we have

$$h(x) = \max\{f(x), g(x)\} \geq f(x).$$

We analogously have $h \geq g$. If $p \in E$ satisfies $p \geq f$ and $p \geq g$ then for any $x \in S$ we have that $p(x) \geq f(x)$ and $p(x) \geq g(x)$ and therefore $p \geq h$. This proves that $h = f \vee g$. We can similarly prove that the function $k\colon S \to \mathbb{R}$ defined by $k(x) = \min\{f(x), g(x)\}$ is $f \wedge g$.

2. Let X be a nonempty topological space and let $C(X)$ be the vector space of all continuous functions $f\colon X \to \mathbb{R}$. Let $\leq$ be a partial order on $C(X)$ defined in the same way as in the previous example, that is, by

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \text{ for all } x \in X.$$

Then $C(X)$ is a vector lattice. The proof is the same as in the previous example.

3. Let (Ω, Σ, μ) be a measure space and let E be the vector space of all Lebesgue measurable functions $f\colon \Omega \to \mathbb{R}$ (two functions in E that differ at a point are considered different as elements of E). Let $\leq$ be a partial order on E defined in the same way as in the previous examples. Then E is a vector lattice since the pointwise maximum and minimum of two measurable functions are again measurable.

4. Let (Ω, Σ, μ) be a measure space, and let $L_0(\Omega, \Sigma, \mu)$ be the vector space of all (equivalence classes of) measurable functions $f\colon \Omega \to \mathbb{R}$ (we identify two functions that are equal almost everywhere). We define a partial order $\leq$ on $L_0(\Omega, \Sigma, \mu)$ by

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \text{ for almost all } x \in X.$$

It is not hard to see that with this ordering, $L_0(\Omega, \Sigma, \mu)$ is an ordered vector space. Indeed, reflexivity of $\leq$ is obvious. To show antisymmetry, choose $f, g \in L_0(\Omega, \Sigma, \mu)$ such that $f \leq g$ and $g \leq f$. Let $A = \{x \in \Omega : f(x) \leq g(x)\}$ and $B = \{x \in \Omega : g(x) \leq f(x)\}$. Then the complement of $A \cap B$ has measure 0 and $f(x) = g(x)$ on $A \cap B$. Hence $f = g$ in $L_0(\Omega, \Sigma, \mu)$ and $\leq$ is antisymmetric. The proof of transitivity is similar, and is therefore omitted.

To show that $L_0(\Omega, \Sigma, \mu)$ is a vector lattice, let $f, g \in L_0(\Omega, \Sigma, \mu)$. We define $h, k \in L_0(\Omega, \Sigma, \mu)$ by

$$h(x) = \max\{f(x), g(x)\} \qquad \text{and} \qquad k(x) = \min\{f(x), g(x)\}.$$

Since h and k are measurable functions that are finite almost everywhere, by a similar argument as above one can see that $h = f \vee g$ and $k = f \wedge g$.

In the following theorem we gather basic and important equalities that we need throughout the thesis.

Theorem 2.4. *For elements x, y and z of a vector lattice and $c \in \mathbb{R}^+$, the following identities hold:*

(a) $x \vee y = -[(-x) \wedge (-y)]$ *and* $x \wedge y = -[(-x) \vee (-y)]$*;*

(b) $x + y = x \wedge y + x \vee y$*;*

(c) $x + (y \vee z) = (x + y) \vee (x + z)$ *and* $x + (y \wedge z) = (x + y) \wedge (x + z)$*;*

(d) $c(x \vee y) = cx \vee cy$ *and* $c(x \wedge y) = cx \wedge cy$*.*

Proof. (a) Let $z = x \vee y$ and $w = -[(-x) \wedge (-y)]$. From $z \geq x$ and $z \geq y$ we have that $-z \leq -x$ and $-z \leq -y$ and hence $-z \leq (-x) \wedge (-y)$ which implies $z \geq w$. On the other hand from $-w \leq -x$ and $-w \leq -y$ we have that $w \geq x$ and $w \geq y$ and hence $w \geq x \vee y = z$. Therefore $z = w$. The second statement can be proven analogously.

(b) Let $z = x + y - x \wedge y$. Then $z \geq x + y - x = y$ and $z \geq x + y - y = x$, so that $z \geq x \vee y$. This proves that

$$x + y \geq x \wedge y + x \vee y$$

holds for all $x, y \in E$. If we put $-x$ and $-y$ instead of x and y, respectively, and use (a), we get

$$-x - y \geq -x \vee y - x \wedge y$$

or equivalently

$$x + y \leq x \wedge y + x \vee y.$$

This proves (b).

(c) Let $u = x + y \vee z$ and $w = (x + y) \vee (x + z)$. Then we have that $u \geq x + y$ and $u \geq x + z$ imply $u \geq w$. On the other hand, $w - x \geq x + y - x = y$ and $w - x \geq x + z - x = z$ from where it follows that $w - x \geq y \vee z$ and this proves that $w \geq u$, which finishes the proof of the first equality. The second equality can be proven analogously.

(d) If $c = 0$ then there is nothing to prove. Assume that $c > 0$ and let $z = c(x \vee y)$ and $w = (cx) \vee (cy)$. From $\frac{1}{c}z \geq x$ and $\frac{1}{c}z \geq y$ we have that $z \geq cx$ and $z \geq cy$ which imply that $z \geq (cx) \vee (cy) = w$. On the other hand from $w \geq cx$ and $w \geq cy$ we have that $\frac{1}{c}w \geq x$ and $\frac{1}{c}w \geq y$ which imply that $\frac{1}{c}w \geq x \vee y$. This immediately implies that $w \geq c(x \vee y) = z$ which finishes the proof. The other statement can be proven analogously. □

Remark 2.5. By Theorem 2.4, in order to show that an ordered vector space is a vector lattice it is enough to show that for any two elements their supremum exists.

Given an element x of a vector lattice E and a subset $A \subseteq E$ we define $x + A = \{x + a : a \in A\}$, $x \wedge A := \{x \wedge a : a \in A\}$, $x \vee A := \{x \vee a : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$.

Lemma 2.6. *Let E be a vector lattice, $x \in E$ and $A, B \subseteq E$ be nonempty subsets. The following assertions hold:*

(a) if $\sup A$ *exists then* $\sup(x + A)$ *exists and* $\sup(x + A) = x + \sup A$*;*

(b) if $\inf A$ *exists then* $\inf(x + A)$ *exists and* $\inf(x + A) = x + \inf A$*;*

(c) if $\sup A$ *exists then* $\sup(x \wedge A)$ *exists and* $\sup(x \wedge A) = x \wedge \sup A$*;*

(d) if $\inf A$ *exists then* $\inf(x \vee A)$ *exists and* $\inf(x \vee A) = x \vee \inf A$*;*

(e) if $\sup A$ *and* $\sup B$ *exist then* $\sup(A+B)$ *exists and* $\sup(A+B) = \sup A + \sup B$.

Proof. (a) Let $y = \sup A$. For all $a \in A$ we have that $a \leq y$, which implies that $x + a \leq x + y$, and therefore $x + y$ is an upper bound of $x + A$. If $z \in E$ is an upper bound of $x + A$ then $x + a \leq z$ for all $a \in A$ from where it follows that $a \leq z - x$ for all $a \in A$. This shows that $z - x$ is an upper bound of A and hence $y \leq z - x$, which implies that $x + y \leq z$. We conclude that $x + y = \sup(x + A)$.

(b) The proof is similar to the proof of (a).

(c) Suppose that $y := \sup A$ exists. Then $x \wedge a \leq x \wedge y$ for all $a \in A$ so that $x \wedge y$ is an upper bound for $x \wedge A$. Let z be any upper bound of $x \wedge A$. Then for all $a \in A$ we have that $a = x \wedge a + x \vee a - x \leq z + x \vee y - x$ so that $y \leq z + x \vee y - x$. This implies that $z \geq x + y - x \vee y = x \wedge y$. Hence, we have that $x \wedge y = \sup(x \wedge A)$.

(d) This can be proven similarly as (c).

(e) We have that $\sup A + \sup B \geq a + b$ for all $a \in A$ and $b \in B$ which implies that $\sup A + \sup B$ is an upper bound of $A + B$. Suppose that $u \in E$ is an upper bound of $A+B$. For a fixed $a \in A$ we have that $u \geq \sup(a+B) = a + \sup B$ and therefore $u - \sup B \geq a$ for all $a \in A$. By taking the supremum over all $a \in A$ we obtain $u \geq \sup A + \sup B$. Therefore $\sup(A + B) = \sup A + \sup B$. □

Let x be an element of a vector lattice. We define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$, and we call x^+, x^- and $|x|$ the *positive part*, the *negative part*, and the *absolute value* or *modulus* of x, respectively.

Theorem 2.7. *For an element* x *of a vector lattice the following statements hold:*

(a) $x = x^+ - x^-$*;*

(b) $|x| = x^+ + x^-$*;*

(c) $x^+ \wedge x^- = 0$.

The decomposition in statement (c) is unique in the sense that if $x = y - z$ *with* $y \wedge z = 0$ *then* $y = x^+$ *and* $z = x^-$.

Proof. (a) Using Theorem 2.4 we have that

$$x^+ - x^- = x \vee 0 - ((-x) \vee 0) = x \vee 0 + x \wedge 0 = x + 0 = x.$$

(b) By (a) we have that

$$x^+ + x^- = x^+ - x^- + 2x^- = x + 2((-x) \vee 0) = x + (-2x) \vee 0 = (-x) \vee x = |x|.$$

(c) Using (a) and (b) we have that

$$x^+ \wedge x^- = \frac{1}{2}((|x|+x)\wedge(|x|-x)) = \frac{1}{2}(|x|+x\wedge(-x)) = \frac{1}{2}(|x|-(x\vee(-x))) = 0.$$

Now assume that $x = y - z$ with $y \wedge z = 0$. Then we have

$$0 = y \wedge z = (x+z) \wedge z = x \wedge 0 + z,$$

and hence $z = -(x \wedge 0) = -x \vee 0 = x^-$. Since $y - z = x = x^+ - x^-$ we conclude that $y = x^+$. □

Lemma 2.8. *If x and y are elements of a vector lattice then:*

(a) $|x+y| \leq |x| + |y|$*;*

(b) $|x^+ - y^+| \leq |x-y|$ *and* $|x^- - y^-| \leq |x-y|$*;*

(c) $x \vee y = \frac{1}{2}(x+y+|x-y|)$ *and* $x \wedge y = \frac{1}{2}(x+y-|x-y|)$*.*

Proof. (a) By adding the inequalities $x \leq |x|$ and $y \leq |y|$ we get that $x+y \leq |x|+|y|$. If we replace x and y in the last equality with $-x$ and $-y$, respectively, we obtain $-x-y \leq |x|+|y|$ which together with the previous inequality implies that $|x+y| \leq |x|+|y|$.

(b) Observe that

$$\begin{aligned} x^+ - y^+ &= x \vee 0 - y \vee 0 = x \vee 0 + (-y) \wedge 0 = (x + ((-y) \wedge 0)) \vee ((-y) \wedge 0) \\ &= ((x-y) \wedge x) \vee ((-y) \wedge 0). \end{aligned}$$

As $|x-y| \geq x-y \geq (x-y) \wedge x$ and $|x-y| \geq 0 \geq (-y) \wedge 0$ we have that $|x-y| \geq x^+ - y^+$. By symmetry we have $|y-x| \geq y^+ - x^+$, which proves the first statement. The second statement follows from the first one by putting $-x$ and $-y$ for x and y, respectively.

(c) The first statement follows from

$$\frac{1}{2}(x+y+|x-y|) = \frac{1}{2}(x+y+(x-y)\vee(y-x)) = \frac{1}{2}((2x)\vee(2y)) = x \vee y.$$

We omit the proof of the second statement since it is similar. □

Remark 2.9. From Lemma 2.8 it follows that in order to prove that an ordered vector space V is a vector lattice, it is enough to show that the absolute value of any element in V exists.

The following is an important decomposition property for vector lattices. It is going to be used in the proof of Theorem 2.21 on the existence of the modulus of a linear operator between vector lattices.

Lemma 2.10. *Let $x, y_1, \ldots, y_n$ be elements of a vector lattice such that*

$$|x| \leq |y_1 + \cdots + y_n|.$$

Then there exist elements $x_1, \ldots, x_n$ such that

$$x = x_1 + \cdots + x_n$$

and $|x_i| \leq |y_i|$ for all $i = 1, \ldots, n$. Furthermore, if x is positive then the elements x_i can be chosen to be positive.

Proof. Using induction, it is enough to prove the statement for $n = 2$. Suppose $|x| \leq |y_1 + y_2|$ and define $x_1 = (x \vee (-|y_1|)) \wedge |y_1|$. Then $x_1 \leq |y_1|$ and $-x_1 = ((-x) \wedge |y_1|) \vee (-|y_1|)$. From $(-x) \wedge |y_1| \leq |y_1|$ it follows that $-x_1 \leq |y_1|$. Therefore, $|x_1| \leq |y_1|$, and if x is positive we also have $0 \leq x_1 \leq x$.

Now we put $x_2 = x - x_1$ and note that

$$x_2 = x + (-x_1) = x + ((-x) \wedge |y_1|) \vee (-|y_1|) = (0 \wedge (x + |y_1|)) \vee (x - |y_1|).$$

Since $|x| \leq |y_1 + y_2|$ we have $|x| \leq |y_1| + |y_2|$ or equivalently

$$-|y_1| - |y_2| \leq x \leq |y_1| + |y_2|.$$

Therefore

$$-|y_2| = (-|y_2|) \wedge 0 \leq (x + |y_1|) \wedge 0 \leq x_2 \leq 0 \vee (x - |y_1|) \leq |y_2|,$$

and hence $|x_2| \leq |y_2|$. If x is positive then due to $0 \leq x_1 \leq x$ we have that x_2 is positive too. □

Two elements x, y of a vector lattice E are called *disjoint* if $|x| \wedge |y| = 0$. If x and y are disjoint we write $x \perp y$. For $A, B \subseteq E$ we write $A \perp B$ if $x \perp y$ for all $x \in A$ and $y \in B$. Also, we define the *disjoint complement* A^d of A by

$$A^d := \{x \in E : x \perp y \text{ for all } y \in A\}.$$

From the definition of the disjoint complement it follows that $A \cap A^d = \{0\}$.

Now we define the notion of a net which is a generalization of the notion of a sequence. Let A be a partially ordered set with the property that for any $x, y \in A$ there exists $z \in A$ such that $x \leq z$ and $y \leq z$. Then A is called a *directed set*. If M is a set we call a mapping $s \colon A \to M$ a *net*. In particular, the identity mapping from A to itself is a net. For $\alpha \in A$ we denote the value $s(\alpha)$ by s_α and we denote the net s by (s_α), $(s_\alpha)_\alpha$ or $(s_\alpha)_{\alpha \in A}$. We say that a net (x_α) in a vector lattice E is *increasing* if $\alpha \leq \beta$ implies $x_\alpha \leq x_\beta$ and we write $x_\alpha \uparrow$. If $x_\alpha \uparrow$ and $x_\alpha \leq x$ for all α then we write $x_\alpha \uparrow \leq x$. Similarly if $x_\alpha \uparrow$ and $x \leq x_\alpha$ for all α then we write $x \leq x_\alpha \uparrow$. If $\sup_\alpha x_\alpha = x$ we write $x_\alpha \uparrow x$. We analogously define the meaning of a *decreasing* net, $x_\alpha \downarrow \leq x$, $x \leq x_\alpha \downarrow$ and $x_\alpha \downarrow x$.

A subset A of E is said to be *directed upward* if for every $x, y \in A$ there exists $z \in A$ such that $x \leq z$ and $y \leq z$. If A is directed upward then we write $A \uparrow$. If furthermore $y \leq x$ for all $y \in A$ we write $A \uparrow \leq x$ and if $x \leq y$ for all $y \in A$ we write $x \leq A \uparrow$. Also, if $\sup A = x$ then we write $A \uparrow x$. We analogously define the meaning of *downward directed* set, $A \downarrow$, $A \downarrow \leq x$, $x \leq A \downarrow$ and $A \downarrow x$.

We prove the following useful short lemma concerning nets.

Lemma 2.11. *If $x_\alpha \uparrow x$ and $y_\alpha \uparrow y$ then $(x_\alpha + y_\alpha) \uparrow x + y$.*

Proof. Obviously $(x_\alpha + y_\alpha) \uparrow \leq x + y$. Let u be an upper bound of $(x_\alpha + y_\alpha)_\alpha$. If α_0 is some index then $u \geq x_\alpha + y_{\alpha_0}$ for all α and so $u \geq x + y_{\alpha_0}$. As α_0 was arbitrary we get that $u \geq x + y$ and therefore $x_\alpha + y_\alpha \uparrow x + y$. □

A vector lattice E is called *(σ-)Dedekind complete* if every nonempty (countable) subset of E which is bounded from above has a supremum. The following lemma gives alternative equivalent formulations of Dedekind completeness and has an analogue for σ-Dedekind completeness.

Lemma 2.12. *The following statements are equivalent for a vector lattice E:*

(a) E is Dedekind complete;

(b) every bounded from below subset of E has an infimum;

(c) if $0 \leq x_\alpha \uparrow \leq x$ in E then there exists $y \in E$ such that $x_\alpha \uparrow y$.

Proof. (a) $\Rightarrow$ (b) Suppose that E is Dedekind complete and let $A \subseteq E$ be a subset of E that is bounded from below. Then the set $-A = \{-x : x \in A\}$ is bounded from above and has the supremum v. Now it is easy to check that $-v$ is the infimum of A.

(b) $\Rightarrow$ (c) Suppose that every subset of E which is bounded from below has an infimum. Suppose that $0 \leq x_\alpha \uparrow \leq x$. If $A = \{x_\alpha : \alpha\}$ then A is bounded from above, and hence $-A$ is bounded from below. If y is the infimum of $-A$ then it is not hard to see that $-y$ is the supremum of A.

(c) $\Rightarrow$ (a) Suppose that (c) holds and let $A \subseteq E$ be a set which is bounded from above by z. By considering the set $\{x - y : x \in A\}$ for some $y \in A$ instead of A, if necessary, we can assume without loss of generality that $0 \in A$. Let $B = \{x_1 \vee \ldots \vee x_n : x_1, \ldots, x_n \in A, n \in \mathbb{N}\}$ and $C = B \cap E^+$. We have that C is a directed set with the ordering inherited from E, and so the identity map on C is a net. Also C is nonempty since $0 \in A$ and C is bounded from above by z. By the assumption, C has a supremum z_0. It is easy to see that z_0 is also the supremum of B and A. ☐

Remark 2.13. An analogous lemma to Lemma 2.12 can be proven for σ-Dedekind completeness.

A vector lattice is said to be *Archimedean* if $\frac{1}{n}x \downarrow 0$ for every $x \in E^+$. Since all vector lattices in Example 2.3 are function lattices, the Archimedean property of these vector lattices translates to the Archimedean property of $\mathbb{R}$, which is obviously true.

Lemma 2.14. *Every σ-Dedekind complete vector lattice is Archimedean.*

Proof. Let E be a σ-Dedekind complete vector lattice and pick $x \in E^+$. By Remark 2.13 the set $\{\frac{1}{n}x : n \in \mathbb{N}\}$ has the infimum $y \in E$. It is easy to see that $y \geq 0$. For every $n \in \mathbb{N}$ we have that $y \leq \frac{1}{2n}x$ which implies that $2y \leq \frac{1}{n}x$ from where it follows that $2y$ is a lower bound of $\{\frac{1}{n}x : n \in \mathbb{N}\}$. Since y is the infimum of $\{\frac{1}{n}x : n \in \mathbb{N}\}$ and $2y \geq y$ we have that $2y = y$ or equivalently $y = 0$. ☐

We end this section with a lemma that will be used in the proof of Theorem 2.40.

Lemma 2.15. *For an Archimedean vector lattice E, if $0 \leq x_\alpha \uparrow \leq x$ then the set $D = \{y \in E, x_\alpha \leq y \text{ for all } \alpha\}$ is directed downward and*

$$\inf\{y - x_\alpha : y \in D, \alpha\} = 0.$$

Proof. If $y, z \in D$ then clearly $y \wedge z \in D$, implying that D is directed downward. Also D is nonempty since $x \in D$. Suppose that $0 \leq u \leq y - x_\alpha$ for all $y \in D$ and α, and choose any $y \in D$. Then $y - u \geq x_\alpha$ for all α and so $y - u \in D$. By an induction argument we conclude that $y - nu \in D$ for all $n \in \mathbb{N}$. Hence, we have that $0 \leq nu \leq y$ for all $n \in \mathbb{N}$, which is possible only for $u = 0$ since E is Archimedean. □

2.2 Ideals and Bands

A *vector sublattice* of E is a vector subspace G of E such that G with the induced (restricted) partial order from E is a vector lattice and the supremum and infimum of $f, g \in G$ in G coincide with their supremum and infimum in E, respectively. A subset A of a vector lattice E is called *solid* if for $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. A vector subspace of E which is solid is called an *ideal*, and it is always a vector sublattice of E, since the supremum of x and y in E is in absolute value less than $|x| + |y|$, which belongs to the ideal. A vector sublattice G of E is said to be *order dense* in E if for every $0 \neq x \in E^+$ there exists $0 \neq y \in G^+$ such that $y \leq x$.

Theorem 2.16. *Let E be an Archimedean vector lattice and let G be a sublattice of E. Then G is order dense in E if and only if for every $x \in E^+$ we have that* $\sup\{y \in G : 0 \leq y \leq x\} = x$.

Proof. Clearly, if for every $x \in E^+$ we have that $\sup\{y \in G : 0 \leq y \leq x\} = x$ then G is order dense in E.

To show the converse, let G be order dense in E and let $x \in E^+$. Suppose that x is not the supremum of $\{y \in G : 0 \leq y \leq x\}$. Then there exists some $z \in E^+$ such that z is an upper bound of $\{y \in G : 0 \leq y \leq x\}$ but $z \not\geq x$. By replacing z by $z \wedge x$ if necessary, we can assume that $z < x$. Since G is order dense in E there exists $u \in G^+$ such that $0 < u \leq x - z$. Since $u \leq x$ we have that $u \leq z$ and therefore $2u = u + u \leq z + x - z = x$. By induction we have that $nu \leq x$ for every $n \in \mathbb{N}$. But this contradicts the Archimedean property of E. □

We say that a net $(x_\alpha)_\alpha$ in a vector lattice E *converges to x in order* if there exists a net $(y_\alpha)_\alpha$ in E with the same index set as $(x_\alpha)_\alpha$ such that $|x_\alpha - x| \leq y_\alpha$ for each α and $y_\alpha \downarrow 0$. If $(x_\alpha)_\alpha$ converges to x in order we write $x_\alpha \xrightarrow{o} x$.

A subset $A \subseteq E$ is called *order closed* if $(x_\alpha)_\alpha \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$. An order closed ideal is called a *band*.

Lemma 2.17. *A solid subset A of E is order closed if and only if $(x_\alpha) \subseteq A$ and $0 \leq x_\alpha \uparrow x$ imply that $x \in A$.*

Proof. Let $A \subseteq E$ be solid and such that $(x_\alpha) \subseteq A$ and $0 \leq x_\alpha \uparrow x$ imply that $x \in A$. Suppose that $x_\alpha \xrightarrow{o} x$ with $(x_\alpha) \subseteq A$. Then there is a net (y_α) in E such that $|x_\alpha - x| \leq y_\alpha \downarrow 0$. Then $|x| - |x_\alpha| \leq |x - x_\alpha| \leq y_\alpha$ for all α and therefore $|x| - y_\alpha \leq |x_\alpha|$ for all α which implies that $(|x| - y_\alpha)^+ \leq |x_\alpha|$ for all α. As $(x_\alpha) \subseteq A$ and A is solid we have that $(|x| - y_\alpha)^+ \in A$ for all α. From $y_\alpha \downarrow 0$ we have that $|x| - y_\alpha \uparrow |x|$ and since $|x| \geq (|x| - y_\alpha)^+ \geq |x| - y_\alpha \uparrow |x|$ we have that $(|x| - y_\alpha)^+ \uparrow |x|$. This implies that $|x| \in A$ and since A is solid we have that $x \in A$. □

Lemma 2.18. *If A is an ideal in E then the vector subspace $A \oplus A^d$ is an order dense ideal of E.*

Proof. It is not hard to see that A^d is an ideal. Pick some element $x = y+z \in A \oplus A^d$ with $y \in A$ and $z \in A^d$ and pick $u \in E$ with $|u| \leq |x|$. Then by Lemma 2.10 there exist $v, w \in E$ with $u = v + w, |v| \leq |x|$ and $|w| \leq |y|$. As A and A^d are ideals, it follows that $v \in A$ and $w \in A^d$. Therefore $u = v + w \in A \oplus A^d$.

To show that $A \oplus A^d$ is order dense in E, let $0 \neq x \in E^+$ and suppose that there is no $y \in A \oplus A^d$ with $0 < y \leq x$. Then since $A \oplus A^d$ is an ideal we have that $x \wedge |y| = 0$ for all $y \in A \oplus A^d$. In particular, $x \wedge |y| = 0$ holds for all $y \in A$ and all $y \in A^d$. Therefore $x \in A^d$ and $x \in A^{dd}$ which implies that $x = 0$. This contradiction shows that there exists $0 < y \in A \oplus A^d$ such that $x \wedge y \neq 0$ and as $x \wedge y \in A \oplus A^d$, we have that $A \oplus A^d$ is indeed order dense in E. □

Lemma 2.19. *For a subset A of an Archimedean vector lattice E, the following statements hold:*

(a) A^d is a band;

(b) if A is a band then $A = A^{dd}$.

Proof. (a) We have already observed that A^d is an ideal. To show that A^d is order closed, let $0 \leq x_\alpha \uparrow x$ with $(x_\alpha) \subseteq A^d$. Fix $y \in A$. By Lemma 2.6 we have that $x \wedge |y| = \sup_\alpha (x_\alpha \wedge |y|) = 0$ since $x_\alpha \wedge |y| = 0$ for all α. As $y \in A$ was arbitrary we get that $x \in A^d$. Therefore A^d is order closed.

(b) Clearly $A \subseteq A^{dd}$. We will first show that A is order dense in A^{dd}. Pick arbitrary $0 < x \in A^{dd}$. If $y \wedge x = 0$ for all $y \in A^+$ then $x \in A^d$ and hence $x = 0$. Therefore there exists $y \in A^+$ such that $0 < y \wedge x \leq x$. Since A is an ideal we have that $y \wedge x \in A$. This shows that A is order dense in A^{dd}.

Let $0 \leq x \in A^{dd}$. By Theorem 2.16 $\sup\{y \in A : 0 \leq y \leq x\} = x$ and therefore there is a net in A that increases to x and hence is order convergent to x. As A is order closed, we conclude that $x \in A$. This implies that $A^{dd} \subseteq A$. □

2.3 Linear Operators

The theory of linear operators between vector lattices is important for the thesis especially since it is applied later to the study of linear functionals and dual spaces. Positive maps will play an important role since an operator or a functional is often decomposable as the difference of its positive and negative parts, in an analogous way as a measurable function is decomposable as the difference of its positive and negative parts. If E and F are vector lattices and $T \colon E \to F$ is a linear map, we say that T is *positive* if Tx is positive for all $x \in E^+$.

Theorem 2.20 (Kantorovič)**.** *Let E and F be vector lattices with F Archimedean and let $T \colon E^+ \to F^+$ be an additive map. Then T extends uniquely to a positive operator $S \colon E \to F$. Furthermore,*

$$Sx = Tx^+ - Tx^-$$

holds for any $x \in E$.

Proof. We define $S\colon E \to F$ by $Sx = Tx^+ - Tx^-$. Clearly, S is well-defined as a map from E to F since the decomposition $x = x^+ - x^-$ is unique. Pick $x, y \in E$ and observe that the equality

$$(x+y)^+ - (x+y)^- = x + y = x^+ - x^- + y^+ - y^-$$

immediately yields

$$(x+y)^+ + x^- + y^- = (x+y)^- + x^+ + y^+.$$

As T is additive, we have that

$$T(x+y)^+ + Tx^- + Ty^- = T(x+y)^- + Tx^+ + Ty^+$$

from where it follows that

$$S(x+y) = T(x+y)^+ - T(x+y)^- = Tx^+ - Tx^- + Ty^+ - Ty^- = Sx + Sy.$$

This proves that S is additive.

To show homogenuity of S, we first note that if $x \in E$ and $n, m \in \mathbb{N}$, from the additivity of S we have that

$$mS(\frac{n}{m}x) = S(m\frac{n}{m}x) = S(nx) = nSx$$

from where it follows that $S(\frac{n}{m}x) = \frac{n}{m}Sx$. Now, let $x \in E^+$ and $c \in \mathbb{R}^+$. Let (r_n) and (s_n) be nonnegative sequences of rational numbers such that $r_n \uparrow c$ and $s_n \downarrow c$. Since for every $n \in \mathbb{N}$ we have

$$r_n Sx = S(r_n x) \leq S(cx) \leq S(s_n x) = s_n Sx,$$

by taking the limit as $n \to \infty$ and using the Archimedean property of F, we get that $S(cx) = cSx$. If $c \in \mathbb{R}$ is negative, we use additivity of S to conclude that

$$S(cx) = -S((-c)x) = -(-c)Sx = cSx.$$

If $c \in \mathbb{R}$ and $x \in E$ are arbitrary then by using the previously proven identities we have that

$$\begin{aligned} S(cx) &= S(cx^+ - cx^-) = S(cx^+) - S(cx^-) \\ &= cSx^+ - cSx^- = c(Sx^+ - Sx^-) = cSx. \end{aligned}$$

We have now proven that S is linear. To show uniqueness of S, we note that any linear extension $\widetilde{T}$ of T to E satisfies

$$\widetilde{T}x = \widetilde{T}x^+ - \widetilde{T}x^- = Tx^+ - Tx^- = Sx.$$ □

We denote the real vector space of all linear operators from E to F by $\mathcal{L}(E, F)$. If we equip $\mathcal{L}(E, F)$ with the ordering $\leq$ where

$$T \leq S \quad \Leftrightarrow \quad Tx \leq Sx \;\text{ for all }\; x \in E^+$$

then $\mathcal{L}(E, F)$ becomes an ordered vector space. Indeed, reflexivity and transitivity of $\leq$ are obvious. If $T \leq S$ and $S \leq T$ then $Tx = Sx$ for all $x \in E^+$ which implies

$$Tx = Tx^+ - Tx^- = Sx^+ - Sx^- = Sx$$

for any $x \in E$. Therefore, $\leq$ is indeed a partial order. The two conditions in the definition of an ordered vector space are easily seen to be true.

Theorem 2.21. *Let $T\colon E \to F$ be a linear operator between vector lattices such that* $\sup\{|Ty| : |y| \le x\}$ *exists for every $x \in E^+$. Then the modulus of T exists and it satisfies*

$$|T|x = \sup\{|Ty| : |y| \le x\}$$

for every $x \in E^+$. In particular, we have that $|Tx| \le |T||x|$ for all $x \in E$.

Proof. By assumption the mapping $S\colon E^+ \to F^+$ defined by

$$Sx = \sup\{|Ty| : |y| \le x\}$$

is well-defined. We will prove that S is additive. Pick $x, y \in E^+$ and observe that

$$\begin{aligned} Sx + Sy &= \sup\{|Tu| : |u| \le x\} + \sup\{|Tv| : |v| \le y\} \\ &= \sup\{Tu \vee T(-u) : |u| \le x\} + \sup\{Tv \vee T(-v) : |v| \le y\}. \end{aligned}$$

We claim that $\sup\{Tu \vee T(-u) : |u| \le x\} = \sup\{Tu : |u| \le x\}$. Indeed, as $Tu \vee T(-u) \ge Tu$ for all $u \in E$, we have $\sup\{Tu \vee T(-u) : |u| \le x\} \ge \sup\{Tu : |u| \le x\}$, and as $\sup\{Tu : |u| \le x\} \ge Tu \vee T(-u)$ where $u \in E$ is such that $|u| \le x$, we have that $\sup\{Tu : |u| \le x\} \ge \sup\{Tu \vee T(-u) : |u| \le x\}$. Similarly we can prove that $\sup\{Tv \vee T(-v) : |v| \le y\} = \sup\{Tv : |v| \le y\}$. Hence by Lemma 2.6 we have that

$$\begin{aligned} Sx + Sy &= \sup\{Tu : |u| \le x\} + \sup\{Tv : |v| \le y\} \\ &= \sup\{Tu + Tv : |u| \le x, |v| \le y\} = \sup\{T(u+v) : |u| \le x, |v| \le y\}. \end{aligned}$$

As $|u+v| \le |u| + |v| \le x + y$, we have that the latter supremum is less than or equal to

$$\sup\{Tw : |w| \le x + y\} = S(x+y)$$

from where it follows that $Sx + Sy \le S(x+y)$.

To prove the converse inequality, let $w \in E$ be such that $|w| \le x + y$. Then by Lemma 2.10 we can find u and v such that $w = u + v$ and $|u| \le x$ and $|v| \le y$. Then

$$Tw = Tu + Tv \le Sx + Sy$$

yields

$$S(x+y) = \sup\{Tw : |w| \le x + y\} \le Sx + Sy,$$

which completes the proof that S is additive. By Theorem 2.20 S extends uniquely to a positive linear operator $S\colon E \to F$.

To see that $|T| = S$, we first note that $Sx \ge Tx$ and $Sx \ge (-T)x$ for every $x \in E^+$, so that $S \ge T$ and $S \ge -T$. Now let $R\colon E \to F$ be such that $R \ge T$ and $R \ge -T$. We first prove that R is positive. Indeed, if $x \in E^+$ then $Rx \ge Tx$ and $Rx \ge -Tx$, and so $Rx \ge \frac{1}{2}(Tx - Tx) = 0$. Now, if $|y| \le x \in E^+$ then

$$Ty = Ty^+ - Ty^- \le Ry^+ + Ry^- = R|y| \le Rx,$$

so that by definition of S we conclude that $S \le R$. $\square$

We call a linear operator $T\colon E \to F$ *order bounded* if it maps order bounded sets of E to order bounded sets of F. It is easy to see that the set of all order bounded operators from E to F is a vector subspace of $\mathcal{L}(E,F)$. Indeed, if T and S are order bounded operators from E to F, and $[x,y]$ is an interval in E then there exist $u_1, v_1, u_2, v_2 \in F$ such that $T[x,y] \subseteq [u_1, v_1]$ and $S[x,y] \subseteq [u_2, v_2]$, so that $(T+S)[x,y] \subseteq [u_1+u_2, v_1+v_2]$. Similarly we can prove that cT is order bounded for $c \in \mathbb{R}$. We denote the vector space of all order bounded operators by $\mathcal{L}_b(E,F)$. We next prove that $\mathcal{L}_b(E,F)$ is a Dedekind complete vector lattice whenever F is. We also prove the so called Riesz-Kantorovič formulas for its lattice operations.

Theorem 2.22 (Riesz-Kantorovič)**.** *Let E and F be vector lattices with F Dedekind complete. Then the vector space $\mathcal{L}_b(E,F)$ is a Dedekind complete vector lattice. Its lattice operations satisfy*

$$(S \vee T)x = \sup\{Sy + Tz : y, z \in E^+, y + z = x\}$$

and

$$(S \wedge T)x = \inf\{Sy + Tz : y, z \in E^+, y + z = x\}$$

for every $x \in E^+$. We also have that

$$T_\alpha \downarrow 0 \text{ in } \mathcal{L}_b(E,F) \text{ if and only if } T_\alpha x \downarrow 0 \text{ for all } x \in E^+.$$

Proof. Choose any $T \in \mathcal{L}_b(E,F)$. As T is order bounded and F is Dedekind complete, we have that

$$\begin{aligned}\sup\{|Ty| : |y| \le x\} &= \sup\{Ty \vee T(-y) : |y| \le x\}\\ &= \sup\{Ty : |y| \le x\} = \sup T[-x,x]\end{aligned}$$

exists in F. Hence, by Theorem 2.21 we have that $|T|$ exists, and by Remark 2.5 we get that $\mathcal{L}_b(E,F)$ is a vector lattice.

Let us now take any $S, T \in \mathcal{L}_b(E,F)$ and $x \in E^+$. We have (by Lemma 2.8)

$$\begin{aligned}(S \vee T)x &= \frac{1}{2}(Sx + Tx + |S-T|x)\\ &= \frac{1}{2}(Sx + Tx + \sup\{(S-T)y : |y| \le x\})\\ &= \frac{1}{2}\sup\{Sx + Tx + Sy - Ty : |y| \le x\}\\ &= \sup\{S(\tfrac{1}{2}(x+y)) + T(\tfrac{1}{2}(x-y)) : |y| \le x\}.\end{aligned}$$

Since for $u, v \in E^+$ we have that $u + v = x$ if and only if there exists y with $|y| \le x$ such that $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{2}(x-y)$, the latter supremum equals to

$$\sup\{Su + Tv : u, v \in E^+, u + v = x\}$$

which proves the formula for $S \vee T$. The other formula can be proven analogously.

To prove that $\mathcal{L}_b(E,F)$ is Dedekind complete, suppose that $T_\alpha \uparrow\, \le T$. We will show that the map S defined by $S(x) = \sup_\alpha T_\alpha x$ for all $x \in E^+$ extends uniquely to a positive operator from E to F which equals $\sup_\alpha T_\alpha$. For $x, y \in E^+$ we have that $S(x+y) = \sup_\alpha T_\alpha(x+y) = \sup_\alpha (T_\alpha x + T_\alpha y)$. We claim that $\sup_\alpha (T_\alpha x + T_\alpha y) = \sup_\alpha T_\alpha x + \sup_\alpha T_\alpha y$. By fixing an index β and using Lemma 2.6, we see that

$$\sup_\alpha (T_\alpha x + T_\alpha y) \geq \sup_\alpha (T_\alpha x + T_\beta y) = T_\beta y + \sup_\alpha T_\alpha x.$$

As β was arbitrary, we get that $\sup_\alpha (T_\alpha x + T_\alpha y) \geq \sup_\alpha T_\alpha x + \sup_\alpha T_\alpha y$. On the other hand, we have that $\sup_\alpha T_\alpha x + \sup_\alpha T_\alpha y \geq T_\alpha x + T_\alpha y$ for every index α, which implies that $\sup_\alpha T_\alpha x + \sup_\alpha T_\alpha y \geq \sup_\alpha (T_\alpha x + T_\alpha y)$. Therefore $S(x+y) = \sup_\alpha T_\alpha x + \sup_\alpha T_\alpha y = Sx + Sy$ from where it follows that S is additive. By Theorem 2.20, S uniquely extends to a positive linear operator $\widetilde{S} \colon E \to F$.

From the definition of S one can conclude that $T_\alpha \uparrow \widetilde{S}$, and hence $\mathcal{L}_b(E, F)$ is Dedekind complete. It follows that for $0 \leq T \in \mathcal{L}_b(E, F)$ we have that $T_\alpha \uparrow T$ if and only if $T_\alpha x \uparrow Tx$ for all $x \in E^+$. From this it follows that $T_\alpha \downarrow 0$ if and only if for a fixed β we have $(T_\beta - T_\alpha) \uparrow_{\alpha \geq \beta} T_\beta$. The latter statement holds if and only if $(T_\beta - T_\alpha)x \uparrow_{\alpha \geq \beta} T_\beta x$ for every $x \in E^+$, and this is true if and only if $T_\alpha x \downarrow 0$ for every $x \in E^+$. □

If $T \colon E \to F$ is an order bounded operator and F is Dedekind complete then from Theorem 2.22 we can derive the following important equalities that hold for every $x \in E^+$:

$$T^+ x = (T \vee 0)x = \sup\{Ty : y, z \in E^+, y + z = x\} = \sup\{Ty : 0 \leq y \leq x\},$$

and

$$T^- x = (-T)^+ x = \sup\{-Ty : 0 \leq y \leq x\} = -\inf\{Ty : 0 \leq y \leq x\}.$$

We conclude this section with the following theorem which will be used in the proof of Theorem 2.29.

Theorem 2.23. *Let $T \colon E \to F$ be a positive operator between vector lattices with F σ-Dedekind complete. Then for every $x \in E^+$ there exists a positive operator $S \colon E \to F$ with the following properties:*

(a) $0 \leq S \leq T$;

(b) $Sx = Tx$;

(c) $Sy = 0$ *for all* $y \perp x$.

Proof. Fix $x \in E^+$ and define $S \colon E^+ \to F^+$ by $Sy = \sup_{n \in \mathbb{N}} T(y \wedge nx)$. This is well-defined since the sequence $(T(y \wedge nx))_{n \in \mathbb{N}}$ is bounded from above by Ty and F is σ-Dedekind complete.

We claim that S is additive. Pick $y, z \in E^+$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} y \wedge nx + z \wedge nx &= (y + (z \wedge nx)) \wedge (nx + (z \wedge nx)) \\ &= ((y+z) \wedge (y + nx)) \wedge (nx + (z \wedge nx)) \geq ((y+z) \wedge nx) \wedge nx \\ &= (y+z) \wedge nx. \end{aligned}$$

Positivity of T implies that $Sy+Sz \geq T(y \wedge nx)+T(z \wedge nx) \geq T((y+z) \wedge nx)$ for every $n \in \mathbb{N}$, and so $Sy+Sz \geq S(y+z)$. On the other hand, for arbitrary $m, n \in \mathbb{N}$ we have that

$$y \wedge nx + z \wedge mx = (y + z \wedge mx) \wedge (nx + z \wedge mx) \leq (y+z) \wedge (nx+mx)$$

and hence $T(y \wedge nx) + T(z \wedge mx) \leq T((y+z) \wedge ((n+m)x))$ which easily implies that $Sy + Sz \leq S(y+z)$. Therefore S is additive.

Let $\widetilde{S}$ be the unique extension of S to E. For every $y \in E^+$ we have that $\widetilde{S}y = Sy = \sup_{n\in\mathbb{N}} T(y \wedge nx) \leq Ty$, and therefore (a) holds for $\widetilde{S}$. Also, $\widetilde{S}x = \sup_{n\in\mathbb{N}} T(x \wedge nx) = \sup_{n\in\mathbb{N}} Tx = Tx$ and so (b) holds for $\widetilde{S}$. To check (c), we note that if $|y| \wedge x = 0$ then for every $n \in \mathbb{N}$ we have that $|y| \wedge nx = 0$ since $\{|y|\}^d$ is a vector subspace of E. This implies that $S|y| = 0$ and so $|\widetilde{S}y| \leq \widetilde{S}|y| = S|y| = 0$. □

2.4 Order Continuous Operators

We now introduce the basic properties of order continuous operators whose definition was introduced by T. Ogasawara.

Definition 2.24. An operator $T\colon E \to F$ between vector lattices is called:

(a) *order continuous* if $Tx_\alpha \xrightarrow{o} 0$ whenever $x_\alpha \xrightarrow{o} 0$;

(b) *σ-order continuous* if $Tx_n \xrightarrow{o} 0$ whenever $x_n \xrightarrow{o} 0$.

We note that a positive operator T is order continuous if and only if $x_\alpha \downarrow 0$ implies $Tx_\alpha \downarrow 0$, and also if and only if $0 \leq x_\alpha \uparrow x$ implies $Tx_\alpha \uparrow Tx$. Analogous statements hold for positive σ-order continuous operators. Obviously, an order continuous operator is also σ-order continuous. However, a σ-order continuous operator does not need to be order continuous as the following example shows.

Example 2.25. Let E be the vector lattice of all Lebesgue integrable functions $f\colon [0,1] \to \mathbb{R}$ (two functions in E that differ at a point are considered different as elements of E). We note that $f_\alpha \uparrow f$ holds in E if and only if $f(x) = \sup_\alpha f_\alpha(x)$ for all $x \in [0,1]$. Let $T\colon E \to \mathbb{R}$ be defined by $T(f) = \int_0^1 f(x)dx$. Then T is a positive linear functional and from Lebesgue's dominated convergence theorem it follows that T is σ-order continuous.

We claim that T is not order continuous. Consider the directed set $\mathcal{F} = \{P \subset [0,1] : P \text{ is finite}\}$ ordered by inclusion. Then (χ_P) is a net in E and $\chi_P \uparrow \mathbf{1}$, where $\mathbf{1}$ is the constant one function on $[0,1]$. Since $T(\chi_P) = 0 \not\uparrow 1 = T(\mathbf{1})$ we have that T is not order continuous.

We denote the subset of $\mathcal{L}_b(E,F)$ of order continuous operators and σ-order continuous operators by $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$, respectively. The rest of the section is devoted to determining the vector lattice structure of $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$.

Proposition 2.26. *The sets of operators $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are vector subspaces of $\mathcal{L}_b(E,F)$.*

Proof. Choose $T, S \in \mathcal{L}_n(E,F)$ and $c, d \in \mathbb{R}$ and suppose that $x_\alpha \xrightarrow{o} 0$. Then there exist nets (y_α) and (z_α) that satisfy $y_\alpha \downarrow 0, z_\alpha \downarrow 0, |Tx_\alpha| \leq y_\alpha$ and $|Sx_\alpha| \leq z_\alpha$ for all α. For each α we have

$$\begin{aligned} |(cT + dS)x_\alpha| &= |(cT)x_\alpha + (dS)x_\alpha| \leq |(cT)x_\alpha| + |(dS)x_\alpha| \\ &= |c||Tx_\alpha| + |d||Sx_\alpha| \leq |c|y_\alpha + |d|z_\alpha \downarrow 0 \end{aligned}$$

which implies that $cT + dS \in \mathcal{L}_n(E,F)$. The fact that $\mathcal{L}_c(E,F)$ is a vector space can be proven similarly. □

Theorem 2.27. *Let $T: E \to F$ be an order bounded operator between vector lattices with F Dedekind complete. The following statements are equivalent:*

(a) T is order continuous;

(b) If $x_\alpha \downarrow 0$ in E then $Tx_\alpha \xrightarrow{o} 0$;

(c) If $x_\alpha \downarrow 0$ in E then $\inf_\alpha |Tx_\alpha| = 0$;

(d) T^+ and T^- are order continuous;

(e) $|T|$ is order continuous.

An analogous statement holds for σ-order continuous operators.

Proof. The statements (a) ⇒ (b) and (b) ⇒ (c) are easy to check.

To show (c) ⇒ (d), we first prove that T^+ is order continuous. Let $x_\alpha \downarrow 0$ in E. Since F is Dedekind complete there exists $0 \leq z \in F$ such that $T^+x_\alpha \downarrow z$. We need to prove that $z = 0$.

Let $x = x_\beta$ for some fixed β. Since for every $0 \leq y \leq x$ and $\alpha \geq \beta$ we have

$$\begin{aligned} y \wedge (x - x_\alpha) + y \wedge x_\alpha &= (y + y \wedge x_\alpha) \wedge (x - x_\alpha + y \wedge x_\alpha) \\ &\geq y \wedge ((x - x_\alpha + y) \wedge x) \geq y \wedge x, \end{aligned}$$

we obtain that

$$0 \leq y - y \wedge x_\alpha = y \wedge x - y \wedge x_\alpha \leq y \wedge (x - x_\alpha) \leq x - x_\alpha.$$

Then

$$\begin{aligned} Ty - T(y \wedge x_\alpha) &= T(y - y \wedge x_\alpha) = T^+(y - y \wedge x_\alpha) - T^-(y - y \wedge x_\alpha) \\ &\leq T^+(y - y \wedge x_\alpha) \leq T^+(x - x_\alpha) = T^+x - T^+x_\alpha, \end{aligned}$$

which implies that

$$0 \leq z \leq T^+x_\alpha \leq T^+x + |T(y \wedge x_\alpha)| - Ty.$$

Since $y \wedge x_\alpha \downarrow_{\alpha \geq \beta} 0$, it follows that $\inf_{\alpha \geq \beta} |T(y \wedge x_\alpha)| = 0$, so that $0 \leq z \leq T^+x - Ty$. Since $0 \leq y \leq x$ was arbitrary and $T^+x = \sup\{Ty : 0 \leq y \leq x\}$, we have that $\inf\{T^+x - Ty : 0 \leq y \leq x\} = 0$ and therefore $z = 0$. Since $T^- = (-T)^+$, we obtain that T^- is also order continuous.

The implication (d) ⇒ (e) is obvious since $|T| = T^+ + T^-$ is the sum of two order continuous operators.

Lastly, the implication (e) ⇒ (a) follows from the facts that $|Tx| \leq |T||x|$ for all $x \in E$ and that $x_\alpha \xrightarrow{o} 0$ implies $|x_\alpha| \xrightarrow{o} 0$. □

Theorem 2.28 (Ogasawara)**.** *If E and F are vector lattices with F Dedekind complete then $\mathcal{L}_n(E,F)$ and $\mathcal{L}_c(E,F)$ are bands of $\mathcal{L}_b(E,F)$.*

Proof. We only prove that $\mathcal{L}_n(E,F)$ is a band since the proof for $\mathcal{L}_c(E,F)$ is similar. To show that $\mathcal{L}_n(E,F)$ is an ideal, pick $T \in \mathcal{L}_n(E,F)$ and $S \in \mathcal{L}_b(E,F)$ such that $|S| \leq |T|$. If $x_\alpha \downarrow 0$ in E then $|T|x_\alpha \downarrow 0$ and since $0 \leq |S|x_\alpha \leq |T|x_\alpha$ we have that $|S|x_\alpha \downarrow 0$. This shows that $|S| \in \mathcal{L}_n(E,F)$ and by Theorem 2.27 we have $S \in \mathcal{L}_n(E,F)$.

To show that $\mathcal{L}_n(E,F)$ is order closed, pick $(T_\alpha)_\alpha \subseteq \mathcal{L}_n(E,F)$ and $T \in \mathcal{L}_b(E,F)$ such that $0 \leq T_\alpha \uparrow T$, and pick a net $(x_\beta)_\beta \subseteq E$ such that $0 \leq x_\beta \uparrow x$. Then for a fixed index α we have that $Tx_\beta \geq T_\alpha x_\beta$ for all β which implies that $\sup_\beta Tx_\beta \geq \sup_\beta T_\alpha x_\beta = T_\alpha x$. Therefore we have that $\sup_\beta Tx_\beta \geq Tx$, and since $Tx_\beta \leq Tx$ for every β, we obtain that $Tx_\beta \uparrow Tx$. $\square$

2.5 Linear Functionals

Let E be a vector lattice. We recall that the vector space $\mathcal{L}(E,\mathbb{R})$ of all linear functionals on E is an ordered vector space with the ordering

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \text{ for all } x \in E^+.$$

Since $\mathbb{R}$ is a Dedekind complete vector lattice, by Theorem 2.22 we have that the vector space of all order bounded linear functionals $\mathcal{L}_b(E,\mathbb{R})$ is a Dedekind complete vector lattice. We denote $\mathcal{L}_b(E,\mathbb{R})$ by $E^\sim$ and call it the *order dual* of E.

A subset $A \subseteq \mathcal{L}(E,\mathbb{R})$ of functionals is said to *separate the points* of E if for any two different points $x, y \in E$ there is a functional $f \in A$ such that $f(x) \neq g(x)$. Obviously A separates the points of E if and only if for every $x \in E$, $x \neq 0$ implies that there exists a functional $f \in A$ with $f(x) \neq 0$.

Theorem 2.29. *If E is a vector lattice then the following statements hold:*

(a) The order dual $E^\sim$ separates the points of E if and only if for every $0 < x \in E$ there exists $0 < f \in E^\sim$ such that $f(x) \neq 0$;

(b) If $E^\sim$ separates the points of E then for $x \in E$, $x \geq 0$ if and only if $f(x) \geq 0$ for every $0 \leq f \in E^\sim$.

Proof. (a) Suppose that for every $0 < x \in E^+$ there exists $0 < f \in E^\sim$ with $f(x) \neq 0$. Since $0 \neq x \in E$ then at least one of x^+ and x^- is nonzero, say the former. Then there exists $0 \leq f \in E^\sim$ such that $f(x^+) \neq 0$. By Theorem 2.23 there exists $0 \leq g \in E^\sim$ such that $g(x^+) = f(x^+) \neq 0$ and $g(x^-) = 0$. Hence, $g(x) \neq 0$ and $E^\sim$ separates the points of E. Conversely, if $E^\sim$ separates the points of E, then for $x \in E^+$ there exists $f \in E^\sim$ such that $f(x) \neq 0$. To conclude the proof, observe that $|f|(x) \geq |f(x)| > 0$.

(b) If $x \geq 0$ then clearly $f(x) \geq 0$ for every $0 \leq f \in E^\sim$. Conversely, suppose that $f(x) \geq 0$ for every $0 \leq f \in E^\sim$. If $x^- \neq 0$ then as $E^\sim$ separates the points of E, there exists $f \in E^\sim, f \geq 0$ such that $f(-x^-) \neq 0$. Then by Theorem 2.23 there exists $g \in E^\sim, g \geq 0$ such that $g(x^-) = f(x^-) \neq 0$ and $g(x^+) = 0$.

Then $g(x) = g(x^+) - g(x^-) = g(x^-) < 0$ which contradicts our assumption. Therefore $x^- = 0$ and $x \geq 0$.

□

Let E be a vector lattice. The space $\mathcal{L}_n(E, \mathbb{R})$ of all order continuous linear functionals on E is denoted by $E_n^\sim$ and is called the *order continuous dual* of E. The space $\mathcal{L}_c(E, \mathbb{R})$ of all σ-order continuous functionals on E is denoted by $E_c^\sim$ and is called the *σ-order continuous dual* of E. We recall that by Theorem 2.28 both $E_n^\sim$ and $E_c^\sim$ are bands of $E^\sim$.

If $f \in E^\sim$ is an order bounded functional, the set

$$N_f = \{x \in E : |f|(|x|) = 0\}$$

is called the *null ideal* of f.

Lemma 2.30. *If $f \in E^\sim$ then N_f is an ideal.*

Proof. If $x, y \in N_f$ and $c, d \in \mathbb{R}$ then $||f|(cx + dy)| \leq |f|(|cx + dy|) \leq |f|(|c||x| + |d||y|) = |c||f|(|x|) + |d||f|(|y|) = 0$ and so $cx + dy \in N_f$, which implies that N_f is a vector subspace of E. For $x \in E$ and $y \in N_f$ with $|x| \leq |y|$ we have that $|f|(|x|) \leq |f|(|y|) = 0$, from where it follows that $x \in N_f$. □

The set $C_f := N_f^d$ is called the *carrier* of f. From Lemma 2.19 it follows that the carrier of f is always a band. Next we give a characterization of disjointness of two functionals $f, g \in E_n^\sim$ in terms of their null ideals and carriers. It is easy to see that for $f, g \in E^\sim$, we have that $f \perp g$ if and only if $|f| \perp |g|$ and the null ideals and the carriers of f and $|f|$ coincide, since in the definitions of disjointness and a null ideal only the absolute values of all elements are used.

Theorem 2.31 (Nakano)**.** *If E is Archimedean, then for $f, g \in E_n^\sim$ the following statements are equivalent:*

(a) $f \perp g$;

(b) $C_f \subseteq N_g$;

(c) $C_g \subseteq N_f$;

(d) $C_f \perp C_g$.

Proof. Without loss of generality we assume that f and g are positive.

(a) ⇒ (b) Suppose that $f \perp g$ and pick $0 \leq x \in C_f$. We need to show that $g(x) = 0$. Pick $\epsilon > 0$. Since

$$0 = (f \wedge g)(x) = \inf\{f(y) + g(z) : y, z \in E^+, y + z = x\},$$

there exists a sequence $(x_n)_n \subseteq E^+$ such that $x_n \leq x$, $f(x_n) \leq \frac{\epsilon}{2^n}$ and $g(x - x_n) \leq \frac{\epsilon}{2^n}$ for each $n \in \mathbb{N}$.

We claim that the sequence $(y_n)_n$ where $y_n = x_1 \wedge \ldots \wedge x_n$ satisfies $y_n \downarrow 0$. Obviously $y_n \downarrow$. Let $y > 0$ be such that $y \leq y_n$ for all n. Then $f(y) \leq f(y_n) \leq$

$f(x_n) \le \frac{\epsilon}{2^n}$ and hence $f(y) = 0$, so that $y \in N_f$. Since $y \le x$ and C_f is a band we have that $y \in C_f$ which implies that $y \in N_f \cap C_f = \{0\}$.

From $y_n \downarrow 0$ we get that $(x - y_n) \uparrow x$ and as g is order continuous, we have that $g(x - y_n) \uparrow g(x)$. On the other hand, for all $n \in \mathbb{N}$ we have that

$$\begin{aligned} g(x - y_n) &= g(x - (x_1 \wedge \ldots \wedge x_n)) = g((x - x_1) \vee \ldots \vee (x - x_n)) \\ &\le g(x - x_1) + \cdots + g(x - x_n) \le \epsilon \end{aligned}$$

implies that $g(x) \le \epsilon$, and since ϵ was arbitrary, we obtain that $g(x) = 0$.

(b) $\Rightarrow$ (c) Since f is order continuous, by Theorem 2.27 we have that $|f|$ is order continuous. We claim that N_f is order closed. By Lemma 2.17 it is enough to show that $(x_\alpha) \subseteq N_f$ and $0 \le x_\alpha \uparrow x$ imply that $x \in N_f$. Suppose that $(x_\alpha) \subseteq N_f$ and $0 \le x_\alpha \uparrow x$. Then $(x - x_\alpha) \downarrow 0$ and as $|f|$ is order continuous, we have that

$$|f|(x) = |f|(x) - |f|(x_\alpha) = |f|(x - x_\alpha) \downarrow 0.$$

Therefore N_f is a band and by applying Lemma 2.19 we obtain

$$C_g = N_g^d \subseteq C_f^d = N_f^{dd} = N_f.$$

(c) $\Rightarrow$ (d) From $C_f = N_f^d \subseteq C_g^d$ it follows that $C_f \perp C_g$.

(d) $\Rightarrow$ (a) From $C_f \perp C_g$ it follows that $C_g \subseteq C_f^d = N_f^{dd} = N_f$. We first prove that $f \wedge g = 0$ on the order dense ideal $N_g \oplus C_g$ (Lemma 2.18). It suffices to show that $(f \wedge g)(x) = 0$ for every $x \in N_g^+$ and every $x \in C_g^+$. If $x \in N_g^+$ then $0 \le (f \wedge g)(x) \le g(x) = 0$ and if $x \in C_g^+$ then $0 \le (f \wedge g)(x) \le f(x) = 0$. As $f \wedge g$ is order continuous and $N_g \oplus C_g$ is order dense in E, we easily get that $f \wedge g = 0$ on E.

□

2.6 Banach Lattices

We shall now study vector lattices equipped with a norm that is compatible with its order structure. A seminorm (norm) p on a vector lattice E is called a *lattice seminorm (norm)* if $|x| \le |y|$ implies $p(x) \le p(y)$ for all $x, y \in E$. If $\|\cdot\|$ is a lattice norm on E, we call E a *normed vector lattice*, and if E is also norm complete then E is called a *Banach lattice*. We notice that $\||x|\| = \|x\|$ and from Lemma 2.8 we easily get that $\|x^+ - y^+\| \le \|x - y\|$, $\|x^- - y^-\| \le \|x - y\|$ and $\||x| - |y|\| \le \|x - y\|$. These inequalities imply that the lattice operations on E are uniformly continuous. A net (x_α) in E is called *norm Cauchy* if for all $\epsilon > 0$ there exists α_0 such that $\beta, \gamma \ge \alpha_0$ implies $\|x_\beta - x_\gamma\| \le \epsilon$.

If E is a normed vector lattice then we denote the vector space of all norm continuous linear functionals on E by E^*. It turns out that every norm continuous functional on E is also order bounded.

Theorem 2.32. *The norm dual E^* of a normed vector lattice E is an ideal of $E^\sim$.*

Proof. We first prove that $E^* \subseteq E^\sim$. Let $f \in E^*$ and $x, y, z \in E$ such that $x \le y \le z$. We have that $|y| \le z \vee (-x) = |z \vee (-x)|$, which implies that $\|y\| \le \|z \vee (-x)\|$ and hence $|f(y)| \leq \|f\| \, \|y\| \leq \|f\| \, \|z \vee (-x)\|$. Therefore the image of f of the order interval $[x, z]$ is bounded in $\mathbb{R}$.

To show that E^* is an ideal, let $f \in E^\sim$ and $g \in E^*$ be such that $|f| \leq |g|$. Since for each $x \in E$ we have

$$|f(x)| \leq |f|(|x|) \leq |g|(|x|) = \sup_{|y| \leq |x|} |g(y)| \leq \sup_{|y| \leq |x|} \|g\| \, \|y\| \leq \|g\| \, \|x\|$$

we immediately conclude that $f \in E^*$. □

Theorem 2.33. *The norm dual E^* of a normed vector lattice E is a Banach lattice.*

Proof. Since E^* is a Banach space, we only need to show that the norm on the dual space E^* is a lattice norm. Let $|f| \leq |g|$ in E^*. By using the Riesz-Kantorovič formulas we have that

$$|f(x)| \leq |f|(|x|) \leq |g|(|x|) = \sup_{|y| \leq |x|} |g(y)| \leq \sup_{|y| \leq |x|} \|g\| \, \|y\| \leq \|g\| \, \|x\|$$

for all $x \in E$ from where it follows that $\|f\| \leq \|g\|$. □

For a positive linear functional $f \in E^*$ where E is a normed vector lattice we have that $|f(x)| \leq f(|x|)$ which implies that $\|f\| = \sup\{f(x) : x \geq 0 \text{ and } \|x\| = 1\}$. Similarly, for $x \in E^+$ by using the Hahn-Banach theorem and from the fact that $|f|(x) \geq f(x)$ for every $f \in E^*$ we get that $\|x\| = \sup\{f(x) : 0 \leq f \in E^* \text{ and } \|f\| = 1\}$.

Next we prove an interesting and important property of positive operators between Banach lattices.

Theorem 2.34. *A positive operator $T \colon E \to F$ where E is a Banach lattice and F is a normed lattice is continuous.*

Proof. Assume that T is not norm bounded. Then there exists a sequence of vectors $(x_n)_n$ in E such that $\|x_n\| = 1$ and $\|Tx_n\| \geq n^3$. Since the series $\sum_{n=1}^{\infty} \frac{\|x_n\|}{n^2}$ converges, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}|x_n|$ converges in E to some element x. From the continuity of the map $y \to y^-$ on E, it follows that the limit of a sequence of positive elements is positive. From this we easily get that $\frac{1}{n^2}|x_n| \leq x$ for all $n \in \mathbb{N}$. Since T is positive we obtain $\|Tx\| \geq \left\|T(\frac{1}{n^2}|x_n|)\right\| \geq \frac{1}{n^2}\left\|T(x_n)\right\| \geq n$ for all $n \in \mathbb{N}$ which is impossible. Therefore T is continuous. □

An interesting consequence of Theorem 2.34 is that all lattice norms that make a vector lattice a Banach lattice are equivalent. This was proved by Goffman [6].

Corollary 2.35. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two lattice norms on a vector lattice E such that $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are Banach lattices then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

Proof. The identity maps $I_1 \colon (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$ and $I_2 \colon (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$ are positive, and by Theorem 2.34 they are continuous. This implies that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Corollary 2.36. *For a Banach lattice E we have that $E^* = E^\sim$.*

Proof. Since every order bounded functional f is the difference of its positive and negative parts, by Theorem 2.34 we have that f is continuous, and so $E^\sim \subseteq E^*$. By Theorem 2.32 we have that $E^* \subseteq E^\sim$ which implies that $E^* = E^\sim$. □

We end this section with the introduction of the Fatou properties which will be important for the main result of the thesis.

Definition 2.37. Let p be a lattice seminorm on a vector lattice E. We say that p has the:

(a) *strong Fatou property* if $x_\alpha \uparrow$ in E^+ and $\sup_\alpha p(x_\alpha) < \infty$ imply the existence of $x \in E^+$ such that $x_\alpha \uparrow x$ and $p(x_\alpha) \uparrow p(x)$;

(b) *Fatou property* if $x_\alpha \uparrow x$ in E^+ implies that $p(x_\alpha) \uparrow p(x)$;

(c) *weak Fatou property* if $x_\alpha \uparrow x$ in E^+ and $\sup_\alpha p(x_\alpha) < \infty$ imply that $p(x) < \infty$.

If in Definition 2.37 we substitute nets with sequences, we obtain the definition of *σ-(strong/weak) Fatou property*. If p is a lattice seminorm on E and p has the (strong/weak) Fatou property then we also say that the E has the (strong/weak) Fatou property. It is easy to see that the (σ-)strong Fatou property implies the (σ-)Fatou property which implies the (σ-)weak Fatou property.

Theorem 2.38. *If E is a Banach lattice then $E^\sim, E_n^\sim$ and $E_c^\sim$ have the strong Fatou property.*

Proof. Suppose that $0 \leq f_\alpha \uparrow$ in $E^\sim$ and $B = \sup_\alpha \|f\|_\alpha < \infty$. Then the map $f\colon E^+ \to \mathbb{R}^+$ given by

$$f(x) = \sup_\alpha f_\alpha(x)$$

for all $x \in E^+$ is well-defined. To show that f is additive, observe that for any $x, y \in E^+$ by Lemma 2.11 we have that

$$f(x+y) = \sup_\alpha f_\alpha(x+y) = \sup_\alpha (f_\alpha(x)+f_\alpha(y)) = \sup_\alpha f_\alpha(x)+\sup_\alpha f_\alpha(y) = f(x)+f(y).$$

We can therefore extend f uniquely to a positive linear functional on E by Theorem 2.20. For any $x \in E$ we have that

$$\begin{aligned} |f(x)| = |f(x^+) - f(x^-)| &\leq \max(f(x^+), f(x^-)) = \max(\sup_\alpha f_\alpha(x^+), \sup_\alpha f_\alpha(x^-)) \\ &\leq \max(B\left\|x^+\right\|, B\left\|x^-\right\|) \leq B\left\|x\right\| \end{aligned}$$

which implies that f is bounded. Since $f_\alpha(x) \uparrow f(x)$ for all $x \in E^+$ by Theorem 2.22 we easily obtain that $f_\alpha \uparrow f$.

We have already proven that $\|f\| \leq B$. Pick any $\epsilon > 0$ and for any α pick $x \in E$ with $\|x\| = 1$ such that $|f_\alpha(x)| > \|f_\alpha\| - \epsilon$. Then $f(|x|) \geq f_\alpha(|x|) \geq |f_\alpha(x)| > \|f_\alpha\| - \epsilon$. By taking supremum over α we get that $\|f\| \geq B - \epsilon$ which implies that $\|f\| \geq B$ since ϵ was arbitrary.

We have proven that $E^\sim$ has the strong Fatou property. Since $E_n^\sim$ and $E_c^\sim$ are bands of $E^\sim$ (Theorem 2.28) we easily get that they have the strong Fatou property. □

2.7 Banach Lattices with Order Continuous Norms

The last section of this chapter is a brief introduction to order continuous norms and order continuous Banach lattices. Order continuity of norms is similar to order continuity of linear operators.

Definition 2.39. A lattice seminorm p on a vector lattice E is called:

(a) *order continuous* whenever $x_\alpha \downarrow 0$ implies $p(x_\alpha) \downarrow 0$;

(b) *σ-order continuous* whenever $x_n \downarrow 0$ implies $p(x_n) \downarrow 0$.

If p is a (σ-)order continuous norm on E then we call E (σ-)order continuous.

Theorem 2.40. *For a Banach lattice $(E, \|\cdot\|)$, the following statements are equivalent:*

(a) $\|\cdot\|$ is order continuous;

(b) $0 \leq x_n \uparrow \leq x$ in E implies that $(x_n)_n$ is norm convergent;

(c) E is σ-Dedekind complete and $\|\cdot\|$ is σ-order continuous.

Proof. (a) $\Rightarrow$ (b) Suppose that $0 \leq x_n \uparrow \leq x$ in E, and let $\epsilon > 0$. Pick some $z \in E^+$ such that $\|z\| \leq \frac{\epsilon}{2}$. By Lemma 2.15 there exists $y \in E$ and $n_0 \in \mathbb{N}$ such that for all $m \geq n_0$ we have $0 \leq y - x_m \leq z$ which implies that $\|y - x_m\| \leq \|z\| \leq \frac{\epsilon}{2}$. If $m, l \geq n_0$ then $\|x_m - x_l\| \leq \|x_m - y\| + \|y - x_l\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which shows that $(x_n)_n$ is norm Cauchy and therefore convergent.

(b) $\Rightarrow$ (c) Suppose that $0 \leq x_n \uparrow \leq x$ in E. By (b), the sequence (x_n) is norm Cauchy which implies the existence of $y \in E$ such that $x_n \to y$ in norm. Since $y - x_n \downarrow$ and $\|y - x_n\| \to 0$, we easily get that $y \geq x_n$ for all $n \in \mathbb{N}$. Indeed, if $y \not\geq x_m$ for some m, then from $y - x_n \leq y - x_m$ for all $n \geq m$ we have that $|y - x_n| \geq (y - x_n)^- \geq (y - x_m)^-$ for all $n \geq m$. It follows that $\|y - x_n\| \geq \|(y - x_m)^-\| > 0$ for all $n \geq m$, which is a contradiction. If $z \in E$ is such that $y \geq z \geq x_n$ for all $n \in \mathbb{N}$, then $\|z - x_n\| \leq \|y - x_n\| \to 0$, and so $z = y$. This proves that $x_n \uparrow y$, and so E is σ-Dedekind complete.

Suppose that $x_n \downarrow 0$ in E. Then $(x_1 - x_n) \uparrow x_1$ and so $(x_1 - x_n)_{n \in \mathbb{N}}$ is norm Cauchy which implies that $(x_n)_n$ is also norm Cauchy. Let y be the norm limit of $(x_n)_n$. Suppose that $0 < z \leq |y - x_n|$ for some $z \in E$ and all $n \in \mathbb{N}$. Then $\|z\| \leq \|y - x_n\| \downarrow 0$ and so $z = 0$. This implies that $\inf_n |y - x_n| = 0$. Now we have that $|y| \leq |y - x_n| + x_n$ for all $n \in \mathbb{N}$ from which, by using a similar argument as in the proof of Lemma 2.11, we get that $|y| \leq \inf_n(|y - x_n| + x_n) = \inf_n(|y - x_n|) + \inf_n x_n = 0$. Therefore $\|x_n\| \downarrow 0$.

(c) $\Rightarrow$ (a) Suppose that $x_\alpha \downarrow 0$ in E. If (x_α) is not norm Cauchy, then there exists $\epsilon > 0$ and an increasing sequence of indices (α_n) such that $\left\|x_{\alpha_n} - x_{\alpha_{n+1}}\right\| \geq \epsilon$ for all $n \in \mathbb{N}$. As $x_{\alpha_n} \downarrow$ and E is σ-Dedekind complete, there exists $x \in E^+$ such that $x_{\alpha_n} \downarrow x$. Then $x_{\alpha_n} - x \downarrow 0$ and by the hypothesis $\|x_{\alpha_n} - x\| \downarrow 0$ which is in contradiction with (x_{α_n}) not being a norm Cauchy sequence. Therefore (x_α) is norm convergent to some $y \in E$ and similarly as before we can show that $y = 0$. This proves that E has order continuous norm. $\square$

3 Normed Function Spaces

In this chapter we study normed function spaces which are normed spaces consisting of measurable functions. The case of sequence spaces (i.e. when the underlying measure is the counting measure on a countably infinite set) has been studied by G.Köthe and O.Toeplitz, and for this reason the normed spaces that are introduced are sometimes called normed Köthe spaces. We introduce the Riesz-Fischer property, saturated function seminorms, and the associate space to a normed function space the latter of which is involved in the main result of the thesis. We mostly follow the treatment in [8, Chapter 15] by Zaanen.

3.1 Normed and Banach Function Spaces; Function Norms and Seminorms

Let (Ω, Σ, μ) be a measure space, and let M be the set of all real valued measurable functions on Ω (we identify two functions that are equal almost everywhere). If we equip M with the ordering $\leq$ defined by $f \leq g$ if and only if $f(x) \leq g(x)$ for almost all $x \in \Omega$, then by Example 2.3 we have that M is a vector lattice. Note that the lattice operations $\vee, \wedge$ and $|\cdot|$ are given by the pointwise maximum, the pointwise minimum, and the usual pointwise absolute value of a function, respectively. We now define the notions of a function seminorm and a function norm on M.

Definition 3.1. A *function seminorm* is a mapping $\rho\colon M^+ \to [0, \infty]$ that satisfies the following conditions for all $u, v \in M^+$:

(a) if $u = 0$ then $\rho(u) = 0$;

(b) $\rho(cu) = c\rho(u)$ for all $c \in \mathbb{R}^+$;

(c) $\rho(u + v) \leq \rho(u) + \rho(v)$;

(d) if $u \leq v$ then $\rho(u) \leq \rho(v)$.

If, furthermore, $\rho(u) = 0$ implies that $u = 0$ then ρ is called a *function norm*.

We are mainly interested in function norms. Given a function norm ρ, we can extend the domain of ρ to all of M by defining $\rho(f) = \rho(|f|)$ for every $f \in M$. We define $L_\rho = \{f \in M : \rho(f) < \infty\}$.

Lemma 3.2. *The set L_ρ is an ideal of M and ρ is a norm on L_ρ.*

Proof. Since ρ is monotone and satisfies the triangle inequality we get that L_ρ is closed under addition and from the positive homogenuity of ρ we get that L_ρ is closed under multiplication with scalars. The fact that L_ρ is an ideal follows immediately from the monotonicity of ρ.

We have that ρ satisfies $\rho(f) = 0$ if and only if $f = 0$, $\rho(|cf|) = |c|\rho(|f|)$ and $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in M$ and $c \in \mathbb{R}$. □

We proceed with very well-known examples of function spaces.

Example 3.3. Let $1 \le p < \infty$ and define the function norm ρ by

$$\rho(f) = (\int_\Omega |f|^p \, d\mu)^{\frac{1}{p}}$$

for all $f \in M$. Then we have that $L_\rho = L^p(\mu)$. For the case $p = \infty$ we define $\rho(f) = \operatorname{ess\,sup} f$ for all $f \in M^+$ and observe that $L_\rho = L^\infty(\mu)$.

Normed spaces L_ρ are usually called *normed function spaces*. They need not be norm complete as the following example shows.

Example 3.4. Let $\Omega = \mathbb{N}, \Sigma = \mathcal{P}(\mathbb{N})$ and μ be the counting measure on (Ω, Σ). For all $f \in M^+$ we define

$$\rho(f) = \limsup_{n\to\infty} f(n) + \sum_{n=1}^{\infty} \frac{f(n)}{2^n}.$$

We will show that the sequence $(f_k)_k$ defined by $f_k = (\underbrace{1, 1, \dots, 1}_{k}, 0, 0, \dots)$ is Cauchy which does not converge. For $m > n$ we have that

$$f_m - f_n = (\underbrace{0, 0, \dots, 0}_{n}, \underbrace{1, 1, \dots, 1}_{m-n}, 0, 0, \dots),$$

and so $\rho(f_m - f_n) < \frac{1}{2^n}$ which implies that $(f_k)_k$ is Cauchy. If $(f_k)_k$ converges in norm to some $f \in M$ then for all $n \in \mathbb{N}$ we have that $\frac{1}{2^n}|f(n) - f_k(n)| \le \rho(f - f_k) \to 0$ as $k \to \infty$ from where it follows that $f(n) = 1$ for all $n \in \mathbb{N}$. Then $\rho(f - f_k) \ge \limsup_{n\to\infty}(|f(n) - f_k(n)|) = 1$ which implies that $(f_k)_k$ does not converge to f and hence is not convergent.

We finish this section with some general results which will be used later.

Lemma 3.5. *If $(\rho_\tau)_\tau$ is a family of seminorms on M^+ then the map $\rho\colon M^+ \to \mathbb{R}^+$ defined by*

$$\rho(f) = \sup_\tau \rho_\tau(f)$$

is a seminorm as well.

Proof. As $\rho_\tau(0) = 0$ for all τ we have that $\rho(0) = 0$. Positive homogenuity and monotonicity of ρ follow immediately from the positive homogenuity and monotonicity of the seminorms $\{\rho_\tau\}$. If $f, g \in M^+$ then

$$\begin{aligned}\rho(f + g) &= \sup_\tau \rho_\tau(f + g) \le \sup_\tau(\rho_\tau(f) + \rho_\tau(g)) \le \sup_\tau \rho_\tau(f) + \sup_\tau \rho_\tau(g) \\ &= \rho(f) + \rho(g).\end{aligned}$$

□

Lemma 3.6. *For any net $(f_\alpha)_\alpha \in M^+$ and function $g \in M^+$ the following statements hold:*

(a) If $f_\alpha \uparrow f$ in M^+ then $f_\alpha g \uparrow fg$ in M^+;

(b) If $f_\alpha \downarrow 0$ in M^+ then $f_\alpha g \downarrow 0$ in M^+.

Proof. (a) Suppose that $f_\alpha \uparrow f$ in M^+. It is easy to see that $f_\alpha g \uparrow \leq fg$. Suppose that $h \geq f_\alpha g$ for all α. Then for every α we have that $h/g \geq f_\alpha$ on $\mathrm{supp}(g)$. Let $h' \in M^+$ equal h/g on $\mathrm{supp}(g)$ and f elsewhere. Then for all α we have that $h' \geq f_\alpha$ and since $f_\alpha \uparrow f$ we have that $h' \geq f$. Therefore $h/g \geq f$ on $\mathrm{supp}(g)$ and so $h \geq fg$ on $\mathrm{supp}(g)$. Since $fg = 0$ outside $\mathrm{supp}(g)$ we have that $h \geq fg$ on Ω.

(b) Suppose that $f_\alpha \downarrow 0$ in M^+. Pick an index α_0 and note that $f_{\alpha_0} - f_\alpha \uparrow_{\alpha \geq \alpha_0} f_{\alpha_0}$. By (a) we have that $f_{\alpha_0} g - f_\alpha g \uparrow_{\alpha \geq \alpha_0} f_{\alpha_0} g$ and this easily implies that $f_\alpha g \downarrow_{\alpha \geq \alpha_0} 0$.

□

Lemma 3.7. *The space $L^\infty(\mu)$ is a sublattice of M.*

Proof. If $f \in L^\infty(\mu)$ is arbitrary then f is finite almost everywhere which implies that there is a real measurable function g with $g = f$ almost everywhere. This shows that $L^\infty(\mu)$ is a subset of M. We define the partial order on $L^\infty(\mu)$ to be the partial order on M restricted to $L^\infty(\mu)$. It is obvious that $L^\infty(\mu)$ is an ordered vector space and if $f, g \in L^\infty(\mu)$ then it is easy to check that $f \vee g$ and $f \wedge g$ belong to $L^\infty(\mu)$. □

3.2 The Riesz-Fischer Property

We recall that Example 3.4 provides a normed function space that is not norm complete. The norm completeness of a normed function space is equivalent to the Riesz-Fischer property which is simpler to check.

Definition 3.8. The function norm ρ is said to have the *Riesz-Fischer* property if for all sequences $(f_n)_n \subseteq (L_\rho)^+$ such that $\sum\limits_{n=1}^{\infty} \rho(f_n) < \infty$ we have that $\sum\limits_{n=1}^{\infty} f_n \in L_\rho$.

Remark 3.9. The convergence of the series $\sum_{n=1}^\infty f_n$ that appears in Definition 3.8 is pointwise almost everywhere.

Theorem 3.10. *If the function norm ρ has the Riesz-Fischer property then for all sequences $(f_n)_n \subseteq (L_\rho)^+$ such that $\sum\limits_{n=1}^{\infty} \rho(f_n) < \infty$ we have that $\rho(\sum\limits_{n=1}^{\infty} f_n) \leq \sum\limits_{n=1}^{\infty} \rho(f_n)$.*

Proof. Suppose that ρ has the Riesz-Fischer property but that there exists a sequence $(f_n)_n \subseteq (L_\rho)^+$ such that $\sum\limits_{n=1}^{\infty} \rho(f_n) < \infty$ and $\rho(\sum\limits_{n=1}^{\infty} f_n) > \sum\limits_{n=1}^{\infty} \rho(f_n)$. Then there exists $\epsilon > 0$ such that $\rho(\sum\limits_{n=1}^{\infty} f_n) > \epsilon + \sum\limits_{n=1}^{\infty} \rho(f_n)$. By multiplying (f_n) with $\frac{k}{\epsilon}$ for every $k \in \mathbb{N}$, we obtain a sequence $(f_{k,n})_n$ for every $k \in \mathbb{N}$ such that $\rho(\sum\limits_{n=1}^{\infty} f_{k,n}) >$

$k + \sum_{n=1}^{\infty} \rho(f_{k,n})$. For all $k, r \in \mathbb{N}$ by the triangle inequality we have

$$\rho(\sum_{n=r+1}^{\infty} f_{k,n}) \geq \rho(\sum_{n=1}^{\infty} f_{k,n}) - \rho(\sum_{n=1}^{r} f_{k,n}) \geq \rho(\sum_{n=1}^{\infty} f_{k,n}) - \sum_{n=1}^{r} \rho(f_{k,n})$$
$$\geq k + \sum_{n=1}^{\infty} \rho(f_{k,n}) - \sum_{n=1}^{r} \rho(f_{k,n}) = k + \sum_{n=r+1}^{\infty} \rho(f_{k,n}).$$

For every $k \in \mathbb{N}$ we choose $r = r_k$ such that $\sum_{n=r+1}^{\infty} \rho(f_{k,n}) \leq \frac{1}{k^2}$ and thus by reindexing we obtain a sequence $(f_{n,k})_n$ for every $k \in \mathbb{N}$ such that $\rho(\sum_{n=1}^{\infty} f_{k,n}) > k$ and $\sum_{n=1}^{\infty} \rho(f_{k,n}) \leq \frac{1}{k^2}$. If $(g_m)_m$ is the sequence of all functions $f_{n,k}$ where $n, k \in \mathbb{N}$ then $\sum_{m=1}^{\infty} \rho(g_m) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ and so $\sum_{m=1}^{\infty} g_m \in L_\rho$ by the Riesz-Fischer property. On the other hand, we also have that $\rho(\sum_{m=1}^{\infty} g_m) \geq \rho(\sum_{n=1}^{\infty} f_{k,n}) > k$ for all $k \in \mathbb{N}$ which is impossible. □

Theorem 3.11. *The normed vector lattice L_ρ is norm complete if and only if ρ has the Riesz-Fischer property.*

Proof. Assume that ρ has the Riesz-Fischer property and choose a norm Cauchy sequence $(f_n)_n \subseteq L_\rho$. Since $\rho(f_m - f_n) \to 0$ when $m, n \to \infty$, there exists a subsequence $(f_{n_k})_k$ of (f_n) such that $\rho(f_{n_{k+1}} - f_{n_k}) \leq \frac{1}{2^k}$ for all $k \in \mathbb{N}$. Since

$$\sum_{k=1}^{\infty} \rho(f_{n_{k+1}} - f_{n_k}) \leq 1,$$

by the Riesz-Fischer property we have that the function

$$g = |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$$

belongs to L_ρ. In particular, g is finite almost everywhere, which implies that the series $f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ converges pointwise almost everywhere to a function f that is finite almost everywhere. From

$$f - f_{n_k} = (f_{n_1} + \sum_{m=1}^{\infty}(f_{n_{m+1}} - f_{n_m})) - (f_{n_1} + \sum_{m=1}^{k-1}(f_{n_{m+1}} - f_{n_m})) = \sum_{m=k}^{\infty}(f_{n_{m+1}} - f_{n_m})$$

and Theorem 3.10 we get that $\rho(f - f_{n_k}) \leq \sum_{m=k}^{\infty} \rho(f_{n_{m+1}} - f_{n_m}) \to 0$ as $k \to \infty$, which implies that $(f_{n_k})_k$ converges to f with respect to the norm ρ. Since $(f_n)_n$ is Cauchy, we have that $(f_n)_n$ converges to f too.

Assume now that L_ρ is norm complete and suppose that $(f_n)_n \subseteq L_\rho^+$ is such that $\sum\limits_{n=1}^{\infty} \rho(f_n) < \infty$. If $g_n = f_1 + \cdots + f_n$ for all $n \in \mathbb{N}$ then (g_n) is norm Cauchy and so by the assumption it converges to some $f \in L_\rho$. From $f^- \leq |f - g_n|$ for all $n \in \mathbb{N}$ we get that $\rho(f^-) \leq \rho(f - g_n) \to 0$ as $n \to \infty$ and so $f^- = 0$. Also, from $|g_k - f \wedge g_k| = |g_n \wedge g_k - f \wedge g_k| \leq |g_n - f|$ for all $k, n \in \mathbb{N}, n \geq k$, we get that $\rho(g_k - f \wedge g_k) \leq \rho(g_n - f) \to 0$ as $n \to \infty$, and so $g_k = f \wedge g_k$ or equivalently $g_k \leq f$ for all $k \in \mathbb{N}$. This shows that $\sum\limits_{n=1}^{\infty} f_n \leq f$ and so $\rho(\sum\limits_{n=1}^{\infty} f_n) \leq \rho(f) < \infty$. □

Remark 3.12. Note that in the second part of the proof of Theorem 3.11 we differentiate between convergence of the series $\sum_{n=1}^{\infty} f_n$ in the norm ρ and convergence of the same series poinwise almost everywhere.

The theorems in this section are due to I.Halperin and W.A.J. Luxemburg.

3.3 Saturated Function Seminorms

Let ρ be a function seminorm on M^+. If either $\rho(f) = 0$ for every $f \in M^+$ or $\rho(f) = \infty$ for every $0 < f \in M^+$ we call ρ a *trivial* function seminorm. Therefore, ρ is nontrivial if and only if there exists a function $0 < f \in M^+$ such that $0 < \rho(f) < \infty$. If ρ is nontrivial there may still exist measurable sets $E \subseteq \Omega$ with $\mu(E) > 0$ such that not only $\rho(\chi_E) = \infty$ but also $\rho(\chi_F) = \infty$ for every subset $F \subseteq E$ with $\mu(F) > 0$. A subset $E \subseteq \Omega$ that has this property is called a *ρ-purely infinite set*. Clearly, finite and countable unions of ρ-purely infinite sets are again ρ-purely infinite.

Proposition 3.13. *A subset $E \subseteq \Omega$ with $\mu(E) > 0$ is ρ-purely infinite if and only if all $f \in M^+$ with $\rho(f) < \infty$ vanish on E almost everywhere.*

Proof. Suppose that $E \subseteq \Omega$ is ρ-purely infinite and suppose that $f \in M^+$ satisfies $\rho(f) < \infty$. If f does not vanish on E almost everywhere, then there exists an $\epsilon > 0$ and $F \subseteq E$ with $\mu(F) > 0$ such that $f \geq \epsilon\chi_F$. Consequently $\rho(f) \geq \epsilon\rho(\chi_F) = \infty$ which is a contradiction.

For the proof of the converse implication, assume that $E \subseteq \Omega$ and $\mu(E) > 0$ and suppose that for all $f \in M^+$ we have that $\rho(f) < \infty$ implies that f vanishes on E almost everywhere. If $F \subseteq E$ and $\mu(F) > 0$ then since χ_F obviously does not vanish on E we have that $\rho(\chi_F) = \infty$. □

If ρ is a function norm then by Proposition 3.13 we have that any $f \in L_\rho$ vanishes almost everywhere on any ρ-purely infinite set. Hence, for the purpose of investigating the space L_ρ we may as well remove any ρ-purely infinite set. We next show that if μ is σ-finite then all ρ-purely infinite sets can be removed in one step. We introduce the relation $\sim$ on Σ by defining $A \sim B$ if and only if $\mu(A \setminus B) = \mu(B \setminus A) = 0$.

Lemma 3.14. *The relation $\sim$ is an equivalence relation on Σ.*

Proof. Let $A, B, C \in \Sigma$. We have $\mu(A \setminus A) = \mu(\emptyset) = 0$ which implies that $A \sim A$. If $A \sim B$ then from the definition of $\sim$ it immediately follows that $B \sim A$. Suppose that $A \sim B$ and $B \sim C$. From $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$ it follows that $\mu(A \setminus C) \leq \mu(A \setminus B) + \mu(B \setminus C) = 0$. Analogously we can prove that $\mu(C \setminus A) = 0$. □

Theorem 3.15. *Let ρ be a function seminorm. If μ is σ-finite then there exists a unique equivalence class of ρ-purely infinite sets Γ such that for any $\Omega_\infty \in \Gamma$ the set $\Omega \setminus \Omega_\infty$ has no ρ-purely infinite subsets. If ρ is nontrivial then $\mu(\Omega \setminus \Omega_\infty) > 0$.*

Proof. First we assume that $\mu(\Omega) < \infty$. We have that

$$a := \sup\{\mu(A) : A \subseteq \Omega \text{ and } A \text{ is } \rho\text{-purely infinite}\} < \infty$$

and so there exists a sequence (A_n) of ρ-purely infinite measurable sets such that $\mu(A_n) \uparrow a$. If $\Omega_\infty := \bigcup_{n=1}^{\infty} A_n$ then Ω_∞ is ρ-purely infinite and $\mu(\Omega_\infty) = a$. If there would exist a ρ-purely infinite set $E \subseteq \Omega \setminus \Omega_\infty$ then $\Omega_\infty \cup E$ would be a ρ-purely infinite set of measure greater than a which is impossible.

Now suppose that μ is σ-finite. Then there exists a sequence (Ω_n) of subsets of Ω of finite measure such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. For each $n \in \mathbb{N}$ there exists a ρ-purely infinite set $A_n \subseteq \Omega_n$ such that $\Omega_n \setminus A_n$ has no ρ-purely infinite subsets. Let $\Omega_\infty := \bigcup_{n=1}^{\infty} A_n$ and note that Ω_∞ is ρ-purely infinite. If $E \subseteq \Omega \setminus \Omega_\infty$ is ρ-purely infinite then $E \cap \Omega_n$ has positive measure for some $n \in \mathbb{N}$ and is therefore a ρ-purely infinite subset of $\Omega_n \setminus A_n$ which is impossible.

Let Γ be the equivalence class of Ω_∞. Then the complement of any set in Γ has no ρ-purely infinite subset. To show that Γ is unique, let Ω'_∞ be another ρ-purely infinite set such that $\Omega \setminus \Omega'_\infty$ has no ρ-purely infinite subsets. Hence $\Omega_\infty \cap (\Omega \setminus \Omega'_\infty)$ is not ρ-purely infinite. Since $\Omega_\infty \cap (\Omega \setminus \Omega'_\infty) \subseteq \Omega_\infty$ and Ω_∞ is ρ-purely infinite we conclude that $0 = \mu(\Omega_\infty \cap (\Omega \setminus \Omega'_\infty)) = \mu(\Omega_\infty \setminus \Omega'_\infty)$. Similarly, we can conclude that $\mu(\Omega'_\infty \setminus \Omega_\infty) = 0$, and therefore $\Omega'_\infty \in \Gamma$.

If ρ is nontrivial then there exists a function $0 < f \in M^+$ such that $0 < \rho(f) < \infty$. Then f is positive on some set of positive measure and by Proposition 3.13 f vanishes on Ω_∞. This implies that $\mu(\Omega \setminus \Omega_\infty) > 0$. □

Definition 3.16. The function seminorm ρ is called *saturated* if there do not exist any ρ-purely infinite sets.

If ρ is a saturated function norm then we say that L_ρ is a saturated normed function space. For a saturated function seminorm ρ it may happen that $\rho(\chi_\Omega) = \infty$. We later prove (Theorem 3.19) that if μ is σ-finite and ρ is saturated then there always exists a sequence of measurable sets (Ω_n) of finite measure with $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$ and $\rho(\chi_{\Omega_n}) < \infty$ for all $n \in \mathbb{N}$.

Definition 3.17. Let (Ω_n) be an increasing (with respect to inclusion) sequence of measurable sets such that $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$. The measurable set $E \subseteq \Omega$ is called $\{\Omega_n\}$*-bounded* if $E \setminus \Omega_n$ has measure 0 for some $n \in \mathbb{N}$.

In the following lemma we denote by (P) any property that a measurable set may or may not possess. We assume that all sets of measure 0 simultaneously possess (P) or do not possess (P).

Lemma 3.18 (Exaustion lemma)**.** *Let $\mu \neq 0$ and (Ψ_n) be an increasing sequence of sets such that $\Omega = \bigcup\limits_{n=1}^{\infty} \Psi_n$ and $\mu(\Psi_n) < \infty$ for all $n \in \mathbb{N}$. Let (P) be a property that a measurable set may or may not possess. Assume furthermore that:*

(a) if A and B possess (P) then $A \cup B$ possesses (P);

(b) if A possesses (P) then any measurable subset of A possesses (P);

(c) any $\{\Psi_n\}$-bounded set of positive measure has a subset of positive measure possessing (P).

Then there exists an increasing sequence of measurable sets (Ω_n) such that $\Omega = \bigcup\limits_{n=1}^{\infty} \Omega_n$, $\Omega_n \subseteq \Psi_n$ for all $n \in \mathbb{N}$ and every $\{\Omega_n\}$-bounded set has property (P).

Proof. Let E be a $\{\Psi_n\}$-bounded set. If

$$a = \sup\{\mu(A) : A \subseteq E, A \text{ has property } (P)\}$$

then there exists a sequence $(A_n) \subseteq E$ such that A_n has property (P) for all $n \in \mathbb{N}$ and $\mu(A_n) \uparrow a$. Due to (a), the sequence (A_n) can be assumed to be increasing. We define $A := \bigcup_{n=1}^{\infty} A_n$ and since (A_n) is increasing we conclude that $\mu(A) = a$. If $a < \mu(E)$ then $\mu(E \setminus A) > 0$ and by (c) there exists a set $B \subseteq E \setminus A$ with property (P) such that $\mu(B) > 0$. Then we have that $A_n \cup B$ has property (P) and $\mu(A_n \cup B) > a$ for n large enough which contradicts the definition of a. This proves that $\mu(E) = a$. Therefore, for each $n \in \mathbb{N}$ there exists $\Omega_n \subseteq \Psi_n$ such that Ω_n has property (P) and $\mu(\Omega_n) > \mu(\Psi_n) - \frac{1}{n}$. Due to (a) it can be assumed that the sequence (Ω_n) is increasing. If $\Omega' := \bigcup_{n=1}^{\infty} \Omega_n$ then

$$\mu(\Omega \setminus \Omega') = \mu(\bigcup_{n=1}^{\infty}(\Psi_n \setminus \Omega')) = \lim_{n\to\infty} \mu(\Psi_n \setminus \Omega') \leq \lim_{n\to\infty} \mu(\Psi_n \setminus \Omega_n) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

We replace Ω_n by $\Omega_n \cup (\Psi_n \cap (\Omega \setminus \Omega'))$ for every $n \in \mathbb{N}$. It is easy to see that $(\Omega_n)_n$ is increasing, $\Omega_n \subseteq \Psi_n$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. From $\mu \neq 0$, (c) and (b) it follows that all measurable sets of measure 0 have property (P). This implies that Ω_n has property (P) for all $n \in \mathbb{N}$.

If A is a $\{\Omega_n\}$-bounded set then as $A \subseteq \Omega_n$ except for a μ-null set for some $n \in \mathbb{N}$ and Ω_n has property (P), it follows that A has property (P). Therefore (Ω_n) has the desired properties. □

Theorem 3.19. *If μ is σ-finite and ρ is a function seminorm then the following statements are equivalent:*

(a) ρ is saturated;

(b) For any increasing sequence (Ψ_n) of measurable sets such that $\Omega = \bigcup_{n=1}^{\infty} \Psi_n$ there exists an increasing sequence (Ω_n) such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, $\Omega_n \subseteq \Psi_n$ and $\rho(\chi_{\Omega_n}) < \infty$ for all $n \in \mathbb{N}$.

Proof. (a) ⇒ (b) Let (Φ_n) be an increasing sequence of sets of finite measure such that $\Omega = \bigcup_{n=1}^{\infty} \Phi_n$. By substituting $\Psi_n \cap \Phi_n$ for Ψ_n for all $n \in \mathbb{N}$ (if necessary) we can assume that Ψ_n has finite measure for all $n \in \mathbb{N}$. Let (P) be the property $\rho(\chi_E) < \infty$ of a measurable set E. Since ρ is saturated and the sequence (Ψ_n) and property (P) satisfy the assumptions of Lemma 3.18, we can apply it to immediately obtain (b).

(b) ⇒ (a) By choosing $\Psi_n = \Omega$ for all n we obtain a sequence (Ω_n) such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and $\rho(\chi_{\Omega_n}) < \infty$ for all $n \in \mathbb{N}$. To see that ρ is saturated pick a measurable set $A \subseteq \Omega$ such that $\rho(\chi_A) > 0$. Then $\mu(A) > 0$ and there exists $n \in \mathbb{N}$ such that $\mu(A \cap \Omega_n) > 0$. Then $\rho(\chi_{A\cap\Omega_n}) \leq \rho(\chi_{\Omega_n}) < \infty$ so that A is not ρ-purely infinite. □

3.4 The Fatou Properties for Function Seminorms

Here we will investigate the Fatou properties (Definition 2.37) in the context of function spaces.

Proposition 3.20. *If the function norm ρ has the σ-weak Fatou property then ρ has the Riesz-Fischer property.*

Proof. Suppose that $(f_n)_n \subseteq M^+$ and $\sum_{n=1}^{\infty} \rho(f_n) < \infty$. If $g_n = f_1 + \cdots + f_n$ for all $n \in \mathbb{N}$ then

$$\sup_n \rho(g_n) \leq \sup_n(\rho(f_1) + \cdots + \rho(f_n)) = \sum_{n=1}^{\infty} \rho(f_n) < \infty.$$

We now obtain that $\rho(\sum_{n=1}^{\infty} f_n) = \rho(\sup_n g_n) < \infty$ by the σ-weak Fatou property of ρ. □

Proposition 3.21. *Let $(\rho_\tau)_\tau$ be a nonempty family of function seminorms and let $\rho = \sup_\tau \rho_\tau$. We have that:*

(a) if ρ_τ has the strong Fatou property for all τ then ρ has the strong Fatou property;

(b) if ρ_τ has the Fatou property for all τ then ρ has the Fatou property;

(c) if ρ_τ has the σ-Fatou property for all τ then ρ has the σ-Fatou property.

Proof. By Lemma 3.5 ρ is a seminorm. Suppose that ρ_τ has the strong Fatou property for every τ and let $0 \leq f_\alpha \uparrow$ in M be such that $\sup_\alpha \rho(f_\alpha) < \infty$. If τ is some index then $\sup_\alpha \rho_\tau(f_\alpha) \leq \sup_\alpha \rho(f_\alpha) < \infty$ and by the strong Fatou property of ρ_τ there exists $f \in M^+$ such that $f_\alpha \uparrow f$ (due to the uniqueness of the supremum of a net, f is independent of τ). Let $c < \rho(f)$ be arbitrary. By the definition of ρ, there exists τ_0 such that $\rho_{\tau_0}(f) > c$ and by the strong Fatou property of ρ_{τ_0} there exists α_0 such that $\rho_{\tau_0}(f_{\alpha_0}) > c$. Then $\rho(f_{\alpha_0}) > \rho_{\tau_0}(f_{\alpha_0}) > c$ and so $\rho(f_\alpha) \uparrow \rho(f)$. The proof for the ($\sigma$-)Fatou property is similar. □

The following example illustrates the use of Proposition 3.21.

Example 3.22. Let μ be the Lebesgue measure on $\Omega = [0,1]$. For each $h \in (0,1]$ define

$$\rho_h(f) = \frac{1}{h}\int_0^h f\,d\mu$$

for all $f \in M^+$. Obviously ρ_h is a function seminorm and by the monotone convergence theorem ρ_h has the σ-Fatou property for all $h \in (0,1]$. Then by Proposition 3.21 $\rho = \sup_h \rho_h$ has the σ-Fatou property, and consequently by Proposition 3.20 ρ has the Riesz-Fischer property. Since ρ_1 is a norm, ρ is a norm. Then by Theorem 3.11 we have that L_ρ is a Banach lattice.

3.5 Normed Function Spaces over a Localizable Measure

In this section we prove several results that relate the structure of M to the properties of the measure μ. Most of the results can be found in [4, Theorem 64B, Lemma 64C].

Definition 3.23. The measure μ on a measurable space (Ω, Σ) is called:

(a) *semi-finite* if for every measurable set $A \subseteq \Omega$ of positive measure there exists a measurable subset $B \subseteq A$ of positive finite measure;

(b) *localizable* if μ is semi-finite and for any family $\Delta \subseteq \Sigma$ there is a measurable set $C \subseteq \Omega$ such that:

  (i) $\mu(A \setminus C) = 0$ for all $A \in \Delta$;

  (ii) if C' is a measurable set such that $\mu(A \setminus C') = 0$ for all $A \in \Delta$ then $\mu(C \setminus C') = 0$.

Theorem 3.24. *If μ is localizable then $L^\infty(\mu)$ is Dedekind complete.*

Proof. Suppose that μ is localizable. Let $0 \leq f_\alpha \uparrow$ be bounded in $L^\infty(\mu)$ and put $a = \sup_\alpha \|f_\alpha\|_\infty$. We put $c_n = 2^{-n}a$ for all $n \in \mathbb{N}_0$ and we define an increasing bounded sequence $(g_n)_n \subseteq L^\infty(\mu)$ inductively as follows. Put $g_0 = 0$ and, having defined g_n for some $n \in \mathbb{N}$, set

$$A_n^\alpha = \{x \in \Omega : f_\alpha(x) - g_n(x) \geq c_{n+1}\}$$

for every α. Since μ is localizable, there exists a set A_n such that $\mu(A_n^\alpha \setminus A_n) = 0$ for all α and if A_n' is a measurable set such that $\mu(A_n^\alpha \setminus A_n') = 0$ for all α then $\mu(A_n \setminus A_n') = 0$. Set

$$g_{n+1} = g_n + c_{n+1}\chi_{A_n}.$$

Obviously $(g_n)_n$ is increasing and bounded. We prove by induction on n that $g_n \leq u$ for every upper bound $u \in L^\infty(\mu)$ of (f_α). For $n = 0$ the claim is obvious. Suppose that the claim holds for some n, that is, $g_n \leq u$ for all upper bounds $u \in L^\infty(\mu)$ of (f_α), and assume that it does not hold for $n+1$. Then there is an upper bound u of (f_α) such that $g_{n+1} > u$ on some measurable set B with $\mu(B) > 0$. From the fact that $g_{n+1} = g_n + c_{n+1}\chi_{A_n}$ we have that $g_{n+1} = g_n$ outside A_n. As $g_{n+1} > u \geq g_n$ on B it follows that that $\mu(B \setminus A_n) = 0$. We claim that $\mu(B \cap A_n^\alpha) > 0$ for some α. Indeed, if $\mu(B \cap A_n^\alpha) = 0$ for all α then by putting $A_n' = A_n \setminus B$ from the fact that $\mu(A_n^\alpha \setminus A_n) = 0$ for all α we conclude that $\mu(A_n^\alpha \setminus A'_n) = 0$ for all α. Then from the definition of A_n it follows that $\mu(A_n \setminus A'_n) = 0$ which is impossible since $A'_n = A_n \setminus B$, $\mu(B \setminus A_n) = 0$ and $\mu(B) > 0$. Therefore $\mu(B \cap A_n^{\alpha_0}) > 0$ for some index α_0. Observe that by the definition of $A_n^{\alpha_0}$ we have that

$$f_\alpha \geq g_n + c_{n+1}\chi_{A_n} = g_{n+1} > u$$

on $A_n^{\alpha_0} \cap B$ which contradicts the fact that u is an upper bound of (f_α).

Therefore $g = \sup_n g_n$ satisfies $g \leq u$ for all upper bounds u of (f_α). We prove by induction on n that $g_n \geq f_\alpha - c_n$ for all α and $n \in \mathbb{N}$. For $n = 0$ the claim is obvious since $c_0 = a$ is an upper bound of f_α. Suppose that the claim holds for some $n \in \mathbb{N}$. Then

$$g_{n+1} = g_n + c_{n+1}\chi_{A_n} \geq f_\alpha - c_n + c_{n+1}\chi_{A_n} = f_\alpha - c_{n+1}\chi_{A_n} - c_n\chi_{\Omega\setminus A_n}$$

for all α. Therefore $g_{n+1} \geq f_\alpha - c_{n+1}\chi_{A_n}$ on A_n. By the definition of A_n we have that (neglecting sets of measure 0) $f_\alpha - g_n < c_{n+1}$ outside A_n which implies that $g_{n+1} \geq g_n > f_\alpha - c_{n+1}$ outside A_{n+1}. Therefore $g_n \geq f_\alpha - c_n$ on Ω for all α and $n \in \mathbb{N}$. As $c_n \to 0$ as $n \to \infty$ we obtain that g is an upper bound of (f_α). Since g is the smallest upper bound of (f_α) we conclude that $\sup_\alpha f_\alpha = g$. □

Theorem 3.25. *If μ is localizable then M is Dedekind complete.*

Proof. Suppose that μ is localizable and assume that $0 \leq f_\alpha \uparrow \leq f$ in M. Since by Theorem 3.24 $L^\infty(\mu)$ is Dedekind complete we can define $h_n = \sup_\alpha f_\alpha \wedge n\mathbf{1}$ for all $n \in \mathbb{N}$. Due to $h_n \leq f$ for all $n \in \mathbb{N}$ the function $h = \sup_n h_n$ exists in M. If u is an upper bound of (f_α) then $h_n \leq u$ for all n and consequently $h \leq u$. For each α we have that $h = \sup_n h_n \geq \sup_n(f_\alpha \wedge n\mathbf{1}) = f_\alpha$. Hence $f_\alpha \uparrow h$. □

We conclude this section with a lemma which will be crucial in the proof of Theorem 4.6.

Lemma 3.26. *Let μ be a localizable measure. If $f_\alpha \uparrow$ in M^+ then either $(f_\alpha)_\alpha$ is bounded in M or there exists $0 < u \in L^\infty(\mu)$ such that $ku = \sup_\alpha(ku \wedge f_\alpha)$ for all $k \in \mathbb{N}$.*

Proof. Assume that $f_\alpha \uparrow$ in M^+ and that $(f_\alpha)_\alpha$ is unbounded. From the Dedekind completeness of M (Theorem 3.25) it follows that $g_n = \sup_\alpha n\mathbf{1} \wedge f_\alpha$ exists in M for all $n \in \mathbb{N}$. We claim that $(g_n)_n$ is unbounded. Indeed, if $(g_n)_n$ is bounded by $g \in M^+$ then $f_\alpha = \sup_n(n \wedge f_\alpha) \leq \sup_n g_n \leq g$ for every α which contradicts the fact that $(f_\alpha)_\alpha$ is unbounded. By choosing some representatives h_n of the equivalence classes of measurable functions g_n for all $n \in \mathbb{N}$ we define the measurable set

$$A = \{x \in \Omega : \sup_n h_n(x) = \infty\}$$

and observe that $\mu(A) > 0$. We prove that $g_n = n$ almost everywhere on A for all $n \in \mathbb{N}$. Assume that $g_n < n$ on some measurable set $B \subseteq A$ with $\mu(B) > 0$. This implies that $f_\alpha < n$ on B for all α and hence $g_m < n$ on B for all $m \geq n$ which contradicts the fact that $(g_m)_m$ increases to ∞ on $B \subseteq A$. Therefore $g_n = n$ almost everywhere on A for all $n \in \mathbb{N}$. Put $u = \chi_A$ and note that by Lemma 3.6 we have that

$$\sup_\alpha(ku \wedge f_\alpha) = \sup_\alpha(k\chi_A \wedge f_\alpha) = \sup_\alpha(k \wedge f_\alpha)\chi_A = (\sup_\alpha(k \wedge f_\alpha))\chi_A = g_k\chi_A = k\chi_A$$
$$= ku. \qquad \square$$

3.6 Associate Function Seminorms

In this and the next section we define the associate function seminorm and associate space to a normed function space.

Definition 3.27. For a function seminorm ρ we define the sequence $\rho^{(n)}, n = 0, 1, 2, \ldots$ inductively as follows. Put $\rho^{(0)} = \rho$ and if $\rho^{(n)}$ is defined for some $n \in \mathbb{N}_0$ then define $\rho^{(n+1)}$ by

$$\rho^{(n+1)}(f) = \sup\{\int_\Omega fg\,d\mu : g \in M^+, \rho^{(n)}(g) \leq 1\}$$

for all $f \in M^+$.

We write ρ', ρ'' and ρ''' for $\rho^{(1)}$, $\rho^{(2)}$ and $\rho^{(3)}$, respectively.

Proposition 3.28. *For the function seminorm ρ and any $n \in \mathbb{N}$, $\rho^{(n)}$ is a function seminorm with the Fatou property.*

Proof. By an induction argument, it suffices to prove the statement for $n = 1$. For any $g \in M^+$ with $\rho(g) \leq 1$, the map ρ_g given by

$$\rho_g(f) = \int_\Omega fg\,d\mu$$

for all $f \in M^+$ is easily seen to be a function seminorm. To show that ρ_g has the Fatou property, let $f_\alpha \uparrow f$ in M^+ and assume that g is not identically equal to 0 (if $g = 0$ then $\rho_g = 0$ obviously has the Fatou property). Observe that by Lemma 3.6 $f_\alpha g \uparrow fg$ and let

$$a = \sup_\alpha \rho_g(f_\alpha).$$

If $a = \infty$ then $\rho_g(f) \geq \sup_\alpha \rho_g(f_\alpha) = \infty$. If $a < \infty$ then let $(\alpha_n)_n$ be an increasing sequence of indices such that $\rho_g(f_{\alpha_n}) \uparrow a$ and let $f_0 = \sup f_{\alpha_n}$. Since $(f_{\alpha_n})_n$ is bounded from above by f we have that the latter supremum exists in M. We then have that $f_{\alpha_n} \uparrow f_0$ and by Lemma 3.6 we have that $f_{\alpha_n}g \uparrow f_0 g$. By the monotone convergence theorem we have that

$$\int_\Omega f_0 g\,d\mu = \sup_n \int_\Omega f_{\alpha_n} g\,d\mu = \sup_n \rho_g(f_{\alpha_n}) = a.$$

We claim that $f_0 \geq f_\alpha$ on $\operatorname{supp} g$ for all α. Suppose the contrary, that is, that there are α and a measurable set $A \subseteq \operatorname{supp}(g)$ of positive measure such that $f_0 < f_\alpha$ on A. Then we have that $f_0 g < f_\alpha g$ on A which implies that $\int_A f_0 g\,d\mu < \int_A f_\alpha g\,d\mu$.

Let $(\beta_n)_n$ be an increasing sequence of indices such that $\beta_n \geq \alpha$ and $\beta_n \geq \alpha_n$ for all $n \in \mathbb{N}$. Then by the monotone convergence theorem we have that

$$
\begin{aligned}
a \geq \int_\Omega f_{\beta_n} g \, d\mu &= \int_A f_{\beta_n} g \, d\mu + \int_{\Omega\setminus A} f_{\beta_n} g \, d\mu \geq \int_A f_\alpha g \, d\mu + \int_{\Omega\setminus A} f_{\alpha_n} g \, d\mu \\
&\xrightarrow[n\to\infty]{} \int_A f_\alpha g \, d\mu + \int_{\Omega\setminus A} f_0 g \, d\mu > \int_A f_0 g \, d\mu + \int_{\Omega\setminus A} f_0 g \, d\mu = \int_\Omega f_0 g \, d\mu = a
\end{aligned}
$$

which is impossible. Therefore we have that $f_0 \geq f$ on $\operatorname{supp}(g)$ which implies that

$$a = \int_\Omega f_0 g \, d\mu \geq \int_\Omega f g \, d\mu = \rho_g(f).$$

Since we also have that $\rho_g(f) \geq \sup_\alpha \rho_g(f_\alpha) = a$ we obtain that $\rho_g(f) = a$ and therefore ρ_g has the Fatou property. Now by Proposition 3.21 ρ' is a seminorm with the Fatou property. □

The function seminorm $\rho^{(n)}$ is called the *n-th associate seminorm* of ρ.

Theorem 3.29. *(a) (Hölder inequality) For $f, g \in M^+$ such that $\rho(f)$ and $\rho'(g)$ are finite we have that*

$$\int_\Omega f g \, d\mu \leq \rho(f)\rho'(g);$$

(b) We have that $\rho'' \leq \rho$ and $\rho^{(n+2)} = \rho^{(n)}$ for all $n \geq 1$.

Proof. (a) Suppose that $\rho(f) > 0$. Then $\rho(\frac{1}{\rho(f)} f) = 1$ and hence by the definition of ρ', $\rho'(g) \geq \int_\Omega g \frac{1}{\rho(f)} f \, d\mu = \frac{1}{\rho(f)} \int_\Omega f g \, d\mu$. Therefore $\int_\Omega f g \, d\mu \leq \rho(f)\rho'(g)$.

Now suppose that $\rho(f) = 0$. If $f = 0$ then the inequality obviously holds. Suppose that $f \neq 0$ and let $E = \{x \in \Omega : f(x) > 0\}$. We will prove that $g = 0$ on E from where the inequality will obviously follow. Pick an arbitrary number $n \in \mathbb{N}$ and let $E_n = \{x \in \Omega : f(x) > \frac{1}{n}\}$. If $g \neq 0$ on E_n then there exists a measurable set $H \subseteq E_n$ of positive measure and $\epsilon > 0$ such that $g(x) > \epsilon$ for all $x \in H$. As $f > \frac{1}{n}\chi_H$ and $\rho(f) = 0$ we have that $\rho(\frac{1}{n}\chi_H) = 0$ and consequently $\rho(k\chi_H) = 0$ for all $k \in \mathbb{N}$. But then by the definition of ρ' we have that $\rho'(g) \geq \int_\Omega g k \chi_H \, d\mu \geq k\epsilon\mu(H)$ for all $k \in \mathbb{N}$ which implies that $\rho'(g) = \infty$, contradicting our hypotheses. Therefore $g = 0$ on E_n for all $n \in \mathbb{N}$ and so $g = 0$ on E.

(b) Let $f \in M^+$. If $\rho(f) = \infty$ then obviously $\rho''(f) \leq \rho(f)$. If $\rho(f) < \infty$ then by the Hölder inequality we have that

$$
\begin{aligned}
\rho''(f) &= \sup\{\int_\Omega f g \, d\mu : g \in M^+, \rho'(g) \leq 1\} \\
&\leq \sup\{\rho(f)\rho'(g) : g \in M^+, \rho'(g) \leq 1\} \leq \rho(f).
\end{aligned}
$$

Replacing ρ by ρ', we obtain $\rho''' \leq \rho'$. From the definition of the associate seminorm it immediately follows that if $\rho_1 \leq \rho_2$ then $\rho_1' \geq \rho_2'$. Hence $\rho'' \leq \rho$ implies that $\rho''' \geq \rho'$ which shows that $\rho^{(n+2)} = \rho^{(n)}$ for all $n \geq 1$. □

Theorem 3.30. *A measurable set E is ρ-purely infinite if and only if it is ρ''-purely infinite.*

Proof. Since $\rho'' \leq \rho$, if E is ρ''-purely infinite then E is ρ-purely infinite. Suppose that E is ρ-purely infinite. For every function $g \in M^+$ supported on E we have that $\rho'(g) = \sup\{\int_E fg\,d\mu : f \in M^+, \rho(f) \leq 1\} = 0$ as all $f \in M^+$ with $\rho(f) \leq 1$ vanish on E by Proposition 3.13. Therefore, if $F \subseteq E$ has positive measure then $\rho'(n\chi_F) = 0$ for all $n \in \mathbb{N}$ and so

$$\rho''(\chi_F) = \sup\{\int_F g\,d\mu : g \in M^+, \rho'(g) \leq 1\} \geq \sup_{n\in\mathbb{N}} \int_F n\chi_F\,d\mu = \infty. \qquad \square$$

Theorem 3.31. *For a function seminorm ρ the following statements are equivalent:*

(a) ρ is saturated;

(b) ρ' is a norm;

(c) ρ'' is saturated.

Proof. From Theorem 3.30 we immediately obtain the equivalence of (a) and (c). Suppose that ρ is saturated and pick $0 \neq g \in M$. There exists a measurable set $E \subseteq \Omega$ with $\mu(E) > 0$ and $\epsilon > 0$ such that $|g| \geq \epsilon\chi_E$. Since ρ is saturated there exists a measurable set $A \subseteq E$ such that $\mu(A) > 0$ and $\rho(\chi_A) < \infty$. If $\rho(\chi_A) > 0$ then $\rho(\frac{1}{\rho(\chi_A)}\chi_A) = 1$ and from the definition of ρ' we get that

$$\rho'(\chi_E) \geq \int_\Omega \chi_E(\frac{1}{\rho(\chi_A)}\chi_A)\,d\mu = \frac{1}{\rho(\chi_A)}\int_\Omega \chi_A\,d\mu > 0.$$

If $\rho(\chi_A) = 0$ then

$$\rho'(\chi_E) \geq \int_\Omega \chi_E\chi_A\,d\mu = \int_\Omega \chi_A\,d\mu > 0.$$

Therefore we have that $\rho'(\chi_E) > 0$ and so

$$\rho'(g) = \rho'(|g|) \geq \rho'(\epsilon\chi_E) = \epsilon\rho'(\chi_E) > 0.$$

If ρ' is a norm and E is a measurable set with positive measure then $\rho'(\chi_E) > 0$ and the definition of ρ' yields the existence of $g \in M^+$ such that $\rho(g) \leq 1$ and $\int_E g\,d\mu = \int g\chi_E\,d\mu > 0$. Then there is a measurable set $A \subseteq E$ with $\mu(A) > 0$ and $\epsilon > 0$ such that $g \geq \epsilon\chi_A$. Hence $\rho(\chi_A) \leq \frac{1}{\epsilon}\rho(g) \leq \frac{1}{\epsilon}$ which shows that E is not ρ-purely infinite and so ρ is saturated. $\square$

Theorem 3.32 (Hölder inequality for a saturated function norm)**.** *If ρ is a saturated function norm then for all $f, g \in M^+$ we have that*

$$\int_\Omega fg\,d\mu \leq \rho(f)\rho'(g).$$

Proof. By Theorem 3.29, the above inequality holds if $\rho(f)$ and $\rho'(g)$ are finite. If $\rho(f)$ and $\rho'(g)$ are strictly positive and one of them is ∞ then the inequality is obvious. If $\rho(f) = 0$ and $\rho'(g) = \infty$ then $f = 0$ since ρ is a norm, and so the inequality follows. Finally, if $\rho(f) = \infty$ and $\rho'(g) = 0$ then by Theorem 3.31 ρ' is a norm since ρ is saturated, and so $g = 0$ which immediately implies the inequality. $\square$

3.7 The Associate Space of a Normed Function Space

In this section we assume that ρ is a saturated function norm. By Theorem 3.31 we have that ρ' is a norm and by Proposition 3.28 ρ' has the Fatou property which by Proposition 3.20 implies that ρ' has the Riesz-Fischer property. Then from Theorem 3.11 we have that $L_{\rho'}$ is a Banach space. We write L'_ρ instead of $L_{\rho'}$ and call L'_ρ the *associate space* of L_ρ. Recall that the norm ρ' on L'_ρ is given by

$$\rho'(f) = \sup\{\int_\Omega |fg|\,d\mu : \rho(g) \leq 1\}$$

for every $f \in L'_\rho$. We next prove that here we may substitute $\int_\Omega |fg|\,d\mu$ with $|\int_\Omega fg\,d\mu|$.

Proposition 3.33. *For every $f \in L'_\rho$ we have that*

$$\rho'(f) = \sup\{\left|\int_\Omega fg\,d\mu\right| : \rho(g) \leq 1\}.$$

Proof. We have that

$$\rho'(f) = \sup\{\int_\Omega |fg|\,d\mu : \rho(g) \leq 1\} \geq \sup\{\left|\int_\Omega fg\,d\mu\right| : \rho(g) \leq 1\}.$$

For the converse inequality, pick any $g \in M$ such that $\rho(g) \leq 1$ and put $g_1 = |g| \operatorname{sgn} f$ where sgn is the sign function. Then $\rho(g_1) = \rho(g) \leq 1$ and $fg_1 \in M^+$ from where it follows that $|\int_\Omega fg_1\,d\mu| = \int_\Omega |fg|\,d\mu$. This implies that $\sup\{\left|\int_\Omega fh\,d\mu\right| : \rho(h) \leq 1\} \geq \int_\Omega |fg|\,d\mu$. As $g \in M$ such that $\rho(g) \leq 1$ was arbitrary, we have that

$$\sup\{\left|\int_\Omega fg\,d\mu\right| : \rho(g) \leq 1\} \geq \sup\{\int_\Omega |fg|\,d\mu : \rho(g) \leq 1\} = \rho'(f). \qquad \square$$

Given $f \in L'_\rho$, from Proposition 3.33 we have that the integral $\int_\Omega fg\,d\mu$ is finite for every $g \in L_\rho$. Therefore, f defines a linear functional Λ_f on L_ρ by

$$\Lambda_f(g) = \int_\Omega fg\,d\mu.$$

Proposition 3.34. *If $f \in L'_\rho$ then Λ_f is bounded and $\|\Lambda_f\| = \rho'(f)$.*

Proof. By Proposition 3.33 we have that

$$\sup\{|\Lambda_f(g)| : \rho(g) \leq 1\} = \sup\{|\int_\Omega fg\,d\mu| : \rho(g) \leq 1\} = \rho'(f). \qquad \square$$

Example 3.35. Suppose that μ is semi-finite. If $1 < p < \infty$ and $\rho = \rho_p$ is defined as in Example 3.3 then L'_ρ turns out to be $L^q(\mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, if $f \in L'_\rho$ then f defines a bounded linear functional on $L_\rho = L^p(\mu)$. Since every bounded functional ϕ on $L^p(\mu)$ is given by $\phi(g) = \int_\Omega f'g\,d\mu$ for some $f' \in L^q(\mu)$, we have that there exists $f' \in L^q(\mu)$ such that

$$\int_\Omega fg\,d\mu = \int_\Omega f'g\,d\mu$$

for all $g \in L^p(\mu)$. This implies that

$$\int_A (f - f')\,d\mu = \int_\Omega (f - f')\chi_A\,d\mu = 0$$

for all measurable sets $A \subseteq \Omega$ of finite measure from where by semi-finiteness of μ it easily follows that $f = f'$ almost everywhere.

Also, if $f \in L^q(\mu)$ then by Proposition 3.33 we have that

$$\infty > \|f\|_q = \sup\{\left|\int_\Omega fg\,d\mu\right| : \|g\|_p = \rho(g) \le 1\} = \rho'(f)$$

and so $f \in L'_\rho$.

From Proposition 3.34 we get that the map $f \mapsto \Lambda_f$ is a positive, hence order preserving, linear isometry from L'_ρ to L^*_ρ. Since L'_ρ is a Banach space, we can thus consider L'_ρ to be a closed subspace of L^*_ρ. The next two theorems together prove that if μ is σ-finite and ρ is saturated then $L'_\rho \cong (L_\rho)^\sim_n$. Indeed, if $f \in L'_\rho$ then by Theorem 3.36 we have that Λ_f is order continuous, that is, $\Lambda_f \in (L_\rho)^\sim_n$. If $\phi \in (L_\rho)^\sim_n$ then by Theorem 3.38 we have that $\phi \in L'_\rho$.

Theorem 3.36. *For $f \in L'_\rho$ the linear functional Λ_f is order continuous.*

Proof. Since Λ_f is bounded we have that it is order bounded by Theorem 2.32. Since $\Lambda_f = \Lambda_{f^+} - \Lambda_{f^-}$ it suffices to prove the theorem only for a positive function f. Therefore assume that f is positive and observe that Λ_f is positive. Suppose that $g_\alpha \downarrow 0$ and note that $fg_\alpha \downarrow 0$ by Lemma 3.6. Put $h_\alpha = fg_\alpha$ for all α and $a = \inf_\alpha \int_\Omega h_\alpha\,d\mu$. Then there exists an increasing sequence of indices $(\alpha_n)_n$ such that $a = \inf_n \int_\Omega h_{\alpha_n}\,d\mu$. Suppose that $a > 0$ and put $h = \inf_n h_{\alpha_n}$. By the dominated convergence theorem we conclude that $h \neq 0$ on Ω and that there exists an index α_0 such that $h_{\alpha_0} < h - \epsilon$ for some $\epsilon > 0$ on some set A of positive measure. Let (β_n) be an increasing sequence of indices such that $\beta_n \ge \alpha_0$ and $\beta_n \ge \alpha_n$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
\int_\Omega h_{\beta_n}\,d\mu &= \int_A h_{\beta_n}\,d\mu + \int_{\Omega\setminus A} h_{\beta_n}\,d\mu \le \int_A h_{\alpha_0}\,d\mu + \int_{\Omega\setminus A} h_{\alpha_n}\,d\mu \\
&\le \int_A h\,d\mu - \epsilon\mu(A) + \int_{\Omega\setminus A} h_{\alpha_n}\,d\mu \xrightarrow[n\to\infty]{} \int_A h\,d\mu - \epsilon\mu(A) + \int_{\Omega\setminus A} h\,d\mu \\
&= \int_\Omega h\,d\mu - \epsilon\mu(A) = a - \epsilon\mu(A)
\end{aligned}$$

which is contradictory to the definition of a. □

Remark 3.37. Let us note that the argument in Theorem 3.36 can be used to prove the order continuity of the integral, that is, if $h_\alpha \downarrow 0$ in M^+ with all h_α integrable then $\int_\Omega h_\alpha\,d\mu \downarrow 0$. Note also that this implies that if $0 \le h_\alpha \uparrow h$ with h integrable then $\int_\Omega h_\alpha\,d\mu \uparrow \int_\Omega h\,d\mu$.

Theorem 3.38. *Suppose that μ is σ-finite and ρ is saturated. If $\phi \in L^*_\rho$ is σ-order continuous then $\phi \in L'_\rho$.*

Proof. Suppose that $0 \leq \phi \in L_\rho^*$ is σ-order continuous. Since μ is σ-finite and ρ is saturated by Theorem 3.19 we can find an increasing sequence of sets $(A_n)_n$ of finite measure such that $\Omega = \bigcup_{n=1}^\infty A_n$ and $\rho(\chi_{A_n}) < \infty$ for all $n \in \mathbb{N}$. We define a map $\nu \colon \Sigma \to [0, \infty]$ by

$$\nu(A) = \sup_k \phi(\chi_{A_k \cap A}).$$

We have that $\nu(\emptyset) = \phi(0) = 0$. If B and C are disjoint measurable sets then since ϕ is positive and hence increasing we have that

$$\begin{aligned}\nu(B \cup C) &= \sup_k \phi(\chi_{A_k \cap (B \cup C)}) = \sup_k \phi(\chi_{(A_k \cap B) \cup (A_k \cap C))}) = \sup_k \phi(\chi_{A_k \cap B} + \chi_{A_k \cap C}) \\ &= \sup_k (\phi(\chi_{A_k \cap B}) + \phi(\chi_{A_k \cap C})) = \sup_k \phi(\chi_{A_k \cap B}) + \sup_k \phi(\chi_{A_k \cap C}) \\ &= \nu(B) + \nu(C)\end{aligned}$$

Let $(B_n)_n$ be an increasing sequence of measurable sets and let $B = \bigcup_{n=1}^\infty B_n$. We have that

$$\begin{aligned}\nu(\bigcup_{n=1}^\infty B_n) &= \sup_k \phi(\chi_{A_k \cap \bigcup_{n=1}^\infty B_n}) = \sup_k \phi(\chi_{\bigcup_{n=1}^\infty A_k \cap B_n}) = \sup_k \phi(\sup_n \chi_{A_k \cap B_n}) \\ &= \sup_k \sup_n \phi(\chi_{A_k \cap B_n}) = \sup_n \sup_k \phi(\chi_{A_k \cap B_n}) = \sup_n \nu(B_n)\end{aligned}$$

since ϕ is σ-order continuous and therefore we can exchange the order between calculating ϕ and sup. We have now obtained that ν is a measure on (Ω, Σ). Moreover, from the definition of ν we easily see that $\nu(A_n) = \phi(\chi_{A_n}) < \infty$ for every $n \in \mathbb{N}$ which, since $\Omega = \bigcup_{n=1}^\infty A_n$, shows that ν is σ-finite. Also, if $\mu(B) = 0$ for some measurable set B then $\nu(B) = \sup_k \phi(\chi_{A_k \cap B}) = 0$ and so ν is absolutely continuous with respect to μ. By the Radon-Nikodym theorem there exists a measurable function $g \colon \Omega \to [0, \infty]$ such that

$$\nu(A) = \int_\Omega g \chi_A \, d\mu$$

for each measurable set $A \subseteq \Omega$. Since $\int_\Omega g \chi_{A_n} \, d\mu = \nu(A_n) = \phi(\chi_{A_n}) < \infty$ for all $n \in \mathbb{N}$, we get that $g < \infty$ almost everywhere. If B is a measurable set such that $\rho(\chi_B) < \infty$ then by σ-order continuity of ϕ we have that

$$\phi(\chi_B) = \phi(\sup_k \chi_{A_k \cap B}) = \sup_k \phi(\chi_{A_k \cap B}) = \nu(B) = \int_\Omega g \chi_B \, d\mu.$$

This implies that $\phi(s) = \int_\Omega gs \, d\mu$ for any positive simple function $s \in L_\rho^+$. We claim that $\phi(f) = \int_\Omega gf \, d\mu$ for any positive function $f \in L_\rho^+$. To see this, pick $f \in L_\rho^+$ and find a sequence of positive simple functions $(s_n)_n$ such that $s_n \uparrow f$. Then again by σ-order continuity of ϕ we have that

$$\phi(f) = \phi(\sup_n s_n) = \sup_n \phi(s_n) = \sup_n \int_\Omega g s_n \, d\mu = \int_\Omega gf \, d\mu.$$

If $f \in L_\rho$ is an arbitrary function then

$$\phi(f) = \phi(f^+) - \phi(f^-) = \int_\Omega gf^+ \, d\mu - \int_\Omega gf^- \, d\mu = \int_\Omega gf \, d\mu.$$

To show that $g \in L'_\rho$ note that

$$\rho'(g) = \sup\{\int_\Omega gf\,d\mu : f \in M^+, \rho(f) \leq 1\} = \sup\{\phi(f) : f \in M^+, \rho(f) \leq 1\}$$
$$= \|\phi\| < \infty.$$

Therefore every positive σ-order continuous functional $\phi \in L^*_\rho$ is represented by a positive function $g = g_\phi \in L'_\rho$ in the sense that $\Lambda_{g_\phi} = \phi$.

Now let $\phi \in L^*_\rho$ be an arbitrary σ-order continuous functional. By Theorem 2.27 we have that $\phi = \phi^+ - \phi^-$ where ϕ^+ and ϕ^- are positive σ-order continuous functionals. Then there exist positive functions $g_{\phi^+}, g_{\phi^-} \in L'_\rho$ such that $\Lambda_{g_{\phi^+}} = \phi^+$ and $\Lambda_{g_{\phi^-}} = \phi^-$. We define $g_\phi = g_{\phi^+} - g_{\phi^+}$ and observe that by the linearity of the map $f \mapsto \Lambda_f$ we have that

$$\Lambda_{g_\phi} = \Lambda_{g_{\phi^+} - g_{\phi^+}} = \Lambda_{g_{\phi^+}} - \Lambda_{g_{\phi^-}} = \phi^+ - \phi^- = \phi. \qquad \square$$

4 Characterizations of the associate space with respect to a semi-finite measure

In this chapter we mainly study the relationship, under several conditions, between the associate space of a given normed or Banach function space E to its dual spaces. We prove, among other results, that for a saturated Banach function space E over a semi-finite measure μ we have $E' \cong E_n^\sim$ if and only if E' has the strong Fatou property. Furthermore, we prove that $E' \cong E_n^\sim$ holds if μ is localizable. We follow the paper [1] by Celia Avalos-Ramos and Fernando Galaz-Fontes.

4.1 Main Results

Throughout this section ρ will be a function norm on M with respect to a measurable space with a positive measure (Ω, Σ, μ) and $E = L_\rho$. In the main results the equalities between the various dual spaces of E are understood to be isometric isomorphisms.

We proceed by defining some useful notation. If $A \subseteq \Omega$ is measurable then let $\Sigma_A = \{X \in \Sigma : X \subseteq A\}$ and μ_A be the restriction of μ on Σ_A. Then (A, Σ_A, μ_A) is a measure space. If $f\colon A \to \mathbb{R}$ is measurable and $A \subseteq B$ with B measurable we define the extension $f^B\colon B \to \mathbb{R}$ by $f^B(x) = f(x)$ if $x \in A$ and $f^B(x) = 0$ otherwise. We then define E_A to be the set of all real measurable functions f on A such that $f^\Omega \in E$ and we define $\rho_A\colon E_A \to \mathbb{R}^+$ by $\rho_A(f) = \rho(f^\Omega)$. We also define $\Sigma^f = \{A \in \Sigma : \mu(A) < \infty\}$.

Lemma 4.1. *If $A \subseteq \Omega$ is measurable then (E_A, ρ_A) is a normed function space.*

Proof. Let N be the vector space of all (equivalence classes of) real measurable functions on (A, Σ_A, μ_A). We extend the definition of ρ_A to N by requiring that $\rho_A(f) = \rho(f^\Omega)$ for all $f \in N$. We need to prove that ρ_A is a function norm on N and $E_A = \{f \in N : \rho_A(f) < \infty\}$.

We have that $\rho_A(f) = 0$ if and only if $\rho(f^\Omega) = 0$ which holds if and only if $f^\Omega = 0$ or equivalently $f = 0$. Monotonicity, positive homogenuity and subadditivity of ρ_A on N^+ are obvious. Since we also have that

$$\rho_A(f) = \rho(f^\Omega) = \rho(|f^\Omega|) = \rho(|f|^\Omega) = \rho_A(|f|)$$

for every $f \in N$, we get that ρ_A is a function norm on N. If $f \in E_A$ then $\rho_A(f) = \rho(f^\Omega) < \infty$ by the definition of E_A. If $f \in N$ and $\rho_A(f) < \infty$ then $\rho(f^\Omega) < \infty$ or equivalently $f \in E_A$. □

Theorem 4.2. *Suppose that E is a saturated Banach function space over a semi-finite measure μ. Then we have that E' is isometrically isomorphic to $E_n^\sim$ if and only if E' has the strong Fatou property.*

Proof. If $E' = E_n^\sim$ then by Theorem 2.38 we have that $E_n^\sim$ has the strong Fatou property. Since the map $f \mapsto \Lambda_f$ is an order preserving linear isometry we have that E' has the strong Fatou property.

Suppose that E' has the strong Fatou property. By Theorem 3.36 we have that $E' \subseteq E_n^\sim$. Let us now prove that $E_n^\sim \subseteq E'$. Since every functional in $E_n^\sim$ is the difference of its positive and negative parts, it suffices to show that $\phi \in E'$ for every $0 \leq \phi \in E_n^\sim$.

Suppose that $0 \leq \phi \in E_n^\sim$ and fix a measurable set $A \in \Sigma^f$. Now ϕ defines a positive linear functional ϕ_A on E_A by $\phi_A(f) = \phi(f^\Omega)$. Moreover, since $f_\alpha \downarrow 0$ in E_A implies that $f_\alpha^\Omega \downarrow 0$ in E we have that ϕ_A is order continuous. Then by Theorem 3.38 there exists a unique positive h_A in E'_A such that $\phi_A(f) = \int_A f h_A \, d\mu$ for all $f \in E_A$. We put $H_A = h_A^\Omega$ and observe that

$$\begin{aligned}
\rho'(H_A) &= \sup\{\int_\Omega f H_A \, d\mu : f \in M^+, \rho(f) \leq 1\} \\
&= \sup\{\int_A f|_A h_A \, d\mu : f \in M^+, \rho(f) \leq 1\} \\
&\leq \sup\{\int_A f h_A \, d\mu : f \in N^+, \rho_A(f) \leq 1\} = \rho'_A(h_A) = \|\phi_A\| \leq \|\phi\| < \infty
\end{aligned}$$

since $f \in M^+$ and $\rho(f) \leq 1$ imply that $\rho_A(f|_A) = \rho(f|_A^\Omega) \leq \rho(f) \leq 1$. This shows that $H_A \in E'$.

If $B \subseteq \Omega^f$ then for $f \in E_{A\cap B}$ we have that $\rho((f^A)^\Omega) = \rho(f^\Omega) < \infty$ which implies that $f^A \in E_A$. Furthermore,

$$\phi_{A\cap B}(f) = \phi(f^\Omega) = \phi_A(f^A) = \int_A f^A h_A \, d\mu = \int_{A\cap B} f h_A|_{A\cap B} \, d\mu$$

for all $f \in E_{A\cap B}$ and therefore by the uniqueness of the representation of $\phi_{A\cap B}$ it follows that $h_{A\cap B} = (h_A)|_{A\cap B}$, or equivalently, $H_{A\cap B} = (H_A)|_{A\cap B}$. Hence $H_A \leq H_{A\cup B}$ and $H_B \leq H_{A\cup B}$. This implies that $(H_C)_C$ where $C \in \Sigma^f$ is an upward directed net. We also have that $\rho'(H_C) \leq \|\phi\| < \infty$ for all $C \in \Sigma^f$ and so the strong Fatou property of E' implies the existence of $H \in E'$ such that $H_C \uparrow H$ and $\rho'(H_C) \uparrow \rho'(H)$.

Let us first prove that $H_C = H\chi_C$ for all $C \in \Sigma^f$. Fix some $C \in \Sigma^f$. Obviously $H_C \leq H$ which implies $H_C \leq H\chi_C$. If $H\chi_C > H_C$ on some subset $D \subseteq C$ of positive measure then by defining $H' = H_C$ on D and $H' = H$ outside D we obtain that $H_{C'} \leq H'$ for all measurable sets $C' \in \Sigma^f$ since H_C and $H_{C'}$ agree on $C \cap C'$ for all $C' \in \Sigma^f$.

Pick $f \in E^+$. We prove that $fH_C \uparrow fH$. Obviously $fH_C \uparrow$ and $fH_C \leq fH$ for all $C \in \Sigma^f$. Suppose that $g \geq fH_C$ for all $C \in \Sigma^f$ and $g < fH$ on some measurable set D of positive measure. Since μ is semi-finite there exists a measurable set $D' \subseteq D$ of positive finite measure. But then $g \geq fH_{D'} = fH$ on D' which is a contradiction. We can similarly show that $f\chi_C \uparrow f$.

By order continuity of ϕ and Remark 3.37 we have that

$$\phi(f) = \phi(\sup_{C\in\Sigma^f} f\chi_C) = \sup_{C\in\Sigma^f} \phi(f\chi_C) = \sup_{C\in\Sigma^f} \int_\Omega f H_C \, d\mu = \int_\Omega f H \, d\mu.$$

This shows that ϕ is represented by $H \in E'$. □

Theorem 4.3. *If E is a σ-order continuous Banach function space then it is order continuous.*

Proof. Suppose that E is σ-order continuous. First we prove that if $f \in E^+$ then $\operatorname{supp} f$ is σ-finite. Pick $f \in E^+$ and let $\mathcal{A}$ be the family of all sets $P \subseteq \Sigma$ that satisfy the following conditions:

(a) If $A \in P$ then $A \subseteq \operatorname{supp} f$ and $0 < \mu(A) < \infty$;

(b) If $A, B \in P$ and $A \neq B$ then $A \cap B = \emptyset$.

The family of sets $\mathcal{A}$ is partially ordered by inclusion, and if $\{P_\beta\} \subseteq \mathcal{A}$ is a chain then by checking the definition of $\mathcal{A}$ it is easy to see that $\bigcup_\beta P_\beta \in \mathcal{A}$ is an upper bound of $\{P_\beta\}$. By Zorn's lemma there exists a maximal element Q in $\mathcal{A}$. Our aim is to prove that Q is countable and that $\mu(\operatorname{supp} f \setminus (\bigcup Q)) = 0$ from where it follows that $\operatorname{supp} f$ is σ-finite.

Suppose on the contrary that Q is uncountable and pick arbitrary $A \in Q$. Since $A \in Q$ and $Q \in \mathcal{A}$ by the definition of $\mathcal{A}$ we have that $\mu(A) > 0$ and $A \subseteq \operatorname{supp} f$. This implies the existence of $\epsilon_A > 0$ and measurable set $A' \subseteq A$ such that $\mu(A') > 0$ and $f \geq \epsilon_A$ on A'. Therefore to every $A \in Q$ we associate a real number ϵ_A and a measurable subset $A' \subseteq A$ of positive measure such that we have

$$f \geq \epsilon_A \chi_{A'}$$

for all $A \in Q$. Since ρ is a norm we have that $\rho(\chi_{A'}) > 0$ for all $A \in Q$. We consider the set of positive real numbers

$$\{\rho(\epsilon_A \chi_{A'}) : A \in Q\}.$$

Since Q is uncountable there exists $\delta > 0$ and a sequence with different elements $(A_n)_n \subseteq Q$ such that $\rho(\epsilon_{A_n} \chi_{A'_n}) \geq \delta$ for all $n \in \mathbb{N}$. We define

$$f_m = \sum_{n=m}^{\infty} \epsilon_{A_n} \chi_{A'_n}$$

for all $m \in \mathbb{N}$. We easily see that $f_m \leq f$ for all $m \in \mathbb{N}$ which implies that $f_m \in E^+$ for all $m \in \mathbb{N}$. It is also easy to see that $f_m \downarrow 0$ which by σ-order continuity of ρ implies that $\rho(f_m) \downarrow 0$. On the other hand we have that

$$\rho(f_m) \geq \rho(\epsilon_{A_m} \chi_{A'_m}) \geq \delta > 0$$

for all $m \in \mathbb{N}$ which is a contradiction. Therefore Q is at most countable.

To show that $\mu(\operatorname{supp} f \setminus \bigcup Q) = 0$, suppose the contrary. Define $B = \operatorname{supp} f \setminus (\bigcup Q)$ and note that $\mu(B) > 0, B \subseteq \operatorname{supp} f$ and $B \cap A = \emptyset$ for all $A \in Q$. Then if $Q' = Q \cup \{B\}$ then $Q \subsetneq Q'$ and $Q' \in \mathcal{A}$ which contradicts the maximality of Q. Therefore $\mu(\operatorname{supp} f \setminus (\bigcup Q)) = 0$ and $\operatorname{supp} f$ is σ-finite.

Now we proceed with showing that E is order continuous. Suppose that $f_\alpha \downarrow 0$ in E. First we prove that for any measurable set $A \in \Sigma^f$ there is an increasing sequence of indices (α_n) such that $f_{\alpha_n} \chi_A \downarrow 0$. Define the map $\Gamma \colon M^+ \to \mathbb{R}^+$ by

$$\Gamma(g) = \int_A \frac{g}{1+g} \, d\mu.$$

It is easy to check that if $g_1, g_2 \in M^+$ and $g_1 \leq g_2$ then $\frac{g_1}{1+g_1} \leq \frac{g_2}{1+g_2}$ which implies that Γ is monotonically increasing. By Lemma 3.6 we have that $f_\alpha \chi_A \downarrow 0$ and since $\frac{f_\alpha \chi_A}{1+f_\alpha \chi_A} \leq f_\alpha \chi_A$ we have that $\frac{f_\alpha \chi_A}{1+f_\alpha \chi_A} \downarrow 0$. By order continuity of the integral (Remark 3.37) we have that $\Gamma(f_\alpha \chi_A) \downarrow 0$. There is an increasing sequence of indices $(\alpha_n)_n$ such that $\Gamma(f_{\alpha_n} \chi_A) \downarrow 0$. Put $f = \inf_n f_{\alpha_n} \chi_A$ and suppose that $f \neq 0$. Then there is a measurable set $B \subseteq A$ with $\mu(B) > 0$ such that f is strictly positive almost everywhere on B. Then by monotonicity of Γ we have that

$$0 = \inf_{n \in \mathbb{N}} \Gamma(f_{\alpha_n} \chi_A) \geq \Gamma(f) \geq \int_B \frac{f}{1+f} \, d\mu > 0$$

which is a contradiction. Therefore $\inf_n f_{\alpha_n} \chi_A = f = 0$.

Now choose some index β and write $\operatorname{supp}(f_\beta) = \bigcup_{n=1}^\infty A_n$ with $\mu(A_n) < \infty$ for each $n \in \mathbb{N}$. For each n there is a sequence of indices $(\alpha_m^n)_m$ such that $f_{\alpha_m^n} \chi_{A_n} \downarrow 0$. Define an increasing sequence of indices $(\beta_n)_n$ such that $\beta_n \geq \alpha_n^i$ for all $n \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, n\}$. Then $f_{\beta_n} \leq f_{\alpha_n^i}$ for all $n, i \in \mathbb{N}$ with $i \leq n$. This implies that $f_{\beta_n} \downarrow 0$ on A_m for all $m \in \mathbb{N}$ which implies that $f_{\beta_n} \downarrow 0$ on $\operatorname{supp}(f_\beta)$. Now if $(\gamma_n)_n$ is an increasing sequence of indices such that $\gamma_n \geq \beta$ and $\gamma_n \geq \beta_n$ for all $n \in \mathbb{N}$ we get that $f_{\gamma_n} \downarrow 0$. Since ρ is σ-order continuous we have that $\rho(f_{\gamma_n}) \downarrow 0$. Therefore $\rho(f_\alpha) \downarrow 0$. ☐

Corollary 4.4. *If E is σ-order continuous Banach function space then $E_n^\sim = E_c^\sim = E^*$.*

Proof. By Corollary 2.36 we have that $E_n^\sim \subseteq E_c^\sim \subseteq E^\sim = E^*$. Pick $0 \leq \phi \in E^*$. If $f_\alpha \downarrow 0$ in E, then the inequality $\phi(f_\alpha) \leq \|\phi\| \, \rho(f_\alpha)$ and order continuity of E (Theorem 4.3) imply that ϕ is order continuous. As every functional in E^* is the difference of its positive and negative parts, the corollary follows. ☐

Corollary 4.5. *Suppose that E is a σ-order continuous saturated Banach function space over a semi-finite measure μ. Then E' is isometrically isomorphic to E^* if and only if E' has the strong Fatou property.*

Proof. Immediate corollary from Theorem 4.2 and Corollary 4.4. ☐

Theorem 4.6. *Suppose that E is a saturated normed function space over a localizable measure μ. Then E' has the strong Fatou property.*

Proof. Suppose that $0 \leq f_\alpha \uparrow$ in E' and that $\sup_\alpha \rho'(f_\alpha) = a < \infty$. If $(f_\alpha)_\alpha$ is unbounded then by Lemma 3.26 there exists $0 < u \in L^\infty(\mu)$ such that $ku = \sup_\alpha ku \wedge f_\alpha$ for all $k \in \mathbb{N}$. Since ρ is saturated, Theorem 3.31 implies that ρ' is a norm and so $\rho'(u) > 0$. Since by Proposition 3.28 ρ' has the Fatou property, we have that for all $k \in \mathbb{N}$,

$$k\rho'(u) = \rho'(ku) = \sup_\alpha \rho'(ku \wedge f_\alpha) \leq \sup_\alpha \rho'(f_\alpha) = a$$

which is impossible. Therefore $(f_\alpha)_\alpha$ is bounded and by the Dedekind completeness of M (Theorem 3.25) we have that $f_\alpha \uparrow f$ for some $f \in M^+$. Now the Fatou property of ρ' implies that $\sup_\alpha \rho'(f_\alpha) = \rho'(f)$. ☐

Corollary 4.7. *Suppose that E is a saturated Banach function space over a localizable measure μ. Then E' is isometrically isomorphic to $E_n^\sim$.*

Proof. Immediate corollary from Theorem 4.2 and Theorem 4.6. □

Corollary 4.8. *Suppose that E is a σ-order continuous saturated Banach function space over a localizable measure μ. Then E' is isometrically isomorphic to E^*.*

Proof. Immediate corollary of Theorem 4.6 and Corollary 4.5. □

4.2 A Banach Function Space E such that $E' \neq E_n^\sim$

In this section we provide an example of a Banach function space where the equalities between dual spaces in the results of the previous section fail. This example is $L^1(\mu)$ where μ is a measure with certain properties. We start with providing an example of a measure that is semi-finite but not localizable. We present an example by D. H. Fremlin ([5, Example 216D]).

Theorem 4.9. *There exists a semi-finite measure μ which is not localizable.*

Proof. Let $I = \omega_1$ and $J = \omega_2$ be the first and second uncountable ordinals, respectively. Let $T = \{A \subseteq J : A \text{ or } J \setminus A \text{ is countable}\}$. Then T is a σ-algebra. Define the measure ν on (J, T) which equals 0 on countable sets and 1 on uncountable sets. Let $\Omega = J \times J$ and for every $A \subseteq \Omega$ and $x \in J$ define $A_x = \{y \in J : (x, y) \in A\}$ and $A^y = \{x \in J : (x, y) \in A\}$. We set

$$\Sigma = \{A \subseteq \Omega : A_x \in T \text{ and } A^x \in T \text{ for all } x \in J\}$$

and

$$\mu(A) = \sum_{x \in J} \nu(A_x) + \sum_{y \in J} \nu(A^y)$$

for all $A \in \Sigma$.

We will prove that (Ω, Σ, μ) is the required measure space. It is easy to see that $\emptyset \in \Sigma$ and $J \in \Sigma$. If $(A_n)_n \subseteq \Sigma$ then it is not hard to see that $(\bigcup_{n=1}^\infty A_n)_x = \bigcup_{n=1}^\infty (A_n)_x$ and $(\bigcup_{n=1}^\infty A_n)^y = \bigcup_{n=1}^\infty (A_n)^y$ for all $x, y \in J$. This immediately implies that $\bigcup_{n=1}^\infty A_n \in \Sigma$. Similarly, if $A \in \Sigma$ then $(J \setminus A)_x = J \setminus A_x$ and $(J \setminus A)^y = J \setminus A^y$ imply that $J \setminus A \in \Sigma$. We have therefore proven that Σ is a σ-algebra. As for μ, we obviously have that $\mu(\emptyset) = 0$ and if $(A_n)_n \subseteq \Sigma$ then since ν is a measure we have that

$$\begin{aligned}
\mu(\bigcup_{n=1}^{\infty} A_n) &= \sum_{x\in J} \nu((\bigcup_{n=1}^{\infty} A_n)_x) + \sum_{y\in J} \nu((\bigcup_{n=1}^{\infty} A_n)^y) \\
&= \sum_{x\in J} \nu(\bigcup_{n=1}^{\infty} (A_n)_x) + \sum_{y\in J} \nu(\bigcup_{n=1}^{\infty} (A_n)^y) \\
&= \sum_{x\in J} \sum_{n=1}^{\infty} \nu((A_n)_x) + \sum_{y\in J} \sum_{n=1}^{\infty} \nu((A_n)^y) \\
&= \sum_{n=1}^{\infty} \sum_{x\in J} \nu((A_n)_x) + \sum_{n=1}^{\infty} \sum_{y\in J} \nu((A_n)^y) \\
&= \sum_{n=1}^{\infty} (\sum_{x\in J} \nu((A_n)_x) + \sum_{y\in J} \nu((A_n)^y)) = \sum_{n=1}^{\infty} \mu(A_n)
\end{aligned}$$

which proves that μ is a measure on (Ω, Σ).

To show that μ is semi-finite, pick a measurable set $A \subseteq \Omega$ of positive measure. We have that one of $\sum_{x\in J} \nu(A_x)$ and $\sum_{x\in J} \nu(A^x)$ is positive, say the former. Then there exists $x \in J$ such that $\nu(A_x) > 0$. Since the measure ν can have value either 0 or 1, this implies that $\nu(A_x) = 1$ and it is now easy to see that $\{x\} \times A_x \subseteq A$ and

$$\mu(\{x\}\times A_x) = \sum_{z\in J} \nu((\{x\}\times A_x)_z) + \sum_{y\in J} \nu((\{x\}\times A_x)^y) = \nu((\{x\}\times A_x)_x) = \nu(A_x) = 1.$$

To show that μ is not localizable define $A[x] = \{x\} \times J$ for all $x \in J$. Suppose that there exists a measurable set $C \subseteq \Omega$ such that $\mu(A[x]\setminus C) = 0$ for all $x \in J$, and if D is a measurable set such that $\mu(A[x] \setminus D) = 0$ for all $x \in J$ then $\mu(C \setminus D) = 0$. For every $z \in J$ we have that $\mu(A[z] \setminus C) = 0$ which by the definition of μ implies that $\nu((A[z]\setminus C)_z) = 0$. Now, from the definition of ν we get that the set $(A[z]\setminus C)_z$ is countable. It is easy to check that $A[z] \setminus C = \{z\} \times (A[z] \setminus C)_z$ and therefore $A[z]\setminus C$ is countable for every $z \in J$. Since the set I has a strictly smaller cardinality than J we have that the set $\bigcup_{z\in I}(A[z] \setminus C)$ has a strictly smaller cardinality than J. This implies that the set

$$\{y \in J : (x, y) \in \bigcup_{z\in I} (A[z] \setminus C) \text{ for some } x \in I\}$$

has a strictly smaller cardinality than J, and therefore there exists $y \in J$ that does not belong to it. This means that $(x, y) \notin \bigcup_{z\in I}(A[z] \setminus C)$ for all $x \in I$, or equivalently,

$$(x, y) \in \Omega \setminus \bigcup_{z\in I} (A[z] \setminus C) = \bigcap_{z\in I} ((\Omega \setminus A[z]) \cup C)$$

for all $x \in I$. Therefore for any $x \in I$ we have that $(x, y) \in (\Omega \setminus A[x]) \cup C$ and as $(x, y) \in A[x]$ we have that $(x, y) \in C$. This implies that $I \times \{y\} \subseteq C$. We put $K = J\times\{y\}$ and observe that $I\times\{y\} \subseteq C\cap K$ which implies that $\mu(C\cap K) = 1$ since I is uncountable. If $D = C\setminus K$ then $A[x]\setminus D = A[x]\setminus(C\setminus K) = (A[x]\setminus C)\cup(A[x]\cap K)$

for all $x \in J$. Since $\mu(A[x] \setminus C) = 0$ we have that $A[x] \setminus C$ is countable. As $A[x] \cap K$ contains exactly one point, we have that $A[x] \setminus D$ is countable which implies that $\mu(A[x] \setminus D) = 0$. But as $\mu(C \setminus D) \geq \mu(C \cap K) = 1$ this is contradictory to the definition of C. □

The following theorem can be found in [5, Theorem 243G].

Theorem 4.10. *If the canonical map from $L^\infty(\mu)$ to $(L^1(\mu))^*$ is an isometric isomorphism then μ is localizable.*

Proof. Suppose that the canonical map from $L^\infty(\mu)$ to $(L^1(\mu))^*$ is an isometric isomorphism. To prove that μ is semi-finite pick a measurable set $A \subseteq \Omega$ with $\mu(A) > 0$ and note that $\|\chi_A\|_\infty = 1$. Since by assumption the canonical map from $L^\infty(\mu)$ to $(L^1(\mu))^*$ is an isometric isomorphism we have that the functional $\phi \in (L^1(\mu))^*$ defined by $\phi(f) = \int_\Omega f\chi_A \, d\mu$ for all $f \in L^1(\mu)$ has norm 1. Hence there exists a function $0 \leq g \in L^1(\mu)$ such that

$$\int_A g \, d\mu = \int_\Omega g\chi_A \, d\mu = \phi(g) > 0.$$

Since $0 \leq g \in L^1(\mu)$ we have that $0 < \int_A g \, d\mu < \infty$. Since g is not equal to 0 almost everywhere there exists $\epsilon > 0$ and a measurable set $B \subseteq A$ such that $\mu(B) > 0$ and $g \geq \epsilon$ on B. Then

$$\infty > \int_A g \, d\mu \geq \int_B \epsilon \, d\mu = \epsilon\mu(B),$$

and so $0 < \mu(B) < \infty$.

To show that μ is localizable let $\mathcal{A}$ be some family of subsets of Σ and let $\mathcal{B}$ consist of all finite unions of sets in $\mathcal{A}$. We observe that for any measurable set $C \subseteq \Omega$ we have that $\mu(A \setminus C) = 0$ for all $A \in \mathcal{A}$ if and only if $\mu(B \setminus C) = 0$ for all $B \in \mathcal{B}$. For each $0 \leq f \in L^1(\mu)$ we define

$$\phi(f) = \sup\{\int_B f \, d\mu : B \in \mathcal{B}\}.$$

Since $f \in L^1(\mu)$ we have that $\int_B f \, d\mu \leq \int_\Omega f \, d\mu < \infty$ for every $B \in \mathcal{B}$ which implies that the above supremum exists. If $0 \leq f, g \in L^1(\mu)$ then it is easy to see that

$$\begin{aligned}\phi(f+g) &= \sup\{\int_B (f+g) \, d\mu : B \in \mathcal{B}\} = \sup\{\int_B f \, d\mu + \int_B g \, d\mu : B \in \mathcal{B}\} \\ &\leq \sup\{\int_B f \, d\mu : B \in \mathcal{B}\} + \sup\{\int_B g \, d\mu : B \in \mathcal{B}\} = \phi(f) + \phi(g).\end{aligned}$$

To prove the opposite inequality, fix some $B_0 \in \mathcal{B}$. For every $B \in \mathcal{B}$ we have that $B_0 \cup B \in \mathcal{B}$ and hence we get that

$$\begin{aligned}\sup\{\int_B f \, d\mu + \int_B g \, d\mu : B \in \mathcal{B}\} &\geq \sup\{\int_{B_0} f \, d\mu + \int_B g \, d\mu : B_0 \subseteq B \in \mathcal{B}\} \\ &= \int_{B_0} f \, d\mu + \sup\{\int_B g \, d\mu : B_0 \subseteq B \in \mathcal{B}\} \\ &= \int_{B_0} f \, d\mu + \sup\{\int_B g \, d\mu : B \in \mathcal{B}\}.\end{aligned}$$

By taking the supremum over all $B_0 \in \mathcal{B}$ we obtain

$$\sup\{\int_B f\,d\mu + \int_B g\,d\mu : B \in \mathcal{B}\} \geq \sup\{\int_B f\,d\mu : B \in \mathcal{B}\} + \sup\{\int_B g\,d\mu : B \in \mathcal{B}\}$$

which proves that $\phi(f+g) \geq \phi(f) + \phi(g)$. Therefore ϕ is additive, and so by Theorem 2.20 ϕ extends uniquely to a positive linear functional on $L^1(\mu)$. Moreover, since

$$|\phi(f)| \leq \phi(|f|) = \sup\{\int_B |f|\,d\mu : B \in \mathcal{B}\} \leq \int_\Omega |f|\,d\mu = \|f\|_1$$

for all $f \in L^1(\mu)$ we have that $\phi \in (L_1(\mu))^*$. Now by our assumption there exists a function $0 \leq h \in L^\infty(\mu)$ such that $\phi(f) = \int_\Omega fh\,d\mu$ for all $f \in L^1(\mu)$. We put $K = \{x \in \Omega : h(x) > 0\}$.

For every $B \in \mathcal{B}$ we have that $\int_B \chi_{B\setminus K}\,d\mu \leq \phi(\chi_{B\setminus K}) = \int_\Omega \chi_{B\setminus K} h\,d\mu = 0$ which implies that $\mu(B \setminus K) = 0$. Suppose that $K' \subseteq \Omega$ is a measurable set such that $\mu(B \setminus K') = 0$ for all $B \in \mathcal{B}$. Then $\int_B \chi_{K\setminus K'}\,d\mu = 0$ for all $B \in \mathcal{B}$ which by the definition of ϕ implies that $\phi(\chi_{K\setminus K'}) = 0$. Then $\int_\Omega \chi_{K\setminus K'} h\,d\mu = 0$ which implies that $\mu((K \setminus K') \cap K) = 0$ and so $\mu(K \setminus K') = 0$. □

Lemma 4.11. *If μ is semi-finite then $L^1(\mu)$ is saturated and $(L^1(\mu))'$ is isometrically isomorphic to $L^\infty(\mu)$.*

Proof. Suppose that μ is semi-finite. To prove that $L^1(\mu)$ is saturated, pick a measurable set $A \subseteq \Omega$ such that $\|\chi_A\|_1 > 0$. Then we have that $\mu(A) > 0$ and since μ is semi-finite there exists a measurable set $B \subseteq A$ with $0 < \mu(B) < \infty$. Then $0 < \|\chi_B\|_1 = \mu(B) < \infty$. This proves that $L^1(\mu)$ is saturated.

To prove that $(L^1(\mu))'$ is isometrically isomorphic to $L^\infty(\mu)$, define $\rho(f) = \int_\Omega |f|\,d\mu$ for all $f \in M$. Note that ρ is a function norm and that $L_\rho = L^1(\mu)$. If $g \in L^\infty(\mu)$ then

$$\rho'(g) = \sup\{\int_\Omega |fg|\,d\mu : 0 \leq f \in L^1(\mu), \rho(f) \leq 1\} \leq \|g\|_\infty .$$

To prove the converse inequality let $(A_n)_n$ be a sequence of measurable sets of positive measure such that $|g| \geq \|g\|_\infty - \frac{1}{n}$ on A_n for all $n \in \mathbb{N}$. Since μ is semi-finite we can find a sequence of sets $(B_n)_n$ such that $B_n \subseteq A_n$ and $0 < \mu(B_n) < \infty$ for all $n \in \mathbb{N}$. Since $\left\|\frac{1}{\mu(B_n)}\chi_{B_n}\right\|_1 = 1$ for every $n \in \mathbb{N}$ we have that

$$\rho'(g) = \rho'(|g|) \geq \int_\Omega (\frac{1}{\mu(B_n)}\chi_{B_n})|g|\,d\mu = \frac{1}{\mu(B_n)}\int_{B_n} |g|\,d\mu \geq \|g\|_\infty - \frac{1}{n}$$

for all $n \in \mathbb{N}$ which shows that $\rho'(g) = \|g\|_\infty$ and hence that $L^\infty(\mu) \subseteq (L^1(\mu))'$.

To prove that $(L^1(\mu))' \subseteq L^\infty(\mu)$ pick $0 \leq g \in (L^1(\mu))'$. For every measurable set $B \subseteq \Omega$ with $0 < \mu(B) < \infty$ we have that

$$\infty > \rho'(g) \geq \int_\Omega (\frac{1}{\mu(B)}\chi_B)g\,d\mu = \frac{1}{\mu(B)}\int_B g\,d\mu.$$

Suppose that $\|g\|_\infty = \infty$. Then for every $k \in \mathbb{N}$ there exists a measurable set $B_k \subseteq \Omega$ such that $\mu(B_k) > 0$ and $g \geq k$ on B_k. Since μ is semi-finite we can choose B_k with $\mu(B_k) < \infty$ for all $k \in \mathbb{N}$. Then

$$\frac{1}{\mu(B_k)} \int_{B_k} g \, d\mu \geq \frac{1}{\mu(B_k)} \int_{B_k} k \, d\mu \geq k$$

for all $k \in \mathbb{N}$ which is a contradiction. Therefore $\|g\|_\infty < \infty$ and so $(L^1(\mu))' \subseteq L^\infty(\mu)$. □

Corollary 4.12. *There exists a measure μ such that $(L^1(\mu))_n^\sim = (L^1(\mu))^*$ and the canonical map from $(L^1(\mu))'$ to $(L^1(\mu))^*$ is not an isometric isomorphism.*

Proof. By Theorem 4.9 there exists a semi-finite measure μ which is not localizable. Since $L^1(\mu)$ is a σ-order continuous Banach function space, by Corollary 4.4 we have that $(L^1(\mu))_n^\sim = (L^1(\mu))^*$. Since μ is semi-finite, by Lemma 4.11 we have that $(L^1(\mu))' = L^\infty(\mu)$. Now from Theorem 4.10 we immediately obtain the latter conclusion of the corollary. □

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Index

5 Razširenji povzetek v slovenščini

5.1 Vektorske mreže

Skoraj vsak klasični Banachov prostor je opremljen z naravno urejenostjo, ki je usklajena z algebrskimi in topološkimi strukturami prostora. To delo preučuje vektorske mreže, ki so delno urejeni vektorski prostori, kjer sta urejenost in algebrska struktura združljiva. V prvem poglavju se razvije osnovna teorija o vektorskih mrežah in linearnih operatorjih. Sledi obravnava [3, Chapter 1, Chapter 4] avtorjev Aliprantis in Burkinshaw. Na samem začetku se predstavijo pojmi delno urejenega vektorskega prostora in vektorske mreže.

Definicija 5.1. *Delno urejen vektorski prostor* je vektorski prostor V, ki je obenem delno urejen s tako relacijo $\leq$, da veljata naslednja pogoja:

(a) če $x \leq y$, potem je $x + z \leq y + z$ za katerekoli $x, y, z \in V$;

(b) če $x \leq y$, potem je $cx \leq cy$ za katerekoli $x, y \in V$ in $c \in \mathbb{R}^+$.

Definicija 5.2. *Vektorska mreža* je delno urejen vektorski prostor E, kjer velja, da ima za vsaka vektorja $x, y \in E$ množica $\{x, y\}$ supremum in infimum.

Osnovni primer vektorske mreže je prostor $L^0(\Omega, \Sigma, \mu)$, ki je sestavljen iz vseh realnih merljivih funkcij na merljivem prostoru (Ω, Σ, μ).

Naj bo V delno urejen vektorski prostor. Vektor $x \in V$, ki zadošča $x \geq 0$, se imenuje *pozitiven* vektor. Množica $\{x \in V : x \geq 0\}$ se imenuje *pozitiven stožec* delno urejenega prostora V in je označena z V^+. Oznaka $x < y$, pomeni, da velja $x \leq y$ in $x \neq y$. Za množico $A \subseteq V$ pravimo, da je A *urejenostno omejena*, če obstaja tak vektor $y \in V$, da je $x \leq y$ za vse $x \in A$. Vektorju y pravimo *zgornja meja* množice A. Najmanjša zgornja meja množice A, če obstaja, se imenuje *supremum* množice A in je označena z oznako $\sup A$. Podobno obrazložimo tudi pojma *spodnje meje* in *infimuma* množice A. Podmnožica množice V, ki je navzgor in navzdol omejena, je *urejenostno omejena.* Za $x, y \in V$ množica $\{z \in V : x \leq z \leq y\}$ imenuje *interval,* ki je označen z oznako $[x, y]$.

Naj bo x element vektorske mreže. Definiramo $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ in $|x| = x \vee (-x)$ in jih zaporedoma imenujemo pozitivni del, negativni del in absolutna vrednost vektorja x.

Pojem posplšenega zaporedja se pojavlja v celotnem delu. Naj bo A delno urejena množica z lastnostjo, da za vse $x, y \in A$ obstaja tak $z \in A$, da je $x \leq z$ in $y \leq z$. Taka množica A se imenuje *navzgor usmerjena množica.* Če je M množica, imenujemo preslikavo $s \colon A \to M$ *posplošeno zaporedje.* V posebnem primeru je identiteta na množici A posplošeno zaporedje. Za $\alpha \in A$ označimo vrednost $s(\alpha)$ z s_α in označimo posplošeno zaporedje s kot (s_α), $(s_\alpha)_\alpha$ ali $(s_\alpha)_{\alpha \in A}$. Pravimo, da posplošeno zaporedje (x_α) v vektorski mreži E *narašča*, če iz $\alpha \leq \beta$ sledi $x_\alpha \leq x_\beta$, kar zapišemo $x_\alpha \uparrow$. Če je $\sup_\alpha x_\alpha = x$, zapišemo $x_\alpha \uparrow x$. Podobno definiramo pomen $x_\alpha \downarrow$ in $x_\alpha \downarrow x$.

Vektorska mreža E se imenuje *(σ-)Dedekindovo polna* če ima vsaka neprazna (števna) navzgor omejena podmnožica množice E supremum. Vektorska mreža je *arhimedska*, če je $\frac{1}{n}x \downarrow 0$ za vsak $x \in E^+$.

Pojma ideala in pasa, ki sta podprostora vektorske mreže s posebnimi lastnostmi, sta zelo pomembna. Podmnožica A vektorske mreže E se imenuje *strnjena* če za $x \in A$ in $y \in E$ iz $|y| \leq |x|$ sledi $y \in A$. Vektorski podprostor E, ki je strnjen, se imenuje *ideal*.

Pravimo, da posplošeno zaporedje $(x_\alpha)_\alpha$ v vektorski mreži E *urejenostno konvergira k* x, če obstaja tako posplošeno zaporedje $(y_\alpha)_\alpha \downarrow 0$ v E z isto indeksno množico, da je $|x_\alpha - x| \leq y_\alpha$ za vsak α. Če $(x_\alpha)_\alpha$ urejenostno konvergira k x, zapišemo $x_\alpha \xrightarrow{o} x$. Podmnožica $A \subseteq E$ se imenuje *urejenostno zaprta*, če iz $(x_\alpha)_\alpha \subseteq A$ in $x_\alpha \xrightarrow{o} x$ sledi $x \in A$. Urejenostno zaprt ideal se imenuje *pas*.

Teorija linearnih operatorjev med vektorskimi mrežami je pomembna za to delo, ker se nanaša na obravnavo linearnih funkcionalov in dualnih prostorov. Pozitivne preslikave imajo pomembno vlogo, saj se operator ali funkcional zelo pogosto razcepi kot razlika med svojim pozitivnim in negativnim delom na podoben način, kot se merljiva funkcija razcepi kot razlika med svojim pozitivnim in negativnim delom. Če sta E in F vektorski mreži in je $T\colon E \to F$ linearna preslikava, pravimo, da je T *pozitivna*, če je Tx pozitiven za vse $x \in E^+$. Obravnavo linearnih operatorjev med vektorskimi mrežami pričnemo s Kantorovičevim razširitvenim izrekom.

Izrek 5.3 (Kantorovič)**.** *Naj bosta E in F vektorski mreži, kjer je F arhimedska, in naj bo $T\colon E^+ \to F^+$ aditivna preslikava. Tedaj se T na en sam način razširi do pozitivnega operatorja $S\colon E \to F$, za katerega velja*

$$Sx = Tx^+ - Tx^-$$

za vsak $x \in E$.

Realni vektorski prostor vseh linearnih operatejev iz E v F označimo z $\mathcal{L}(E, F)$. Če opremimo $\mathcal{L}(E, F)$ z urejenostjo $\leq$, ki je definirana kot

$$T \leq S \;\;\Leftrightarrow\;\; Tx \leq Sx \;\text{ za vsak }\; x \in E^+,$$

$\mathcal{L}(E, F)$ postane delno urejen vektorski prostor.

Linearni operator $T\colon E \to F$ imenujemo *urejenostno omejen*, če preslika urejenostno omejene podmnožice v E v urejenostno omejene podmnožice v F. Zelo enostavno lahko opazimo, da je množica vseh urejenostno omejenih operatorjev iz E v F vektorski podprostor v $\mathcal{L}(E, F)$. Vektorski prostor vseh urejenostno omejenih operatorjev označimo z $\mathcal{L}_b(E, F)$. Dokažemo, da je $\mathcal{L}_b(E, F)$ Dedekindovo polna vektorska mreža, če je F Dedekindovo polna. Prav tako dokažemo Kantorovičeve formule za mrežne operacije.

Izrek 5.4 (Riesz-Kantorovič)**.** *Naj bosta E in F vektorski mreži, kjer je F Dedekindovo polna. Tedaj je vektorski prostor $\mathcal{L}_b(E, F)$ Dedekindovo polna vektorska mreža. Mrežni operaciji zadoščata*

$$(S \vee T)x = \sup\{Sy + Tz : y, z \in E^+, y + z = x\}$$

in

$$(S \wedge T)x = \inf\{Sy + Tz : y, z \in E^+, y + z = x\}$$

za vsak $x \in E^+$. Prav tako imamo

$$T_\alpha \downarrow 0 \text{ v } \mathcal{L}_b(E, F) \text{ natanko tedaj, ko je } T_\alpha x \downarrow 0 \text{ za vsak } x \in E^+.$$

Definicija 5.5. Operator $T\colon E \to F$ med vektorskimi mrežami se imenuje:

(a) *urejenostno zvezen*, če je $Tx_\alpha \xrightarrow{o} 0$ kadarkoli je $x_\alpha \xrightarrow{o} 0$;

(b) *σ-urejenostno zvezen*, če je $Tx_n \xrightarrow{o} 0$ kadarkoli je $x_n \xrightarrow{o} 0$.

Množici urejenostno zveznih operatorjev in σ-urejenostno zveznih operatorjev zaporedoma označimo z $\mathcal{L}_n(E,F)$ in $\mathcal{L}_c(E,F)$. Izkaže se, da sta $\mathcal{L}_n(E,F)$ in $\mathcal{L}_c(E,F)$ pasova v $\mathcal{L}_b(E,F)$.

Nato obravnavamo linearne funkcionale. Naj bo E vektorska mreža. Vektorski prostor $\mathcal{L}(E,\mathbb{R})$ vseh linearnih funkcionalov na E je delno urejen vektorski prostor z urejenostjo

$$f \leq g \quad \Leftrightarrow \quad f(x) \leq g(x) \text{ za vsak } x \in E^+.$$

Ker je množico realnih števil Dedekindovo polna vektorska mreža, je vektorski prostor vseh urejenostno omejenih linearnih funkcionalov $\mathcal{L}_b(E,\mathbb{R})$ Dedekindovo polna vektorska mreža. Označimo ga z $E^\sim$. Prostor $\mathcal{L}_n(E,\mathbb{R})$ vseh urejenostno zveznih linearnih funkcionalov na E označimo z $E_n^\sim$, prostor $\mathcal{L}_c(E,\mathbb{R})$ vseh σ-urejenostno zveznih funkcionalov na E pa označimo z $E_c^\sim$. Prostora $E_n^\sim$ in $E_c^\sim$ sta pasova v $E^\sim$.

Polnorma (norma) p na vektorski mreži E se imenuje *mrežna polnorma (norma)*, če iz $|x| \leq |y|$ sledi $p(x) \leq p(y)$ za vse $x, y \in E$. Vektorsko mrežo, opremljeno z mrežno normo $\|\cdot\|$, imenujemo *normirana mreža.* Če je prostor E poln glede na mrežno normo, potem E imenujemo *Banachova mreža.*

Vektorski prostor vseh zveznih linearnih funkcionalov na normirani mreži E označimo z E^*. Izkaže se, da je vsak zvezni funkcional na E tudi urejenostno omejen. Če je E Banachova mreža, potem drži tudi nasprotno.

Posledica 5.6. *Za Banachovo mrežo E velja $E^* = E^\sim$.*

Definicija 5.7. Naj bo p mrežna polnorma na vektorski mreži E. Pravimo, da ima p:

(a) *krepko Fatoujevo lastnost*, če v primeru, ko velja $x_\alpha \uparrow$ v E^+ in $\sup_\alpha p(x_\alpha) < \infty$, obstaja tak $x \in E^+$, da je $x_\alpha \uparrow x$ in $p(x_\alpha) \uparrow p(x)$;

(b) *Fatoujevo lastnost* če iz $x_\alpha \uparrow x$ v E^+ sledi $p(x_\alpha) \uparrow p(x)$;

(c) *šibko Fatoujevo lastnost* če iz $x_\alpha \uparrow x$ v E^+ in $\sup_\alpha p(x_\alpha) < \infty$ sledi $p(x) < \infty$.

Če je E Banachova mreža, imajo $E^\sim, E_n^\sim$ in $E_c^\sim$ krepko Fatoujevo lastnost.

Zadnji del 2. poglavja je kratka predstavitev urejenostno zveznih norm in urejenostno zveznih Banachovih mrež.

Definicija 5.8. Mrežna polnorma p na vektorski mreži E se imenuje:

(a) *urejenostno zvezna*, če iz $x_\alpha \downarrow 0$ sledi $p(x_\alpha) \downarrow 0$;

(b) *σ-urejenostno zvezna*, če iz $x_n \downarrow 0$ sledi $p(x_n) \downarrow 0$.

5.2 Normirani funkcijski prostori

V 3. poglavju obravnavamo funkcijske prostore, ki so normirani prostori, sestavljeni iz merljivih funkcij. G.Köthe in O.Toeplitz sta obravnavala primer prostorov zaporedij. Zato so normirani funkcijski prostori včasih poimenovani tudi Köthejevi prostori. Predstavimo Riesz-Fischerjevo lastnost, nasičene funkcijske polnorme in pridruženi prostor normiranemu funkcijskemu prostoru, ki je del glavnega rezultata tega dela. V večini sledimo obravnavi [8, Chapter 15] avtorja Zaanen.

Naj bo (Ω, Σ, μ) merljiv prostor in naj bo M množica vseh realnih merljivih funkcij v Ω. Če opremimo M z urejenostjo

$$f \leq g \Leftrightarrow \text{ je } f(x) \leq g(x) \text{ za skoraj vsak } x \in \Omega,$$

potem M postane vektorska mreža.

Definicija 5.9. *Funkcijska polnorma* je preslikava $\rho \colon M^+ \to [0, \infty]$, ki izpolnjuje naslednje pogoje za $u, v \in M^+$:

(a) če je $u = 0$, potem je $\rho(u) = 0$;

(b) $\rho(cu) = c\rho(u)$ za vse $c \in \mathbb{R}^+$;

(c) $\rho(u + v) \leq \rho(u) + \rho(v)$;

(d) če $u \leq v$, potem je $\rho(u) \leq \rho(v)$.

Če iz $\rho(u) = 0$ sledi $u = 0$, se ρ imenuje *funkcijska norma.*

Zanimajo nas predvsem funkcijske norme. Predpis funkcijske norme ρ lahko razširimo na M z $\rho(f) = \rho(|f|)$. Definirajmo $L_\rho = \{f \in M : \rho(f) < \infty\}$.

Lema 5.10. *Množica L_ρ je ideal v M in ρ je norma na L_ρ.*

Normirani prostori oblike L_ρ se imenujejo *normirani funkcijski prostori.* Naj bo ρ funkcijska polnorma na M^+. Če je $\rho(f) = 0$ za vsak $f \in M^+$ ali $\rho(f) = \infty$ za vsak $0 < f \in M^+$, je ρ *trivialna* funkcijska polnorma. Torej, ρ je netrivialna, če in samo če obstaja taka funkcija $0 < f \in M^+$ da je $0 < \rho(f) < \infty$. Če je ρ netrivijalna, lahko še vedno obstajajo merljive podmnožice $E \subseteq \Omega$ z $\mu(E) > 0$, da je $\rho(\chi_F) = \infty$ za vsako podmnožico $F \subseteq E$ z $\mu(F) > 0$. Podmnožica $E \subseteq \Omega$ s to lastnostjo se imenuje *ρ-popolnoma neskončna množica.* Končne in števne unije ρ-popolnoma neskončnih množic so ponovo ρ-popolnoma neskončne.

Trditev 5.11. *Podmnožica $E \subseteq \Omega$ z $\mu(E) > 0$ je ρ-popolnoma neskončna natanko tedaj, ko je vsaka funkcija $f \in M^+$ z katerego velja $\rho(f) < \infty$, skoraj povsod ničelna na E.*

Definicija 5.12. Funkcijska polnorma ρ je *nasičena*, če ne obstaja ρ-popolnoma neskončna množica.

Definicija 5.13. Mera μ na merljivim prostoru (Ω, Σ) se imenuje:

(a) *semi-končna*, če za vsako merljivo množico $A \subseteq \Omega$ s pozitivno mero obstaja merljiva podmnožica $B \subseteq A$ s končno pozitivno mero;

(b) *lokalizabilna*, če je μ semi-končna in za katerokoli družino $\Delta \subseteq \Sigma$ obstaja taka merljiva množica $C \subseteq \Omega$, da velja:

   (i) $\mu(A \setminus C) = 0$ za vsak $A \in \Delta$;

   (ii) če je C' je merljiva množica, za katero velja $\mu(A \setminus C') = 0$ za vsak $A \in \Delta$, potem je $\mu(C \setminus C') = 0$.

Izrek 5.14. *Če je μ lokalizabilna, potem je M Dedekindovo polna.*

Definicija 5.15. Za funkcijsko polnormo ρ definiramo zaporedje funkcijskih polnorma $\rho^{(n)}, (n \in \mathbb{N}_0)$ induktivno, na naslednji način. Naj bo $\rho^{(0)} = \rho$. Če je $\rho^{(n)}$ že definirana, definiramo $\rho^{(n+1)}$ z

$$\rho^{(n+1)}(f) = \sup\{\int_\Omega fg d\mu : g \in M^+, \rho^{(n)}(g) \leq 1\}$$

za vse $f \in M^+$.

Naj bo ρ nasičena funkcijska norma. Tedaj je $L_{\rho'}$ Banachov prostor, ki ga označimo z L'_ρ. Imenujemo ga *pridruženi prostor* prostora L_ρ.

Če je $f \in L'_\rho$, potem f definira linearni funkcional Λ_f na L_ρ s predpisom

$$\Lambda_f(g) = \int_\Omega fg d\mu.$$

Trditev 5.16. *Če je $f \in L'_\rho$, potem je Λ_f omejen in $\|\Lambda_f\| = \rho'(f)$.*

Preslikava $\Lambda\colon f \mapsto \Lambda_f$ je pozitivna linearna izometrija iz L'_ρ v L^*_ρ. Ker je L'_ρ Banachov prostor, lahko L'_ρ smatramo kot zaprt podprostor v L^*_ρ. Če je μ σ-končna in če je ρ nasičena, potem je $L'_\rho \cong (L_\rho)^\sim_n$.

5.3 Glavni rezultati

V 4. poglavju v večini obravnavamo povezavo med pridruženim prostorom normiranega ali Banachovega funkcijskega prostora E in njegovimi dualnimi prostori. To poglavje je povzeto po [1].

Izrek 5.17. *Naj bo E nasičen Banachov funkcijski prostor glede na semi-končno mero μ. Potem je E' izometrično izomorfen $E^\sim_n$ natanko tedaj, ko ima E' krepko Fatoujevo lastnost.*

Posledica 5.18. *Naj bo E σ-urejenostno zveznen nasičen Banachov funkcijski prostor glede na semi-končno mero μ. Potem je E' izometrično izomorfen E^* natanko tedaj, ko ima E' krepko Fatoujevo lastnost.*

Posledica 5.19. *Naj bo E nasičeni Banachov funkcijski prostor glede na lokalizabilno mero μ. Potem je E' izometrično izomorfen $E^\sim_n$.*

Posledica 5.20. *Naj bo E σ-urejenostno zveznen nasičen Banachov funkcijski prostor glede na lokalizabilno mero μ. Potem je E' izometrično izomorfen E^*.*

Na koncu podamo primer Banachovega funkcijskega prostora, kjer enakosti med dualnimi prostori v predhodno omenjenih rezultatih ne veljajo.

Posledica 5.21. *Obstaja taka mera μ, da je $(L^1(\mu))^\sim_n \cong (L^1(\mu))^*$, kanonična preslikava med $(L^1(\mu))'$ in $(L^1(\mu))^*$ pa ni izomorfizem.*