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Automorphism Groups and Elliptic Complex Geometry

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Grupe Avtomorfizmov in Eliptična Kompleksna Geometrija

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Abstract

This dissertation presents the results obtained by the author during the course of his doctoral studies. The first two chapters provide a detailed introduction to the Andersén-Lempert theory which can be used as an introduction to the theory of holomorphic automorphisms of complex Euclidean spaces and more in general of Stein manifolds with the density property. The latter are complex manifolds admitting a large group of holomorphic automorphisms.
Afterwards, we move to the original results concerning parametric jet interpolation by automorphisms. We first provide a complete picture in complex Euclidean spaces and then move to the more technical result for Stein manifolds with the density property. It is at this point that we focus on the topic of tame sets, recalling some of the classical theory developed by Rosay and Rudin and proceeding with the new results. Here, we will focus mostly on linear algebraic Lie groups and the action of holomorphic vector fields. Of particular importance will be the study of the ring of invariant functions for such vector fields.
In the last chapter, we present questions that arose during the described research and suggest future lines of exploration in this field.

Math. Subj. Class. (2010): 32M17, 32M25, 32M05
Keywords: Holomorphic Automorphism, Jet Interpolation, Tame Set

Povzetek

Disertacija vsebuje predstavitev originalnih rezultatov avtorjevega raziskovalnega dela, dobljenih tekom doktorskega študija.

Prvi dve poglavji sta uvodne narave in vsebujeta povzetek nekaterih splošnih pojmov iz teorije Steinovih mnogoterosti ter podroben prikaz teorije Andersén-Lempert, ki obravnava holomorfne avtomorfizme kompleksnih evklidskih prostorov ter sorodnih kompleksnih mnogoterosti z veliko grupo holomorfnih avtomorfizmov.

Tretje poglavje vsebuje avtorjeve originalne rezultate o interpolaciji neizrojenih brstičev holomorfnih preslikav z družinami holomorfnih avtomorfizmov, ki so holomorfno odvisni od parametra v neki Steinovi mnogoterosti. V četrtem poglavju obravnavamo tehnično zahtevne posplošitve omenjenih interpolacijskih rezultatov za avtomorfizme Steinovih mnogoterosti z Varolinovo lastnostjo gostote.

Peto poglavje je posvečeno vpeljavi in obravnavi novega pojma (strogo) pohlevnih diskretnih množic v Steinovih mnogoterostih z lastnostjo gostote. Uvodoma pokažemo, da se na kompleksnih evklidskih prostorih novo vpeljani pojem ujema s klasičnim pojmom pohlevnih diskretnih množic, ki sta jih obravnala Rosay in Rudin leta 1988. V disertaciji se osredotočimo predvsem na pohlevne množice v linearnih algebraičnih kompleksnih Liejevih grupah ter na delovanja kompletnih holomorfnih vektorskih polj na njih. Posebnega pomena je študij kolobarjev invariantnih funkcij takih vektorskih polj.

V zadnjem šestem poglavju predstavimo in diskutiramo vrsto odprtih problemov, ki so se porodili tekom naših raziskav, ter nakažemo nekaj možnih smeri razvoja nadaljnjih raziskav na tem področju.

Math. Subj. Class. (2010): 32M17, 32M25, 32M05

Keywords: holomorfni avtomorfizem, brstič, interpolacija, pohlevna množica

Contents

1

Introduction and Motivation

The present dissertation reports on the progress made by the author in the field of Several Complex Variables during his doctoral studies, more specifically in the Andersén-Lempert theory. The latter is an area of elliptic complex geometry devoted to the study of complex manifolds with large automorphism groups, such as the one of complex Euclidean spaces. Here we provide a coherent account of the original results which have already been published or accepted in peer-reviewed research journals. We also give an introduction to the Andersén-Lempert theory, which can be used to present this field to graduate students.

This dissertation is organized as follows. In this first chapter we begin with a historical tour of the field, then proceed to state the main theorems obtained during the doctoral studies. Chapter 2 offers a detailed introduction to the Andersén-Lempert Theorem. There we state all the results needed to proceed with the following three chapters, which contain the original results and their proofs. It is in Chapter 6 that we discuss new prospects and formulate some unanswered questions related to the presented theorems.

The reader who is already knowledgeable about the Andersén-Lempert theory can skip Chapter 2 and go back to it if necessary, while we suggest the reader interested in learning more about this subject to focus on the first three sections of Chapter 2 and then integrate with [20, Chapter 4]. Another self-contained great source is the survey paper [32] by Kaliman and Kutzchebauch.

1.1 Elliptic Complex Geometry

The term *elliptic complex geometry* has been used in recent years to identify a wide range of results in Complex Geometry pertaining to complex manifolds enjoying one of many properties describing the holomorphic flexibility of the space. The informal term *flexibile* here refers to many different interacting characteristics a space might or might not have. The term *elliptic* is meant to emphasize the contrast with hyperbolic manifolds, which are usually very *rigid* in the holomorphic category. Elliptic complex geometry studies complex manifolds which are natural sources and targets of holomorphic maps. Depending on the specific interest, a manifold considered in this field will admit many automorphisms, many holomorphic functions from it, and/or many holomorphic maps into it. Typical theorems include approximation and interpolation results often depending on a parameter. The addition of the latter is usually not an easy task but it allows to discuss the topology of spaces of maps which are almost always infinite dimensional and very hard to study otherwise.

The first use of the word *elliptic* in this context appeared in the work of Gromov [26], where he says that "*Intuitively, a space Y is elliptic if it contains sufficiently many $\mathbb{C}$-lines that are holomorphic maps $\mathbb{C} \to Y$*" . This property then stands at the opposite side of Brody's hyperbolicity, which requires that there are no non-constant $\mathbb{C}$-lines in the manifold. Gromov's main motivation was the study of the h-principle introduced by Grauert in [22], where he proved that every continuous map from a Stein manifold (see Definition 2.2.2) into a complex Lie group can be deformed to a holomorphic one. The possibility to deform continuous maps to holomorphic ones is in fact an instance of a h-principle; it is usually named *Oka-Grauert principle*. Gromov took this discussion one step further by studying principal holomorphic fiber bundles over Stein spaces with a Lie group as fiber. He described the Oka-Grauert principle with these words: "*Oka's principle (as interpreted by the author) is an expression of an optimistic expectation with regard to the validity of the h-principle for holomorphic maps in the situation where the source manifold is Stein*" [26].

The class of Stein manifolds is a well-understood collection of complex manifolds satisfying many equivalent properties which will be discussed in Section 2.2. Intuitively, a complex manifold X is Stein if it admits sufficiently many holomorphic functions $X \to \mathbb{C}$. Observe that this notion is in some sense dual to the one described by Gromov; the precise nature of this duality has been described by Lárusson in the language of modern homotopy theory [40, 41]. Stein manifolds can be considered as the holomorphic analogue of affine algebraic manifolds; indeed a manifold is Stein if and only if it can be properly embedded into a complex Euclidean space of sufficiently large dimension [51].

In the past twenty years, researchers intensively investigated various holomorphic flexibility properties of complex manifolds and relationships between them. The most popular ones include the *density property* [59], the *Oka property* [15], *dominability* [20], *ellipticity* [26], and *subellipticity* [14]. The following relations are well known [20]:

$$\text{ellipticity} \Rightarrow \text{subellipticity} \Rightarrow \text{Oka property} \Rightarrow \text{strong dominability}$$

All these properties hold on a Stein manifold with the density property. It is a question of general interest which of the above implications can be reversed. Even a brief presentation of each property goes beyond the purpose of this work, hence we will focus only on the ones relevant to us.

1.2 Andersén-Lempert Theory

The study of holomorphic automorphisms of complex Euclidean spaces originated in the early part of the 20th century in the seminal works of Fatou, Bieberbach and others. The highly influential work of Rosay and Rudin [52] from 1988 marked the beginning of a new period in this subject. Soon thereafter, groundbreaking work by Andersén [2] (1990) and Andersén and Lempert [3] (1992) opened a new direction by studying the *infinite dimensional Lie algebra* of the group of holomorphic automorphisms Aut($\mathbb{C}^n$). A more useful formulation of their result was given by Forstnerič and Rosay [19] (1993), where they focused on automorphisms given by flows of complete holomorphic vector fields and their compositions. This work was further developed and applied to concrete problems during 1990's by Forstnerič and his collaborators [8, 12, 13, 16]. The main results have been generalized to *Stein manifolds* satisfying the *density property*, introduced by Varolin in 2000 [58, 59]. The density property was subsequently investigated in a series of works by Tóth and Varolin [55, 56], Kaliman and Kutzschebauch [30, 31, 32, 36], and many other researchers. The Andersén-Lempert theory is currently one of the most active fields in complex analysis. For a more complete account and references we refer to [20, Chapter 4].

While the only holomorphic automorphisms of $\mathbb{C}$ are those of the form $z \mapsto az+b$ for $a, b \in \mathbb{C}, a \neq 0$, the group of holomorphic automorphisms of $\mathbb{C}^n$ for $n > 1$ is very big. It is immediate that maps of the form

$$z = (z_1, \dots, z_n) \mapsto (z_1 + f(z_2, \dots, z_n), z_2, \dots, z_n)$$
$$z \mapsto (z_1 e^{f(z_2,\dots,z_n)}, z_2, \dots, z_n)$$

are holomorphic automorphisms of $\mathbb{C}^n$ for any holomorphic function in $n-1$ complex variables f; it was shown that these maps together with the linear group $\mathrm{GL}_n(\mathbb{C})$ generate a dense subgroup of the group of all holomorphic automorphisms of $\mathbb{C}^n$ [3]. This result is the beginning of the Andersén-Lempert theory. Such maps and their $\mathrm{GL}_n(\mathbb{C})$-conjugates are usually called *shears*. More generally, Forstnerič and Rosay [19] showed that any smooth isotopy of biholomorphic maps between Runge domains in $\mathbb{C}^n$ for $n > 1$ which begins at $t = 0$ with the identity map is approximable by automorphisms of $\mathbb{C}^n$. The precise result (Theorem 2.3.1), together with a more detailed discussion, will be formulated in Section 2.3.

Theorem 2.3.1 does not only apply to complex Euclidean spaces, in fact the Andersén-Lempert theory pertains to the large class of Stein manifolds with the density property, a notion introduced by Varolin in 2001. Recall that a holomorphic vector field is *complete* if its flow exists for all complex values of the time variable.

Definition 1.2.1. [59] A complex manifold has the *density property* if every holomorphic vector field can be approximated (uniformly on compacts) by a Lie combination of complete holomorphic vector fields.

This property alone might not be very significant; for instance, every compact complex manifold has the density property since every vector field is complete. The situation becomes substantially more interesting and nontrivial when we require the manifold to be Stein. Namely, the interaction between the many holomorphic functions and the many complete holomorphic vector fields produces an infinite dimensional group of automorphisms.

Examples of Stein manifolds with the density property include $\mathbb{C}^n$ for $n > 1$ [3], $\mathbb{C}^n \times \mathbb{C}^{*k}$ for n and $k \geq 1$ [59], algebraic Lie groups and their affine-algebraic homogeneous spaces different from $\mathbb{C}$ or $\mathbb{C}^{*n}$ [11, 33, 56], surfaces in $\mathbb{C}^3$ of the form $uv = f(x)$ for holomorphic functions f with reduced zero fiber [31], Gizatullin surfaces [5] and the Koras-Russell cubic [42].

There are two main aspects one can consider when studying the density property. On the one hand, it is important to know which Stein manifolds enjoy the density property; on the other hand, we want to understand which properties of our model space $\mathbb{C}^n$ depend only on the density property, in the sense that we would like to replicate results valid for complex Euclidean spaces to an arbitrary Stein manifold with the density property. This work deals mainly with this second type of results.

1.3 Jet Interpolation

It is elementary that the group $\mathrm{Aut}(\mathbb{C}^n)$ of holomorphic automorphisms of $\mathbb{C}^n$ acts m-transitively for every $m \in \mathbb{N}$ (see for instance [52]).

Definition 1.3.1. Let G be a group acting on a set X and let $m \in \mathbb{N}$. The action is m-transitive if for every pair of m-tuples $\{a_j\}_{j=1}^m, \{b_j\}_{j=1}^m \subset X$ of distinct elements of X, there exists $g \in G$ such that $ga_j = b_j$ for $j = 1, \ldots, m$.

In 1999 Forstnerič sharpened this result by showing that not only we can prescribe a finite number of values of a holomorphic automorphism, but we can also prescribe derivatives up to some finite order.

Theorem 1.3.2. [13, Corollary 2.2] *Let $n > 1$. Given finite subsets $\{a_j\}_{j=1}^m, \{b_j\}_{j=1}^m \subset \mathbb{C}^n$ without repetition and for each $j = 1, \ldots m$ a polynomial $P_j \colon \mathbb{C}^n \to \mathbb{C}^n$ of degree at most $r_j \geq 1$ satisfying $P_j(0) = 0$ and $JP_j(0) \neq 0$, there exists an automorphism $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that for each $j = 1, \ldots, m$ we have $F(a_j) = b_j$ and*

$$F(z) = b_j + P_j(z - a_j) + O(|z - a_j|^{r_j+1}), \quad z \to a_j.$$

Note that we are prescribing the Taylor expansion of F at the points a_j, $j = 1, \ldots, m$ up to the finite order r_j. If we wish to replicate a similar result in an arbitrary Stein manifold with the density property, it is important to understand how to replace the polynomials P_j.

Definition 1.3.3. Let X be a complex manifold and $p \in X$ a point. Given a natural number $r \in \mathbb{N}$ and a pair of holomorphic functions $F, G : U \to X$ in a neighbourhood

U of p, they have the same r-jet at p if $|F(z)-G(z)| = O(|z|^{r+1})$, $z \to 0$ in some (hence in any) local chart centered at p. This defines an equivalence relation for germs and its equivalence classes are called r-jets. We define a jet to be non-degenerate if its linear part (the Jacobian matrix of any representative) has non-zero determinant. We denote by $[F]_p^r$ the equivalence class of F at p and by $J_{p,q}^r(X)$ the space of all non-degenerate r-jets at p with image q (the image of a jet is well-defined as it is an equivalence class of germs).

Once we choose charts for both p and $F(p)$, a r-jet is uniquely identified by a polynomial of degree at most r, representing its Taylor expansion up to that order.

The following result is due to Varolin who realized that the possibility to interpolate jets by automorphisms is not exclusive to complex Euclidean spaces.

Theorem 1.3.4. [58, Theorem 2] *Let X be a Stein manifold with the density property. Given finite subsets $\{a_j\}_{j=1}^m, \{b_j\}_{j=1}^m \subset X$ without repetition and for each $j = 1, \dots m$ a non-degenerate r_j-jet $\gamma_j \in J_{a_j,b_j}^{r_j}(X)$ with $r_j \in \mathbb{N}$, $j = 1, \dots m$, there exists an automorphism $F \in \mathrm{Aut}(X)$ such that for each $j = 1, \dots, m$ we have $[F]_{a_j}^{r_j} = \gamma_j$.*

The proofs of these two results are quite different, even if they present some similarities. In the case of complex Euclidean spaces, one relies on the well-known shear automorphisms of the form $(z, w) \mapsto (z, we^{f(z)} + g(z))$ and their $\mathrm{GL}_n(\mathbb{C})$ conjugates for holomorphic f and g with specific properties obtained via a Runge-Weiestrass theorem. In an arbitrary Stein manifold with the density property, we do not have such explicit automorphisms and we need to use the full strenght of the Andersén-Lempert Theorem (Theorem 2.3.1).

These two basic jet interpolation theorems allowed to construct holomorphic automorphisms with prescribed dynamical behaviour. They were applied in many subsequent works, examples being the theorem of Peters and Wold on non-autonomous basins of attraction of automorphisms [49], the work of Forstnerič, Ivarsson, Kutzschebauch and Prezelj on holomorphic embeddings of Stein manifolds into Euclidean spaces with interpolation on a discrete set [17], and many others.

It is natural to ask what happens when we have an infinite number of points ($m = +\infty$). It is clear that we should restrict ourselves to sequences without accumulation points, otherwise interpolation would not be possible for topological reasons. This question was already considered by Rosay and Rudin in 1988 [52], when they discovered that not all closed infinite discrete sequences in $\mathbb{C}^n$ are equivalent under the action of $\mathrm{Aut}(\mathbb{C}^n)$. In fact, they proved the existence of continuum many equivalence classes of such sequences [52, Corollary 5.3]. Nevertheless, they identified a specific class with good properties with respect to the $\mathrm{Aut}(\mathbb{C}^n)$-action.

Definition 1.3.5. [52] Let e_1 be the first standard basis vector of $\mathbb{C}^n$. A closed infinite discrete sequence $\{a_j\}_{j\geq 1} \subset \mathbb{C}^n$ is called *tame* (in the sense of Rosay and Rudin) if there exists $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that $F(a_j) = j \cdot e_1$ for all $j \geq 1$.

The group of holomorphic automorphisms of $\mathbb{C}^n$ acts transitively on $\mathbb{N} \cdot e_1$, hence it makes sense to talk about tame sets instead of tame sequences. One of the main ways to recognize tame sequences is to look at their projections onto lower dimensional subspaces: every sequence which projects to a discrete set in a lower dimensional subspace is tame. For this reason, it is possible to construct shears with prescribed values at points of a tame sequence. This observation together with a convergence result for sequences of automorphisms (Theorem 2.4.1) [13] led to the following jet interpolation theorem at a tame sequence for holomorphic automorphisms of $\mathbb{C}^n$.

Theorem 1.3.6. [8] *Let $n > 1$. Given tame sets $\{a_j\}_{j\geq 1}, \{b_j\}_{j\geq 1} \subset \mathbb{C}^n$ and for each $j \in \mathbb{N}$ a polynomial $P_j\colon \mathbb{C}^n \to \mathbb{C}^n$ of degree at most $r_j \geq 1$ satisfying $P_j(0) = 0$ and $JP_j(0) \neq 0$, there exists an automorphism $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that for each $j \in \mathbb{N}$ we have $F(a_j) = b_j$ and*

$$F(z) = b_j + P_j(z - a_j) + O(|z - a_j|^{r_j+1}), \quad z \to a_j.$$

As mentioned at the beginning of the chapter, it is important to study these problems in the presence of a parameter in order to better understand the structure of the group of holomorphic automorphisms. In the following theorem, which is the first main result of this dissertation, the parameter will be in a *finite dimensional Stein space*; a Stein space is finite dimensional if its smooth part has finite dimension. Note that this does not imply that it has finite embedding dimension. All the spaces considered in this dissertation are assumed to be reduced.

Theorem 1.3.7. [57] *Let W be a finite dimensional Stein space, $\{a_j\}_{j\in\mathbb{N}}, \{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n$ $(n > 1)$ be tame sequences of points and $r_j \in \mathbb{N}_{>0}$, $j > 0$ be a sequence of natural numbers. For every $j \in \mathbb{N}$, let $P_j\colon W \to J^{r_j}_{a_j,b_j}(\mathbb{C}^n)$ be a holomorphic family of non-degenerate r_j-jets such that $P_j^w(a_j) = b_j$ for all $w \in W$. Then there exists a null-homotopic holomorphic map $F\colon W \to \mathrm{Aut}(\mathbb{C}^n)$ such that*

$$F^w(z) = P_j^w(z) + O(|z - a_j|^{r_j+1}) \quad \text{for } z \to a_j,\ j \in \mathbb{N},\ w \in W$$

if and only if the linear part map $Q_j\colon W \to GL_n(\mathbb{C})$ of P_j at the point a_j is null-homotopic for every $j \in \mathbb{N}$.

As the group $\mathrm{Aut}(\mathbb{C}^n)$ is not a complex manifold, we have to define what it means for the map $F\colon W \to \mathrm{Aut}(\mathbb{C}^n)$ to be holomorphic. Such a map F is holomorphic when the evaluation map

$$\begin{aligned} W \times \mathbb{C}^n &\to \mathbb{C}^n \\ (w, z) &\mapsto F^w(z) \end{aligned}$$

is holomorphic in the usual sense.

The above result will be proved and discussed in Chapter 3. It describes an Oka property of the group Aut($\mathbb{C}^n$). The main results concerning Oka properties of Aut($\mathbb{C}^n$) and many of its subgroups are due to Forstnerič and Lárusson [18].

When we have only a finite number of families of jets, all tangent to the identity map at the points $a_1, \ldots, a_m \in \mathbb{C}^n$, Theorem 1.3.7 was established by Kutzschebauch and Lodin [38, Lemma A.4].

While following the same idea of proof of Theorem 1.3.6, Theorem 1.3.7 is highly non-trivial. In particular, it depends on the solution to the so-called holomorphic Vaserstein problem given by Ivarsson and Kutzschebauch in a spectacular application of Oka theory:

Theorem 1.3.8. [28] *Let W be a finite dimensional Stein space and $f : W \to SL_n(\mathbb{C})$ be a null-homotopic holomorphic mapping. Then there exist an integer $K \in \mathbb{N}$ and holomorphic mappings*

$$G_1, \ldots, G_K : W \to \mathbb{C}^{n(n-1)/2}$$

such that f can be written as a product of upper and lower diagonal unipotent matrix functions of $w \in W$:

$$f(w) = \begin{pmatrix} 1 & 0 \\ G_1(w) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(w) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_K(w) \\ 0 & 1 \end{pmatrix}.$$

In Section 3.2 we consider families of jets fixing a k- dimensional affine complex subspace of $\mathbb{C}^n$ for some $k < n$ and prove a parametric jet interpolation theorem for automorphisms fixing that subspace.

A result analogous to Theorem 1.3.7 for Stein manifolds with the density property will be the main focus of Chapter 4. We do not report it here for the topological condition of being null-homotopic refers to a space of jets which definition, while not hard to understand and imagine, is quite cumbersome and will appear in Section 4.1. The main difference with Theorem 1.3.7 is that we have to again restrict ourselves to a finite number of points, since the notion of tame set in Definition 1.3.5 is not available for an arbitrary Stein manifold.

1.4 Tame Sets

As noted, Definition 1.3.5 is tailor-made for complex Euclidean spaces. Quite surprisingly, an analogous definition for arbitrary complex manifolds has never appeared in the literature. In order to fill this gap, together with Andrist we introduced the following:

Definition 1.4.1. [7] Let X be a complex manifold and let Aut(X) be its group of holomorphic automorphisms. A closed discrete infinite set $A \subset X$ is a *strongly tame set* if for every injective mapping $f : A \to A$ there exists a holomorphic automorphism $F \in$ Aut(X) such that $F_{|_A} = f$.

This definition will be the main topic of Chapter 5. The first thing we will prove there is that Definition 1.4.1 is equivalent to Definition 1.3.5 when $X = \mathbb{C}^n$, therefore this notion generalizes the classical one of Rosay and Rudin. Note that it is possible to enumerate the set A, we will then talk about strongly tame sequences and sets interchangeably.

In 2008, Winkelmann studied a growth condition that ensures a given sequence in $\mathbb{C}^n$ to be tame (Theorem 6.2.11) [61]. He later used this condition as a model for his definition of *weakly tame sets* [63]; it will be presented shortly.

It is not the first time that tame sets have been considered in a setting different than the one proposed by Rosay and Rudin. In [35] Kolarič considered automorphisms of $\mathbb{C}^n$ fixing an algebraic subvariety of codimension at least 2. He used Winkelmann's growth condition and provided many interesting properties of discrete sets which are tame with respect to this subgroup of automorphisms.

The main problem one observes when thinking about this definition is that a strongly tame sequence may not need to exist even in the case of Stein manifolds. It is clear that we need the group $\mathrm{Aut}(X)$ to be particularly large and this is a good reason to consider this definition when X is a Stein manifold with the density property. In [7], we showed the existence of strongly tame sets in any complex linear algebraic Lie group; note that these are known to have the density property [55]. While the density property guarantees the possibility to prescribe the values of an automorphism at a finite number of points, here the main difficulty is to find automorphisms with prescribed values on a closed infinite discrete set. We manage to solve this problem in the algebraic setting thanks to the existence of locally nilpotent derivations (LND'S); that is complete algebraic vector fields with algebraic flows. A careful study of their kernels, which is contained mainly in Section 2.5, will provide us with the required automorphisms.

We now give the definition of *weakly tame set* as proposed by Winkelmann.

Definition 1.4.2. [63] Let X be a complex manifold. An infinite closed discrete subset D is *weakly tame* if for every exhaustion function $\rho : X \to \mathbb{R}$ and every function $\zeta : D \to \mathbb{R}$ there exists an automorphism F of X such that $\rho(F(x)) \geq \zeta(x)$ for all $x \in D$.

Any strongly tame set is also weakly tame, hence their names. Winkelmann proved that for $X = \mathbb{C}^n$, the two notions of tameness coincide [63]. He also proved the existence of weakly tame sets in the special linear group $\mathrm{SL}_2(\mathbb{C})$; there any weakly tame set is also strongy tame. Later, he also showed that the two definitions coincide for linear algebraic Lie groups [62]. The expectation is that the two notions coincide on any Stein manifold with the density property; this result appears to be out of reach at the moment. The manifold $\mathbb{D} \times \mathbb{C}$ is a simple example that shows the two notions are different in general.

These new works reignited the interest in the topic of tame sets; many questions are still unanswered since the seminal work of Rosay and Rudin from 1988 and new ones are arising in view of the recent results. Some of these will be addressed in Chapter 6, together with an outline of the research planned for the future.

2

Technical Tools

In this chapter we develop the tools necessary for the rest of this dissertation. The first three sections aim to give a detailed introduction to the Andersén-Lempert Theorem (Theorem 2.3.1), which can be used to introduce gradute students to the subject. Section 2.4 also covers a basic theorem which is frequently used in this field, yet it is beyond introductory and we point the interested reader to [20, Chapter 4] for a complete account of the Andersén-Lempert theory.

In Section 2.5 we provide a different point of view on complete vector fields and their flows. This section contains most of the technical material needed for Chapter 5 and the techniques needed to prove the results contained in there come mostly from Algebraic Geometry, specifically from Geometric Invariant Theory. For this reason, we do not include the proofs and Section 2.5 should not be considered as part of the introduction to the Andersén-Lempert theory.

2.1 Complete Vector Fields

The use of complete holomorphic vector fields introduced by Andersén and Lempert [2, 3] to study the group of holomorphic automorphisms of $\mathbb{C}^n$ marked the beginning of a completely new field of study in Several Complex Variables.

Definition 2.1.1. Let X be a complex manifold and denote by TX its holomorphic tangent bundle. A holomorphic vector field V is a holomorphic section of TX. We denote the space of holomorphic vector fields by $\mathfrak{X}(X)$.

An important notion linked to a vector field is its flow. Consider the Cauchy problem:

$$\begin{cases} x : \mathbb{D}_\varepsilon \to X \\ \dot{x}(t) = V(x(t)) \\ x(0) = x_0 \in X \end{cases}$$

where $\mathbb{D}_\varepsilon \subset \mathbb{C}$ is an open disk of center 0 and radius $\varepsilon > 0$, x is a holomorphic map and $\dot{x}$ denotes its complex derivative.

It is well known that if V is locally Lipschitz continuous, then the Cauchy problem admits a unique maximal solution for each $x \in X$. In the case of holomorphic vector fields, we can use analytic continuation to extended x to a maximal connected Riemann surface R_{x_0}. Let $\Omega := \{(t, x_0) : x_0 \in X,\ t \in R_{x_0}\}$ be the fundamental domain and define the flow of V as the map

$$\varphi_V : \Omega \to X$$

such that $\varphi_V^t(x_0) = x(t)$, where $x(t)$ is the solution of the Cauchy problem starting at x_0.

Definition 2.1.2. A vector field $V \in \mathfrak{X}(X)$ is complete if $R_{x_0} = \mathbb{C}, \forall x_0 \in X$.

We also recall the semigroup property, that is

$$\varphi_V^{t+s}(x) = \varphi_V^t(\varphi_V^s(x)),$$

for any $s, t \in \mathbb{C}$ for which the expression makes sense.

Note that if $V \in \mathfrak{X}(X)$ is complete, then $\varphi_V^t : X \to X$ is a biholomorphism for each fixed $t \in \mathbb{C}$ thanks to the uniqueness of the solution to the Cauchy problem. More details on holomorphic vector fields and their flows can be found in [12].

In the proof of Theorem 2.3.1, we will also consider time-dependent vector fields.

Definition 2.1.3. Let $\pi : \mathbb{R} \times X \to X$ be the projection onto the second coordinate. A non-autonomous holomorphic vector field $V : \mathbb{R} \times X \to \pi^* TX$ is a smooth section of the pullback bundle $\pi^* TX$ which is holomorphic in $x \in X$ for each fixed $s \in \mathbb{R}$. Its flow $\varphi_V^{t,s}$ is such that

$$\begin{cases} \frac{d}{dt}\varphi_V^{t,s}(x) = V_t(\varphi_V^{t,s}(x)) \\ \varphi_V^{s,s}(x) = x \end{cases}$$

If one considers the flow of a vector field as the curve that is always tangent to that vector field, then the flow of a time-dependent vector field is the curve that at time $t \in \mathbb{C}$ is tangent to V_t at the point $\varphi_V^{t,s}(x)$.

When we deal with differential equations, it is important to recall Gronwall's Lemma; its proof can be found in many textbooks, for instance in [1].

Lemma 2.1.4. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that*

$$f(t) \leq M + L \int_0^t f(\tau) d\tau,$$

for $M, L \in \mathbb{R}$. Then

$$f(t) \leq M e^{Lt}.$$

Since we will repeatedly use it in the upcoming discussion, we also give the Taylor expansion of degree one with respect to t for the flow of a vector field. The complete expression is often called *Lie series*.

Lemma 2.1.5. [20, p. 36] *Let V be a vector field on $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and φ_V^t be its flow. For any $x \in X$ and any t for which the following expression makes sense, it holds that*

$$\varphi_V^t(x) = x + tV(x) + o(t)$$

The key point of the Andersén-Lempert theory is to approximate vector fields in order to approximate their flows. The following few results prove that this is in fact possible and approximating vector fields implies approximation of their flows. We begin by showing that we can approximate the flow of a time-dependent vector field with the composition of flows of time-independent ones. Even if we only consider holomorphic vector fields, these results will be stated and proved under the weaker assumption of local Lipschitz continuity.

Proposition 2.1.6. *Let V_t be a time-dependent locally Lipschitz continuous vector field on a complex manifold X and let $\varphi_V^{t,s}$ be its flow. Let $N \in \mathbb{N}$ and consider the vector fields $\{V_{\frac{k}{N}}\}_{k=0}^{N-1}$ with their respective flows $\{\varphi_k^t\}_{k=0}^{N-1}$. Let $\psi_N^t := \varphi_{N-1}^t \circ \varphi_{N-2}^t \circ \cdots \circ \varphi_0^t$ for $N \geq 1$ and $\psi_0^t = \mathrm{Id}$. Then*

$$\psi_N^{\frac{1}{N}} \xrightarrow{N\to\infty} \varphi_V^{1,0}$$

uniformly on compact subsets of $\Lambda_{1,0} := \{x \in X : \varphi_V^{1,0}(x) \text{ exists}\}$.

Proof. Let K be a compact subset of $\Lambda_{1,0}$ and let L be a uniform Lipschitz constant for V_t on K for $t \in [0,1]$; L exists and is finite as the interval $[0,1]$ is compact. From now on we assume that X is an open subset of $\mathbb{C}^n$ for some $n \in \mathbb{N}$, the claim is obtained by covering the compact set K with a finite number of coordinate charts and applying the following proof to each chart.

Note that for a fixed $N \in \mathbb{N}$ and $j = 0, \ldots, N-1$ we have that

$$\psi_{j+1}^{\frac{1}{N}}(x) = \varphi_j^{\frac{1}{N}} \circ \psi_j^{\frac{1}{N}}(x) = \psi_j^{\frac{1}{N}}(x) + \frac{1}{N} V_{\frac{j}{N}}(\psi_j^{\frac{1}{N}}(x)) + o\left(\frac{1}{N}\right)$$

and

$$\varphi_V^{\frac{j+1}{N},0}(x) = \varphi_V^{\frac{j+1}{N},\frac{j}{N}} \circ \varphi_V^{\frac{j}{N},0}(x) = \varphi_V^{\frac{j}{N},0}(x) + \frac{1}{N} V_{\frac{j}{N}}(\varphi_V^{\frac{j}{N},0}(x)) + o\left(\frac{1}{N}\right)$$

where the last equalities of each line are the Taylor expansions for φ_j^t and $\varphi_V^{t+\frac{j}{N},\frac{j}{N}}$ respectively, valued at $t = \frac{1}{N}$.

Let $0 < \delta < \frac{L}{e^L - 1}$ and choose $N \in \mathbb{N}$ large enough such that the previous approximations have an error of at most $\frac{\delta}{2}$ for all $j = 0, \ldots N$.

For a fixed large $N \in \mathbb{N}$, we prove by induction on $j = 0, \ldots, N$ that

$$||\psi_j^{\frac{1}{N}} - \varphi_V^{\frac{j}{N},0}||_K \leq \frac{\delta}{L}((1 + \frac{L}{N})^j - 1),$$

where $||\cdot||_K$ denotes the uniform norm on K. Note that since δ is arbitrary, the proposition follows from the previous inequality evaluated at $j=N$.

For $j=0$ both functions are the identity, hence the inequality is trivially satisftied.

Computing the uniform norm of the difference on K, we get

$$||\psi_{j+1}^{\frac{1}{N}}-\varphi_V^{\frac{j+1}{N},0}||_K \leq ||\psi_j^{\frac{1}{N}}-\varphi_V^{\frac{j}{N},0}||_K+\frac{L}{N}||\psi_j^{\frac{1}{N}}-\varphi_V^{\frac{j}{N},0}||_K+\delta=$$
$$=\left(1+\frac{L}{N}\right)||\psi_j^{\frac{1}{N}}-\varphi_V^{\frac{j}{N},0}||_K+\delta,$$

where the inequality is due to the previous approximations and the fact that V_t is locally Lipischitz continuous.

Using the induction hypothesis, we obtain that

$$||\psi_{j+1}^{\frac{1}{N}}-\varphi_V^{\frac{j+1}{N},0}||_K \leq \left(1+\frac{L}{N}\right)\frac{\delta}{L}((1+\frac{L}{N})^j-1)+\delta \leq \frac{\delta}{L}((1+\frac{L}{N})^{j+1}-1),$$

hence concluding the proof. The last inequality is due to our choice of $\delta<\frac{L}{e^L-1}$. $\square$

In the next step, we show how to approximate the flow of a vector field V given a map $K_t(x)$, which we call an *algorithm for* V, tangent to V in its initial point $x\in X$. One should think of an algorithm as a first-order approximation of the solution to the Cauchy problem.

Proposition 2.1.7. *Let $V\in\mathfrak{X}(X)$ for a complex manifold X, $t>0$, $x\in\Lambda_t:=\{x\in X:\varphi_V^t(x)$ exists$\}$ and $K:[0,\varepsilon)\times X\to X$ be such that:*

(i) $K_0=\mathrm{Id}_X$;

(ii) $t\to K_t(x)$ *is derivable for all* $x\in X$ *and the derivative is continuous in* (t,x);

(iii) $\frac{\partial}{\partial t}\big|_{t=0^+}K_t(x)=V(x)$.

Then, for $n\in\mathbb{N}$ large enough

$$K_{\frac{t}{n}}^{\circ n}:=\underbrace{K_{\frac{t}{n}}\circ\cdots\circ K_{\frac{t}{n}}}_{n\ times}$$

is defined on each compact subset of Λ_t and converges uniformly on compact sets to the flow $\varphi_V^t(x)$.

Proof. Let $x_0\in X$ and $U\subset X$ be a relatively compact open neighbourhood of x_0 contained in a coordinate chart. From now on, we identify U with the corresponding open set in $\mathbb{C}^N$ for some $N\in\mathbb{N}$. Let $C:=\sup\limits_{t\leq t_0,x\in U}||\frac{\partial}{\partial t}K_t(x)||$, where $t_0>0$ is fixed such that $\mathbb{B}_{2Ct_0}(x_0)\subset U$ and $\mathbb{B}_r(x)$ denotes the ball of radius $r>0$ and center $x\in X$. The constant C is finite because of property (ii). We first prove the statement for $t\leq t_0$.

We use induction to show that $K_{\frac{t}{n}}^{\circ n}(x) \in \mathbb{B}_{2Ct_0}(x_0)$, for every $n \in \mathbb{N}$ and for each $x \in \mathbb{B}_{Ct_0}(x_0)$ and each $t \leq t_0$.
Since

$$||K_t(x)-x|| \leq \int_0^t ||\frac{\partial}{\partial t}K_\tau(x)||d\tau \leq Ct,\ \forall t \leq t_0,$$

we have that

$$||K_t(x)-x_0|| \leq ||K_t(x)-x||+||x-x_0|| \leq 2Ct_0$$

and the first step is proved.
Let $n > 1$, we can then express $K_{\frac{t}{n}}^{\circ n}$ as a telescopic sum in the following way:

$$K_{\frac{t}{n}}^{\circ n}(x)-x = \sum_{j=1}^{n} K_{\frac{t}{n}}(K_{\frac{t}{n}}^{\circ n-j}(x))-K_{\frac{t}{n}}^{\circ n-j}(x).$$

Since

$$K_{\frac{t\frac{n-j}{n}}{n-j}}^{\circ n-j}(x) = K_{\frac{t}{n}}^{\circ n-j}(x)$$

and $t\frac{n-j}{n} \leq t_0$, we can use the induction hypothesis to obtain that $K_{\frac{t}{n}}^{\circ n-j}(x) \in U$ and therefore

$$||K_{\frac{t}{n}}(K_{\frac{t}{n}}^{\circ n-j}(x))-K_{\frac{t}{n}}^{\circ n-j}(x)|| \leq \int_0^{\frac{t}{n}} ||\frac{\partial}{\partial t}K_\tau(K_{\frac{t}{n}}^{\circ n-j}(x))||d\tau \leq C\frac{t}{n}.$$

The latter implies that

$$||K_{\frac{t}{n}}^{\circ n}(x)-x_0|| \leq 2Ct_0.$$

Before proving that $K_{\frac{t}{n}}^{\circ n}(x)$ converges to $\varphi_V^t(x)$, note that

$$\varphi_V^t(x)-K_t(x) = (\varphi_V^t(x)-x-V(x)t)-(K_t(x)-x-V(x)t) = o(t),$$

since φ_t is the flow of V and K satisfies hypothesis (iii).

We now express the difference between $\varphi_V^t(x)$ and $K_{\frac{t}{n}}^{\circ n}$ again as a telescopic sum, using the semi-group law for the flow of a vector field:

$$\varphi_V^t(x)-K_{\frac{t}{n}}^{\circ n}(x) = (\varphi_V^{\frac{t}{n}})^{\circ n}(x)-K_{\frac{t}{n}}^{\circ n}(x) =$$
$$\sum_{j=1}^{n}(\varphi_V^{\frac{t}{n}})^{\circ n-j} \circ \varphi_V^{\frac{t}{n}}(K_{\frac{t}{n}}^{\circ j-1}(x))-(\varphi_V^{\frac{t}{n}})^{\circ n-j} \circ K_{\frac{t}{n}}^{\circ j}(x).$$

Since our vector field is Lipschitz continuous, we can apply Lemma 2.1.4 to each addend and obtain the following expression.

$$||\varphi_V^t(x)-K_{\frac{t}{n}}^{\circ n}(x)|| \leq \sum_{j=1}^{n} e^{t\beta\frac{n-j}{n}}||\varphi_V^{\frac{t}{n}}(K_{\frac{t}{n}}^{\circ j-1}(x))-K_{\frac{t}{n}}(K_{\frac{t}{n}}^{\circ j-1}(x))|| \leq ne^{\beta t}o(\frac{t}{n}),$$

where β is the Lipschitz constant of V on the set U. We have therefore proved the statement for $t \leq t_0$.

Let $T > 0$ and $\Gamma := \{\varphi_V^t(x_0) \colon t \in [0, T]\}$. Since Γ is compact, we can find a neighbourhood $W \subset X$ of Γ and $\delta > 0$ such that $K_{\frac{t}{n}}^{\circ n}$ converges to φ_V^t uniformly on compact sets of W for all $t < \delta$. Choose $u \in \mathbb{N}$ such that $\frac{T}{u} < \delta$, so that

$$\varphi_V^t(x) = (\varphi_V^{\frac{t}{u}})^{\circ u}(x) = \left(\lim_{n\to+\infty} K_{\frac{t}{nu}}^{\circ n}(x)\right)^{\circ u} = \lim_{n\to+\infty} K_{\frac{t}{nu}}^{\circ nu}(x), \quad \forall t \leq T.$$

Let $N \in \mathbb{N}$, then there exist $p, n \in \mathbb{N}$ such that $p < u$ and $N = un + p$. We observe that $\frac{T}{un+p} < \delta$ and $\frac{Tun}{un+p} < T$, hence

$$K_{\frac{T}{N}}^{\circ N}(x) = K_{\frac{T}{un+p}}^{\circ un}(K_{\frac{T}{un+p}}^{\circ p}(x)) = K_{\frac{Tun}{un+p}\frac{1}{un}}^{\circ un}(K_{\frac{T}{un+p}}^{\circ p}(x)) \to \varphi_V^T(x),$$

because $K_{\frac{T}{un+p}}^{\circ p}$ converges to Id_X as n approaches infinity. □

In most situations, the most likely way to find an algorithm for a vector field is to use flows of other vector fields. In order to do this, we observe that $\mathfrak{X}(X)$ is not only a vector space but also a *Lie algebra*.

Definition 2.1.8. A Lie algebra is a finite dimensional vector space $\mathfrak{g}$ equipped with an anticommutative bilinear map $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \forall X, Y, Z \in \mathfrak{g}.$$

The application $[\cdot,\cdot]$ is the Lie bracket of the algebra.

Recall that every Lie group has an associated Lie algebra which can be identified with its tangent space at its neutral element. The usual examples include the common vector spaces of matrices such as $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$ composed of all $n \times n$ matrices and all $n \times n$ matrices with trace zero, respectively. In both these cases the Lie bracket is given by the following expression

$$[A,B] = AB - BA$$

for A, B in the corresponding space.

To define a Lie algebra structure on $\mathfrak{X}(X)$, let us first recall that vector fields act as derivations on the space of holomorphic functions $\mathcal{O}(X)$ according to the following formula.

$$Vf(x) = \frac{d}{dt}_{|_{t=0}} f(\varphi_V^t(x)), \quad V \in \mathfrak{X}(X),\ f \in \mathcal{O}(X)$$

Remark 2.1.9. The vector space $\mathfrak{X}(X)$ is a Lie algebra with the bracket operation

$$[V,W](f) = V(Wf) - W(Vf), \quad V, W \in \mathfrak{X}(X),\ f \in \mathcal{O}(X).$$

When two vector fields commute ($[V,W]=0$), it is not hard to see that the flow of $V+W$ is equal to the flow of V composed with the flow of W.

Lemma 2.1.10. *Let $V,W\in\mathfrak{X}(X)$. The following are equivalent:*

(i) $[V,W]=0$*;*

(ii) the flow of $V+W$ is given by $\varphi_V^t\circ\varphi_W^t$;

(iii) $\varphi_V^t\circ\varphi_W^s=\varphi_W^s\circ\varphi_V^t$ *for any s,t for which this expression makes sense.*

An important tool in the proof of Lemma 2.1.10 is the *Lie derivative*. For $V,W\in\mathfrak{X}(X)$, the Lie derivative L_VW of W with respect to V is the vector field

$$L_VW=\frac{d}{dt}_{|_{t=0}}d\varphi_V^{-t}(W)$$

It holds that $[V,W]=L_VW$ (see for instance [1]).

Proof. $[(i)\Rightarrow(ii)]$ We have

$$0=[V,W]=L_VW,$$

therefore $W=d\varphi_V^{-t}(W)$ for each t for which this expression makes sense. To prove (ii) we derive $\varphi_V^t\circ\varphi_W^t$ to obtain

$$\begin{aligned}\frac{d}{dt}\varphi_V^t\circ\varphi_W^t(x)=V(\varphi_V^t(\varphi_W^t(x)))+d\varphi_V^t(W(\varphi_W^t(x)))=\\ V(\varphi_V^t(\varphi_W^t(x)))+W(\varphi_V^t(\varphi_W^t(x))),\end{aligned}$$

therefore $\varphi_V^t\circ\varphi_W^t$ is the flow of $V+W$.

$[(ii)\Rightarrow(iii)]$ Since $V+W=W+V$, we have that $\varphi_V^t\circ\varphi_W^t=\varphi_W^t\circ\varphi_V^t$. We will prove (iii) for $s,t\in\mathbb{Q}$ and complete the argument by continuity. Let $a,b\in\mathbb{Z}$, $u\in\mathbb{Q}$, such that $\frac{s}{a}=\frac{t}{b}=u$. Then

$$\varphi_V^t\circ\varphi_W^s=(\varphi_V^u)^{\circ b}\circ(\varphi_W^u)^{\circ a}=(\varphi_W^u)^{\circ a}\circ(\varphi_V^u)^{\circ b}=\varphi_W^s\circ\varphi_V^t.$$

$[(iii)\Rightarrow(i)]$ Deriving both sides of $\varphi_V^t\circ\varphi_W^s=\varphi_W^s\circ\varphi_V^t$ with respect to t, we have

$$V(\varphi_V^t\circ\varphi_W^s(x))=d\varphi_W^s(V(\varphi_V^t(x))).$$

Evaluating at $t=0$ we obtain

$$V=d\varphi_W^s(V).$$

Therefore $[V,W]=-[W,V]=-L_WV=0$. □

Lemma 2.1.10 proves that the Lie bracket $[V,W]$ measures the non-commutativity of the vector fields. Even when the vector fields do not commute, the composition of their flows provides an algorithm for the flow of $V+W$; a similar result holds for the Lie bracket.

Lemma 2.1.11. *Let $V, W \in \mathfrak{X}(X)$, then:*

(i) $\varphi_V^t \circ \varphi_W^t$ *is an algorithm for* $V + W$*;*

(ii) $\varphi_W^{-\sqrt{t}} \circ \varphi_V^{-\sqrt{t}} \circ \varphi_W^{\sqrt{t}} \circ \varphi_V^{\sqrt{t}}$ *is an algorithm for* $[V, W]$*.*

Proof. To prove the statement we just need to differentiate on t at $t = 0$:

$$\frac{d}{dt}_{|_{t=0}} \varphi_V^t \circ \varphi_W^t(x) = [\dot{\varphi}_V^t(\varphi_W^t(x)) + d\varphi_V^t(\dot{\varphi}_W^t(x))]_{|_{t=0}} = V(x) + W(x).$$

For the second statement we consider $\varphi_W^{-t} \circ \varphi_V^{-s} \circ \varphi_W^t \circ \varphi_V^s$ and use the Lie series (see for instance [20, p. 36]). □

This fact, together with Proposition 2.1.7, proves the main technical result of this section.

Corollary 2.1.12. *Let $V, W \in \mathfrak{X}(X)$. Let $K \subset X$ a relatively compact subset such that the flow of $V + W$ on K is given by φ_{V+W}^t . Then*

$$\varphi_{V+W}^t(x) = \lim_{n \to +\infty} \left(\varphi_V^{\frac{t}{n}} \circ \varphi_W^{\frac{t}{n}}\right)^{\circ n}(x), \quad \forall x \in K.$$

Let $J \subset X$ a relatively compact subset such that the flow of $[V, W]$ on J is given by $\varphi_{[V,W]}^t$. Then

$$\varphi_{[V,W]}^t(x) = \lim_{n \to +\infty} \left(\varphi_W^{-\sqrt{\frac{t}{n}}} \circ \varphi_V^{-\sqrt{\frac{t}{n}}} \circ \varphi_W^{\sqrt{\frac{t}{n}}} \circ \varphi_V^{\sqrt{\frac{t}{n}}}\right)^{\circ n}(x), \quad \forall x \in J.$$

All limits are uniform on compact sets. Furthermore, the first statement is still true if instead of the flows φ_V^t and φ_W^t we use algorithms for V and W respectively.

If we take V and W to be complete, then their flows are biholomorphisms. Using the above result in the specific case described below, we can show that certain flows can always be approximated by elements of $\mathrm{Aut}(X)$.

Corollary 2.1.13. *Fix $N \in \mathbb{N}$ and let $\{V_i\}_{i\in\{1,\dots,N\}}$ be complete vector fields on the manifold X. Let $V \in Lie(V_1, \dots, V_N)$, the Lie algebra generated by the set $\{V_i\}_{i\in\{1,\dots,N\}}$. Then φ_V^t is the limit of automorphisms of X on each compact set.*

Proof. Since $V \in Lie(V_1, \dots, V_N)$, we can express it as a finite Lie combination of complete vector fields. We'll show by induction that if ϕ^t, $t \in [0, 1]$ is a family of automorphisms approximating the flow of the Lie combination of the first k, $k \in \mathbb{N}$, complete vector fields that appear in the decomposition of V, then we can find another isotopy approximating the flow of the $k+1$-th Lie combination.

Suppose the $k+1$-th operation is an addition of the vector field $\tilde{V} \in \{V_1, \dots, V_N\}$, then we can apply the first part of Corollary 2.1.12 and obtain that $(\phi^{\frac{t}{n}} \circ \varphi_{\tilde{V}}^{\frac{t}{n}})^{\circ n}$ is the desidered approximation for large enough $n \in \mathbb{N}$.

If instead the $k+1$-th operation is a Lie bracket, denote by W the vector field obtained via the first k operations i.e.

$$V = [W, \tilde{V}]$$

for $\tilde{V} \in \{V_1, \dots, V_N\}$. The second part of Corollary 2.1.12 then guarantees that

$$\left(\varphi_W^{-\sqrt{\frac{t}{n}}} \circ \varphi_{\tilde{V}}^{-\sqrt{\frac{t}{n}}} \circ \varphi_W^{\sqrt{\frac{t}{n}}} \circ \varphi_{\tilde{V}}^{\sqrt{\frac{t}{n}}}\right)^{\circ n}$$

approximates the flow of V for large enough $n \in \mathbb{N}$. Since $\varphi_{\tilde{V}}^t$ is an automorphism for any $t \in \mathbb{C}$ and φ_W^t is approximated by ϕ^t, the proof is complete. □

We now know that we can approximate flows of Lie combinations of vector fields by composing flows of single vector fields. Another important step is showing that approximation of vector fields implies approximation of their flows.

Proposition 2.1.14. *[20] Let $\Omega \subset \mathbb{R}^q$ be an open set. Let $V \in \mathfrak{X}(\Omega)$ and $K \subset \Omega$ be compact. Let $t_0 > 0$ such that $\varphi_V^t(x)$ exists for any $x \in K$ and $t \in [0, t_0]$. Let $K_t = \varphi_V^t(K)$ and $K_t(\varepsilon) = \{x \in \mathbb{R}^q \colon dist(x, K_t) < \varepsilon\},\ \varepsilon > 0$. Let L be the Lipschitz constant of V on Ω, $\eta_0 := (1+t_0)e^{Lt_0}$ and fix $\varepsilon_0 > 0$ such that $K(\varepsilon_0\eta_0) \subset \Omega$ is relatively compact.*

If $W \in \mathfrak{X}(\Omega)$ is such that $||V - W||_{K(\varepsilon\eta_0)} \leq \varepsilon$ for a fixed $\varepsilon < \varepsilon_0$, then the flow $\varphi_W^t(x)$ exists for every $x \in K(\varepsilon),\ t \in [0, t_0]$ and

$$||\varphi_V^t - \varphi_W^t||_{K(\varepsilon)} \leq t_0 e^{Lt} ||V - W||_{K(\varepsilon\eta_0)}$$

Proof. Let $x \in K(\varepsilon)$ and $f(t) := |\varphi_V^t(x) - \varphi_W^t(x)|,\ t > 0$. Note that

$$f(t) = |\int_0^t V(\varphi_V^s(x)) - W(\varphi_W^s(x)) ds| \leq$$
$$\int_0^t |V(\varphi_V^s(x)) - V(\varphi_W^s(x))| ds + \int_0^t |V(\varphi_W^s(x)) - W(\varphi_W^s(x))| ds.$$

Therefore, if $\varphi_W^s(x) \in K(\varepsilon\eta_0)$ for all $s < t$ we have

$$f(t) \leq L \int_0^t f(s) ds + t_0 ||V - W||_{K(\varepsilon\eta_0)}$$

and Lemma 2.1.4 provides

$$|\varphi_V^t(x) - \varphi_W^t(x)| \leq t_0 e^{Lt} ||V - W||_{K(\varepsilon\eta_0)}.$$

We will now show that $\varphi_W^s(x) \in K(\varepsilon\eta_0)$ for all $s < t$.

If this is not the case, let $\hat{t} < t$ be such that $\varphi_W^{\hat{t}}(x) \notin K(\varepsilon\eta_0)$ and $\varphi_W^s(x) \in K(\varepsilon\eta_0)$ for all $s < \hat{t}$. For such s, the previous inequalities hold, hence

$$\begin{aligned} \mathrm{dist}(\varphi_W^s(x), K_s) \leq |\varphi_V^s(x) - \varphi_W^s(x)| + \mathrm{dist}(\varphi_V^s(x), K_s) \leq \\ s||V - W||_{K(\varepsilon\eta_0)} e^{Ls} + \varepsilon e^{Ls} \leq e^{Lt_0}\eta_0. \end{aligned}$$

The inequality then holds for $\hat{t}$, reaching a contradiction. □

2.2 Stein Manifolds

This section will provide a short introduction to the theory of Stein manifolds, a central concept in elliptic holomorphic geometry. The main reference for this discussion is the book by Grauert and Remmert [25].

The first idea we will introduce is the one of *holomorphic convexity*, a holomorphic analogue to geometric convexity. Recall that a compact set $K \subset \mathbb{R}^n$ is convex if and only if the segment joining any two points in K is contained in K. We define the convex hull K^{convex} of K to be the smallest convex set containing K; it can be characterized in the following way:

$$K^{\text{convex}} = \{x \in \mathbb{R}^n \colon \lambda(x) \leq \max_{y \in K} \lambda(y), \text{ for all linear functions } \lambda\}.$$

The idea is to replace linear functions with holomorphic functions.

Definition 2.2.1. Let X be a complex manifold and $K \subset X$ a compact subset. The holomorphically convex hull of K is

$$\hat{K}_{\mathcal{O}(X)} := \{x \in X \colon |f(x)| \leq \max_{y \in K} |f(y)|, \ \forall f \in \mathcal{O}(X)\}.$$

The subscript $\mathcal{O}(X)$ represents the ring of holomorphic functions on X; it will be left out when the relevant complex manifold is deducible from context.

Since a compact set is convex if and only if $K^{\text{convex}} = K$, we define $K \subset X$ to be holomorphically convex if $\hat{K} = K$. A complex manifold X is holomorphically convex if $\hat{K} \subset X$ is compact for every compact set $K \subset X$.

This definition gives us information about $\mathcal{O}(X)$. It is indeed equivalent to ask that for any diverging (i.e. leaving every compact set) sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$, there exists a holomorphic function $f \in \mathcal{O}(X)$ such that $\{f(x_k)\}_{k \in \mathbb{N}} \subset \mathbb{C}$ is also diverging.

The definition of *Stein manifold*, introduced by Stein in 1951 [54], describes some properties of $\mathcal{O}(X)$.

Definition 2.2.2. A complex manifold X of complex dimension $n \in \mathbb{N}$ is a Stein manifold if it satisfies the following properties:

(a) X is holomorphically convex;

(b) for every two distinct $x, y \in X$, there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$;

(c) for every $x \in X$, there exist holomorphic functions $\{f_1, \dots, f_n\} \subset \mathcal{O}(X)$ such that their differentials $\{df_1, \dots, df_n\}$ are linearly independent at x.

The first basic example of a Stein manifold is clearly $\mathbb{C}^n$. We also have that $(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$ is a Stein manifold for $0 \leq k \leq n$ ($\mathbb{C}^*$ denotes the punctured complex plane $\mathbb{C} \setminus \{0\}$). If instead of removing hyperplanes we remove just points, we obtain something which is not Stein such as $\mathbb{C}^n \setminus \{(0, \dots, 0)\}$ for $n > 1$. This manifold satisfies property (b) and (c) but it is not holomorphically convex since every holomorphic function extends across the puncture.

It is clear that compact manifolds do not satisfy (b), hence they are also not Stein. For the same reason, Stein manifold do not admit compact complex submanifolds.

An important characteristic of Stein manifolds is given by the Runge approximation property, expressed by the following Oka-Weil theorem.

Theorem 2.2.3. *Let X be a Stein manifold and $K \subset X$ be a holomorphically convex compact set. Then every holomorphic function in a neighbourhood of K can be approximated uniformly on K by holomorphic functions in $\mathcal{O}(X)$.*

The vast literature on Stein manifold produced striking characterization theorems, which we summarize here. Recall that a function $\rho : X \to \mathbb{R}$ is an *exhaustion function* if its sublevel sets are compact and they exhaust X.

Theorem 2.2.4. *For a complex manifold X of dimension $n \in \mathbb{N}$, the following are equivalent:*

(i) X is Stein;

(ii) there exists a proper holomorphic embedding $X \to \mathbb{C}^N$ for some $N \in \mathbb{N}$ [51];

(iii) there exists a proper holomorphic embedding $X \to \mathbb{C}^{2n+1}$ [47];

(iv) there exists a proper holomorphic embedding $X \to Y$ for any Stein manifold Y with the density property and dimension at least $2n+1$ [4];

(v) X admits a strongly plurisubharmonic exhaustion function [10, 23].

Besides (iv), all these characterization date back to the end of the '50s. The recent result (iv) of Andrist, Forstnerič, Ritter and Wold (2016) is one more indicator that Stein manifolds with the density property share many qualities with complex Euclidean spaces. The next section will describe the main shared tool between $\mathbb{C}^n$ and Stein manifolds with the density property.

Property (ii) suggests that Stein manifolds are a suitable holomorphic analogue to affine algebraic manifolds.

Arguably, the most important results about Stein manifolds are Cartan's Theorem A & B [9].

Theorem 2.2.5. *Let $\mathscr{F}$ be a coherent analytic sheaf on a Stein manifold X. Then*

(A) For any $x \in X$, the stalk $\mathscr{F}_x$ is generated by global sections of $\mathscr{F}$ (as a $\mathcal{O}_{X,x}$-module);

(B) The sheaf coholomogy groups satisfy $H^p(X, \mathscr{F}) = 0$ for all $p \geq 1$.

We will not define what is a coherent analytic sheaf, but examples include the sheaf of holomorphic functions $\mathcal{O}_X$, the sheaf of holomorphic functions vanishing on a subvariety, and any sheaf of holomorphic sections of a holomorphic vector bundle.

While the formulation of these theorems requires a basic knowledge of sheaf theory (which can be found in [24]), their corollaries will give even the unexperienced reader a measure of their importance.

Corollary 2.2.6. *Every holomorphic function on a closed complex subvariety of a Stein manifold X extends to a holomorphic function on X.*

Corollary 2.2.7. *Given a finite number of points $\{x_1, \dots x_N\} \subset X$, $N \in \mathbb{N}$ in a Stein manifold, we can find a holomorphic function with prescribed values and derivates up to some finite order at those points..*

2.3 (Parametric) Andersén-Lempert Theorem

The Andersén-Lempert theory began with the study of the group $\mathrm{Aut}(\mathbb{C}^n)$ of holomorphic automorphisms of complex Euclidean spaces. It is known that for $n = 1$, the automorphisms group is a two dimensional Lie group.

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \colon a, b \in \mathbb{C},\ a \neq 0\}.$$

For $n > 1$, the situation is substantially more interesting. Besides the obvious linear transformations in $\mathrm{GL}_n(\mathbb{C})$, we have that the following maps are elements of $\mathrm{Aut}(\mathbb{C}^n)$ for any choice of holomorphic functions $a, b \in \mathcal{O}(\mathbb{C}^{n-1})$:

$$\begin{aligned}(z_1, \dots, z_n) &\mapsto (z_1 + b(z_2, \dots, z_n), z_2, z_3, \dots, z_n),\\ (z_1, \dots, z_n) &\mapsto (e^{a(z_2,\dots,z_n)} z_1, z_2, z_3, \dots, z_n).\end{aligned}$$

Such maps (and their $\mathrm{GL}_n(\mathbb{C})$-conjugates) are called *shears*. It is important to notice that they are time-1 maps of the following vector fields,

$$\begin{aligned}b(z_2, \dots, z_n)\frac{\partial}{\partial z_1},\\ a(z_2, \dots, z_n) z_1 \frac{\partial}{\partial z_1}.\end{aligned}$$

These vector fields (and their $\mathrm{GL}_n(\mathbb{C})$-conjugates) are usually called shear vector fields or vector shears; in particular the first ones are named *additive vector shears* or just *vector shears*, while the second ones are *multiplicative shears* or *overshears*. Outside of this section, we will use the term shear vector field to refer to both instances, as the distinction is not relevant for us. In general, vector shears can be written respectively as

$$\begin{aligned}&f(\lambda z)v\\ &f(\lambda z)\langle z, v\rangle v\end{aligned}$$

for a holomorphic function $f \in \mathscr{O}(\mathbb{C})$, a linear map λ and a vector $v \in \mathbb{C}^n$ such that $\lambda v = 0$, where we wrote $\langle z, v\rangle = \sum_{j=1}^n z_j \bar{v}_j$.

Andersén and Lempert proved that shears generate a dense subgroup of $\mathrm{Aut}(\mathbb{C}^n)$ [3]. Their techniques were later used by Forstnerič and Rosay [19] to give the formulation of the Andersén-Lempert Theorem that is commonly used in applications:

Theorem 2.3.1. *Let $\Omega \subset \mathbb{C}^n$ be an open subset and $\varphi^t : \Omega \longrightarrow \Omega_t := \varphi^t(\Omega) \subset \mathbb{C}^n,\ t \in [0,1]$ be an isotopy of holomorphic maps such that:*

(i) Ω_t is Runge for any $t \in [0,1]$;

(ii) $\varphi^0 = \mathrm{Id}_\Omega$;

(iii) $\varphi^t(z)$ is $\mathscr{C}^1$ in $t \in [0,1]$ and univalent in $z \in \mathbb{C}^n$ for any fixed $t \in [0,1]$.

Then φ^1 is the limit (uniform on compact sets) of composition of shears.

Before proving this theorem, let us show how we can decompose every polynomial into a linear combination of powers of linear functions. The following lemma will be of vital importance in the proof of Theorem 2.3.1, as it will allow us to decompose each algebraic vector field into a Lie combination of complete vector fields.

Lemma 2.3.2. [2] *Let $k, n \in \mathbb{N}$ and let $m \in \mathbb{N}$ be the cardinality of the set of multi-indices $\{I \in \mathbb{N}^n : |I| = k\}$. Then there exist linear maps $\{\lambda_j\}_{j=1}^m,\ \lambda_j : \mathbb{C}^n \longrightarrow \mathbb{C}$ such that any homogeneous polynomial P of degree k is of the form*

$$P(z) = \sum_{j=1}^m c_j(\lambda_j(z))^k, \quad c_j \in \mathbb{C}.$$

Proof. We can always write $\lambda_j(z) = \langle z, \bar{a}_j\rangle$ for a certain $a_j \in \mathbb{C}^n,\ j \in \{1,\dots,m\}$. Then

$$(\lambda_j(z))^k = \sum_{|I|=k} \binom{k}{I} a_j^I z^I,$$

where $z^I := z_1^{I_1} z_2^{I_2} \dots z_n^{I_n}$ and $\binom{k}{I}$ is the standard multinomial coefficient.

By substitution we obtain the equation

$$\sum_{j=1}^m c_j(\lambda_j(z))^k = \sum_{j=1}^m c_j \sum_{|I|=k} \binom{k}{I} a_j^I z^I = \sum_{|I|=k} \binom{k}{I} z^I \sum_{j=1}^m c_j a_j^I.$$

We now write the polynomial P in a similar fashion, so that

$$P(z) = \sum_{|I|=k} P_I z^I.$$

We are hence looking for a solution of

$$\binom{k}{I}\sum_{j=1}^{m} c_j a_j^I = P_I.$$

Consider the $m \times m$ matrix $A = (a_j^I)_{j\in\{1,\dots,m\}}^{|I|=k}$. The previous equation has a solution for $\det A \neq 0$. Choose the entries of the vectors a_j to be multiplicatively independent over $\mathbb{Q}$ i.e. $a_{j_1}^{I_1} = a_{j_2}^{I_2}$ if and only $j_1 = j_2$ and $I_1 = I_2$. With this choice, $\det A$ is a Van der Monde determinant and it is not zero. □

Remark 2.3.3. Note that we have a large degree of freedom in our choice of linear maps. Since it is not identically zero, $\det A$ is a homogeneous polynomial of degree km in mn variables and the set $\{\det A \neq 0\}$ is a non-empty open dense set of $\mathbb{C}^{mn}$. Hence, given any open set $U \subset (\mathbb{C}^n)^*$ (the dual of $\mathbb{C}^n$), we can choose our linear maps to be in U.

We are now ready to give the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. Let $V_t(z) = \dot{\varphi}^t((\varphi^t)^{-1}(z))$ be a time-dependent vector field over $\tilde{\Omega} = \{(t,z) \in [0,1] \times \mathbb{C}^n : z \in \Omega_t\}$ and let $\varphi_V^{t,s}$ be its flow. Let $N \in \mathbb{N}$ and $t_k := \frac{k}{N}$, $k \in \{0,\dots,N\}$. Since $V_{t_k} \in \mathfrak{X}(\Omega_{t_k})$, we can consider its flow φ_k^t. Seeing that $\frac{d}{dt}\big|_{t=0}\varphi_k^t(z) = V_{t_k}(z)$, we have, using Proposition 2.1.6,

$$\varphi_{N-1}^{\frac{1}{N}} \circ \varphi_{N-2}^{\frac{1}{N}} \circ \cdots \circ \varphi_1^{\frac{1}{N}} \xrightarrow{N\to+\infty} \varphi^1(z).$$

We will prove that φ_k^t is a limit of composition of shears for any $k = 1,\dots,N$.

Let $K \subset \Omega_{t_k}$ be a relatively compact set and let $t_0 > 0$ be such that $\varphi_k^t(z)$ exists for $t \leq t_0$, $z \in K$. Let L be a compact subset of Ω_{t_k} such that $\{\varphi_k^t(z) : t \in [0,t_0],\ z \in K\} \subset L$. We want to approximate V_{t_k} on L using polynomial vector fields. Since Ω_{t_k} is Runge, this can always be done. Thanks to Proposition 2.1.14, the flows of the approximating vector fields converge to the flow φ_k^t.

We have reduced the proof to showing that we can approximate the time-1 maps of a polynomial vector field with an automorphism. In order to do this, we will express the polynomial vector field as a sum of complete vector fields. This proves our claim thanks to Lemma 2.1.11.

Any polynomial vector field V is the sum of homogeneous polynomial fields, so that $V = V_0 + \cdots + V_n$ where V is of degree n. We further reduced ourselves to homogeneous polynomial vector fields.

Let V be a homogeneous polynomial vector field of degree k and let $\{\lambda_k\}_{j=1}^m$ be linear maps as in Lemma 2.3.2 for degrees k and $k-1$. By a change of coordinates we can assume that $\lambda_i(e_n) \neq 0,\ \forall i \in \{1,\dots,m\}$. Moreover, we can rescale such that $\lambda_i(e_n) = 1,\ \forall i \in \{1,\dots,m\}$. Since the divergence of V is homogeneous of degree $k-1$, we have

$$\operatorname{div} V = \sum_{j=1}^{m} d_j(\lambda_j(z))^{k-1}, \quad d_j \in \mathbb{C}.$$

Let $v_j \in \operatorname{Ker} \lambda_j,\ j \in \{1, \ldots, m\}$ be such that $||v_j|| = 1,\ \forall j \in \{1, \ldots, m\}$ and

$$V_j(z) := d_j(\lambda_j(z))^{k-1} \langle z, v_j \rangle v_j, \quad j \in \{1, \ldots, m\}$$

so that $\operatorname{div} V_j = d_j(\lambda_j(z))^{k-1}$ and

$$\operatorname{div}(V - \sum_{j=1}^{m} V_j) = 0$$

Note that each $V_j,\ j \in \{1, \ldots, m\}$ is a multiplicative vector shear and its flow is an automorphism, hence we only need to prove that we can express a homogeneous polynomial vector field W of degree k such that $\operatorname{div} W = 0$ as a sum of shears in the following way:

$$W(z) = \sum_{j=1}^{m} c_j(\lambda_j(z))^k v_j, \quad c_j \in \mathbb{C}.$$

Let W_i be the i-th component of W $(W_i = \langle W, e_i \rangle)$. Then

$$W_i(z) = \sum_{j=1}^{m} c_{i,j}(\lambda_j(z))^k, \quad i \in \{1, \ldots, n-1\}.$$

Define $V_{i,j}(z) :\ = c_{i,j}(\lambda_j(z))^k(e_i - \lambda_j(e_i)e_n),\ i = 1, \ldots, m,\ j = 1, \ldots, n-1$. Notice that $V_{i,j}$ is an additive shear vector field, since $\lambda_j(e_i) - \lambda_j(e_i)\lambda_j(e_n) = 0$. Therefore, if we define the vector field U as

$$U := \sum_{j=1}^{m} \sum_{j=1}^{n-1} V_{i,j} = (U_1, \ldots, U_n)$$

we have that $\operatorname{div} U = 0$ and

$$W = U + (W_n - U_n)e_n$$

Since U is a sum of shears, it is sufficient to show that also $(W_n - U_n)e_n$ is a shear. Let us compute

$$0 = \operatorname{div} W = \operatorname{div} U + (W_n - U_n)e_n = \operatorname{div}(W_n - U_n)e_n = \frac{\partial}{\partial z_n}(W_n - U_n)$$

therefore $(W_n - U_n)e_n$ is an additive shear and we are done. □

As already mentioned, the Andersén-Lempert Theorem does not only hold for complex Euclidean spaces but for all Stein manifolds with the density property (Definition 1.2.1). Before formulating the precise result, let us summarize the proof of Theorem 2.3.1 so that it will be clear how to generalize it.

We began by producing a time-dependent vector field, then approximated its flow by composing flows of time-independent ones. Next, we approximated each of these vector fields, defined only on Ω_{t_k}, with vector fields defined on the all $\mathbb{C}^n$. It is here that we used the fact that Ω_t is Runge for all $t \in [0,1]$. The hardest part was to then express these global vector fields as sums of complete ones, a task we managed to fulfill by producing a proper decomposition by homogeneous vector fields.

Varolin's density property is modelled to provide this hardest step. It is then no surprise that it is not a simple task to determine which Stein manifolds enjoy the density property.

Theorem 2.3.4. *Let X be a Stein manifold with the density property. Let $\Omega \subset X$ be an open subset and $\varphi^t : \Omega \longrightarrow \Omega_t := \varphi^t(\Omega) \subset X,\ t \in [0,1]$ be an isotopy of holomorphic maps such that:*

(i) Ω_t *is Runge for any* $t \in [0,1]$*;*

(ii) $\varphi^0 = \mathrm{Id}_\Omega$*;*

(iii) $\varphi^t(x)$ *is* $\mathscr{C}^1$ *in* $t \in [0,1]$ *and univalent in* $x \in X$ *for any fixed* $t \in [0,1]$.

Then φ^1 can be approximated uniformly on compacts by holomorphic automorphisms of X.

Proof. As in the proof of Theorem 2.3.1, we consider the time-dependent vector field $V_t(x) = \dot{\varphi}^t((\varphi^t)^{-1}(x))$, so that we can think of φ^t as its flow. Thanks to Proposition 2.1.7, we can approximate φ^1 by a composition of $\{\varphi_j^{\frac{1}{N}}\}_{j=0}^{N-1}$, $N \in \mathbb{N}$, where $\varphi_j^{\frac{1}{N}}$ is the flow of $V_{\frac{j}{N}}$ on $\Omega_{\frac{j}{N}}$.
Since X is Stein, we can embed it in a complex Euclidean space thanks to Theorem 2.2.4, so that its vector fields are vectors of functions defined on X. Being $\Omega_{\frac{j}{N}}$ Runge, we can approximate $V_{\frac{j}{N}}$ with globally defined holomorphic vector fields (coordinate by coordinate). Hence we only need to approximate global vector fields, but this is provided by the density property. □

The reader can see that the proof of the Andersén-Lempert Theorem for Stein manifolds with the density property is very simple, the reason being that the hardest part of the proof of Theorem 2.3.1 is replaced by the assumption that our Stein manifold enjoys the density property. At present, there is no characterization of Stein manifolds with the density property, so here we present a short list of examples. The best results are for affine algebraic manifolds, since Kaliman and Kutzschebauch [30] provided very strong criteria to determine if one such manifold enjoys the density property.

Theorem 2.3.5. *The following Stein manifolds enjoy the density property:*

(a) $\mathbb{C}^n,\ n \geq 2$ *[3];*

(b) $\mathbb{C}^k \times \mathbb{C}^{*n-k},\ 1 \leq k \leq n$ *[59];*

(c) *affine algebraic homogeneous spaces G/H of algebraic subgroups $H \subset G \subset GL_n(\mathbb{C})$, different from $\mathbb{C}$ or $\mathbb{C}^{*n}$ [11, 33];*

(d) *the Koras-Russell cubic threefold $C = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{C}^4$ [42];*

(e) *surfaces of the form $\{uv = f(z)\} \subset \mathbb{C}^2_{u,v} \times \mathbb{C}^n_z$ for any holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$ with smooth reduced zero fiber [31].*

The complex plane $\mathbb{C}$ does not have the density property as we know the only complete vector fields are the linear ones. It is not known if the product of punctured planes $\mathbb{C}^{*n}$ has the density property. The Koras-Russell threefold is a famous example of an algebraic manifold that is diffeomorphic to $\mathbb{R}^6$ but not algebraically isomorphic to $\mathbb{C}^3$. It is not known whether C is biholomorphic to $\mathbb{C}^3$; a related conjecture was proposed by Tóth and Varolin:

Conjecture 2.3.6. [56] *Let X be a Stein manifold with the density property which is diffeomorphic to $\mathbb{C}^n$. Then X is biholomorphic to $\mathbb{C}^n$.*

The Koras-Russell threefold is to our knowledge the best candidate to disprove or alternatively corroborate this conjecture.

The last variation of the Andersén-Lempert Theorem that we wish to mention is the following parametric version, obtained by Kutzschebauch and Ramos-Peon [39]. It will be fundamental during Chapter 4.

For W and X complex manifolds, denote by $\mathrm{Aut}_W(X)$ be subgroup of $\mathrm{Aut}(W \times X)$ consisting of those automorphisms which act as the identity of the W coordinate. One should think of elements of $\mathrm{Aut}_W(X)$ as holomorphic automorphisms of X depending holomorphically on a parameter in W.

Theorem 2.3.7. *Let W be a Stein manifold and X be a Stein manifold with the density property. Let $U \subset W \times X$ be an open set and $F^t : U \to W \times X$ be a smooth homotopy of injective holomorphic maps acting as the identity on the W coordinate i.e. such that $F^t(w, x) = (w, f^t(w, x))$ and with F^0 being the inclusion map. Suppose $K \subset U$ is a compact set such that $F^t(K)$ is $\mathcal{O}(W \times X)$-convex for each $t \in [0, 1]$. Then there exists a neighbourhood V of K such that for all $t \in [0, 1]$, F^t can be approximated uniformly on compacts of V (with respect to any distance function on X) by automorphisms $\alpha^t \in \mathrm{Aut}_W(X)$ which depend smoothly on t, and moreover we can choose $\alpha^0 = \mathrm{Id}$.*

In order to apply the Andersén-Lempert Theorem in any of its forms, it is required to produced a motion of Runge sets. The following lemma converts motions of holomorphically convex compact sets into motions of Runge sets.

Lemma 2.3.8. [19, Lemma 2.2] *Let $\Omega \subset X$ be an open subset of a Stein manifold and $\varphi^t : \Omega \to X$, $t \in [0, 1]$ a smooth homotopy of injective holomorphic maps such that $\varphi^0 = \mathrm{Id}$. If $K \subset \Omega$ is a compact holomorphically convex set such that $K_t = \varphi^t(K)$ is holomorphically convex for each $t \in [0, 1]$, then there exists a basis of Stein neighbourhoods U of K such that $\varphi^t(U)$ is Runge in X for each $t \in [0, 1]$.*

2.4 Converging Sequences of Automorphisms

The strenght of Theorem 2.3.1 is somewhat limited by the fact that it can provide approximation only on compact subsets, hence we cannot use it directly in applications concerning diverging sequences, such as the ones provided by tame sets. In this section we will formulate and prove a theorem that was first published by Forstnerič in [13]; it describes the limit behaviour of an infinite composition of automorphisms, each approximatively fixing a larger and larger compact set.

Theorem 2.4.1. [13, Proposition 5.1] *Let $D \subset \mathbb{C}^n$ be an open set exhausted by compact sets such that*

$$K_0 \subset K_1 \subset \cdots \subset \bigcup_{j\geq 0} K_j = D$$

and K_j is contained in the interior of K_{j+1} for all $j \in \mathbb{N}$. Let $\varepsilon_j > 0$, $j \in \mathbb{N}$ be a decreasing sequence such that

$$0 < \varepsilon_j < \operatorname{dist}(K_{j-1}, \mathbb{C}^n \setminus K_j), \quad j \in \mathbb{N} \text{ and } \sum_{j=0}^{+\infty} \varepsilon_j < +\infty$$

For each $j \in \mathbb{N}$, let $\varphi_j \in \operatorname{Aut}(\mathbb{C}^n)$ be uniformly ε_j-close to the identity on K_j and let $\phi_j = \varphi_j \circ \varphi_{j-1} \circ \cdots \circ \varphi_0$.

Then the open set $\Omega = \bigcup_{j=0}^{+\infty} \phi_j^{-1}(K_j) \subset \mathbb{C}^n$ is such that $\phi = \lim_{j\to+\infty} \phi_j$ exists on Ω (uniformly on compacts) and ϕ is a biholomorphisms from Ω to D.

Proof. Since $\varepsilon_{j+1} < \operatorname{dist}(K_j, \mathbb{C}^n \setminus K_{j+1})$ and $|\varphi_{j+1}(z) - z| < \varepsilon_{j+1} < \varepsilon_j$ on $\mathring{K}_{j+1} \supset K_j$, we have that $\varphi_{j+1}(K_j)$ is contained in the interior of K_{j+1}. Therefore, we obtain that

$$\phi_{j+1}(\phi_j^{-1}(K_j)) = \varphi_{j+1}(K_j) \subset \mathring{K}_{j+1}$$

and $\phi_j^{-1}(K_j)$ is contained in the interior of $\phi_{j+1}^{-1}(K_{j+1})$. Iterating this procedure, we see that

$$\phi_k(\phi_j^{-1}(K_j)) \subset \mathring{K}_k \quad \text{for all } k \geq j$$

We will use this information to show that $\{\phi_m(z)\}_{m\geq j}$ is a Cauchy sequence for each $z \in \phi_j^{-1}(K_j)$. For such z, let us compute:

$$\begin{aligned}
|\phi_m(z) - \phi_j(z)| &\leq \sum_{k=j+1}^{m} |\phi_k(z) - \phi_{k-1}(z)| \\
&= \sum_{k=j+1}^{m} |\varphi_k \circ \phi_{k-1}(z) - \phi_{k-1}(z)| \\
&< \sum_{k=j+1}^{m} \varepsilon_k,
\end{aligned}$$

where the last inequality is given by the fact that $\phi_{k-1}(\phi_j^{-1}(K_j)) \subset \mathring{K}_{k-1} \subset \mathring{K}_k$ for all $k > j$.

As the estimate is uniform in $z \in \phi_j^{-1}(K_j)$, the limit ϕ exists on $\phi_j^{-1}(K_j)$ for each $j \in \mathbb{N}$ and there it satisfies

$$|\phi(z) - \phi_j(z)| \leq \sum_{k=j+1}^{+\infty} \varepsilon_k < \operatorname{dist}(K_j, \mathbb{C}^n \setminus D).$$

Hence $\phi(\phi_j^{-1}(K_j)) \subset D$ for all $j \in \mathbb{N}$ and $\phi(\Omega) \subset D$.

We now prove that $\phi : \Omega = \bigcup_{j=0}^{+\infty} \phi_j^{-1}(K_j) \to D$ is injective and surjective.

For injectivity, fix $z \in \Omega$ and choose $j \in \mathbb{N}$ and $w \in K_j$ such that $z = \phi_j^{-1}(w)$. For $m \geq j$, we have that

$$|\phi_m(z) - w| = |\phi_m(z) - \phi_j(z)| < \sum_{k=j+1}^{+\infty} \varepsilon_k.$$

Sending m to infinity we obtain that

$$|\phi(\phi_j^{-1}(w)) - w| < \sum_{k=j+1}^{+\infty} \varepsilon_k \quad \text{for } w \in K_j.$$

Therefore $\phi \circ \phi_j^{-1}$ is close to the identity, hence a biholomorphism, on any compact $K \subset D$ for sufficiently large $j \in \mathbb{N}$. Since we already know that ϕ_j is biholomorphic, we proved that ϕ is a biholomorphism on $\phi_j^{-1}(K) \subset \Omega$. Injectivity follows from the fact that ϕ is a non-degenerate limit of injective holomorphic maps.

Let us now prove the surjectivity of ϕ. Fix $j \geq 1$ and choose $m > j$ such that the tail of the series of the ε_k's is small enough:

$$\sum_{k=m+1}^{+\infty} \varepsilon_k < \operatorname{dist}(K_j, \mathbb{C}^n \setminus K_{j+1}).$$

Then, by our previous inequality, we have that

$$|\phi(\phi_m^{-1}(w)) - w| < \operatorname{dist}(K_j, \mathbb{C}^n \setminus K_{j+1}) \quad \text{for all } w \in K_m.$$

Therefore $K_j \subset \phi \circ \phi_m^{-1}(K_m) \subset \phi(\Omega)$. Since this holds for all $j \in \mathbb{N}$, we have that $D \subset \phi(\Omega)$, thereby proving surjectivity. □

For the sake of clarity, we explain the intuitive reasoning behind this result. The main point is that the automorphism φ_j moves the points of K_j (hence of K_{j-1}) by at most $\varepsilon_j < \operatorname{dist}(K_{j-1}, \mathbb{C}^n \setminus K_j)$. In particular this implies that $\varphi_j(K_{j-1})$ is still contained in K_j and we can apply an inductive procedure.

In applications, the compact sets K_j are holomorphically convex and the automorphisms φ_j come from the Andersén-Lempert Theorem. This way, we will be able to inductively prescribe a specific behaviour to our limit biholomorphism ϕ. It will be important for ϕ to be an automorphism of $\mathbb{C}^n$, hence the set D will be the all $\mathbb{C}^n$ and we will need to pay attention that $\Omega = \mathbb{C}^n$ as well, in our constructions.

The Andersén-Lempert Theorem had a parametric analogue for Stein manifolds with the density property (Theorem 2.3.7). The next theorem is the parametric counterpart of Theorem 2.4.1. The density property is not required at this stage, as we already assume the existence of the automorphisms.

Theorem 2.4.2. [39] *Let W be a complex manifold and X be a Stein manifold. Let L_j and K_j, $j \geq 1$ be exhaustions by compact sets of W and X respectively. Let ε_j, $j \in \mathbb{N}$ be such that*

$$0 < \varepsilon_j < \mathrm{dist}(K_{j-1}, X \setminus K_j), \quad j \in \mathbb{N} \text{ and } \sum_{j=1}^{+\infty} \varepsilon_j < +\infty.$$

Let $\varphi_j \in \mathrm{Aut}_W(X)$, $j \in \mathbb{N}$ and for $j \geq m \geq 1$ define $\phi^w_{j,m} = \varphi^w_j \circ \cdots \circ \varphi^w_m \in \mathrm{Aut}_W(X)$. Assume that for any $j \in \mathbb{N}$ and $w \in L_j \setminus L_{j-1}$ (take $L_0 = \emptyset$)

$$\begin{aligned} \mathrm{dist}(\varphi^w_j(z), z) &< \varepsilon_j \quad \text{for } z \in K_j; \\ \mathrm{dist}(\varphi^w_{j+s}(z), z) &< \varepsilon_{j+s} \quad \text{for } z \in K_j \cup \phi^w_{j+s-1,j}(K_{j+s}) \text{ and for all } s \geq 1. \end{aligned}$$

Then $\phi = \lim_{m \to +\infty} \phi_{m,1}$ exists (in the topology of uniform convergence on compacts) and $\phi \in \mathrm{Aut}_W(X)$.

Notice that Theorem 2.4.2 is already stated with all the necessary hypotheses to ensure that our limit biholomorphism is actually a parametrized automorphism of X. This will be of great use in Chapter 4.

2.5 Invariant Functions

This section will require some basic definitions and notation from Algebraic Geometry, which we now recall. In some cases, we will not give the classical definition but an equivalent one. The reason we introduce these definitions will be clear by the end of the section.

Definition 2.5.1. A complex Lie group G is reductive if it is the complexification of a compact (real) Lie group.

Definition 2.5.2. Let R be a commutative ring. The spectrum $\mathrm{Spec} R$ of R is the topological space consisting of all prime ideals of R equipped with the Zariski topology.

We denote by $\mathbb{C}[x_1, \ldots x_n]$ the ring of polynomials in $n \in \mathbb{N}$ variables with coefficients in $\mathbb{C}$ and by $\mathbb{C}[X]$ the algebra of regular functions of an algebraic, affine, or quasi-affine variety X. A variety is quasi-affine if it is isomorphic to a Zariski open set of an affine variety.

As we have seen in the rest of the chapter, we will often consider the case of a group acting on a manifold. In these situations, it is often relevant to study the ring of *invariant functions* on such manifold.

Definition 2.5.3. Let G be a group acting on a manifold X. A function $f : X \to \mathbb{C}$ is *G-invariant* (or just *invariant* when G is clear from context) if $f(gx) = f(x)$ for every $x \in X$ and $g \in G$.

The study of invariant functions is particularly well-developed in the case of an algebraic group acting algebraically on an affine algebraic variety. The field of Geometric Invariant Theory is indeed a very active part of Algebraic Geometry; we refer to the book [44] of Mumford, Fogarty, and Kirwan for more details.

The field of Geometric Invariant Theory dates back to Paul Gordan, the leading figure in *invariant theory* during the second half of the 19^{th} century. His work was vastly generalized by Hilbert, who later proposed the following problem in his famous list from 1900.

Question 2.5.4 (Hilbert's 14^{th} problem)**.** Let $\mathbb{C}(x_1,\dots x_n)$ be the field of rational functions in n variables over $\mathbb{C}$ and let $K \subset \mathbb{C}(x_1,\dots x_n)$ be a subfield. Denote by R the $\mathbb{C}$-algebra $K \cap \mathbb{C}[x_1,\dots x_n]$. Is R finitely generated over $\mathbb{C}$?

In our setting, we will have an affine algebraic variety X and a group G acting algebraically on it. The ring R will consist of all regular invariant functions on X, denoted by $\mathbb{C}[X]^G$. It is known that when G is reductive, $\mathbb{C}[X]^G$ is finitely generated and therefore its spectrum $\operatorname{Spec}\mathbb{C}[X]^G$ is an affine algebraic variety Y whose algebra of regular functions $\mathbb{C}[Y]$ is isomorphic to $\mathbb{C}[X]^G$; this result was proved by Hilbert and it greatly advanced our understanding of invariant functions. The variety Y is usually called the *Geometric Invariant Theory (GIT) quotient* of X.

Hilbert's 14^{th} problem was settled in the negative in 1959 by Nagata, who constructed a counterexample using the invariant functions of a linear algebraic group on a 32-dimensional variety [46].

In Chapter 5 we will consider the following situation. Given an affine algebraic manifold X and a complete algebraic vector field V whose flow φ_V^t is algebraic (a locally nilpotent derivation or LND), we have an algebraic action of the additive group $(\mathbb{C},+)$:

$$\begin{aligned} \mathbb{C} \times X &\to X \\ (t,x) &\mapsto \varphi_V^t(x). \end{aligned}$$

This gives an action on the algebra of regular functions $\mathbb{C}[X]$. Note that the functions invariant under this action are precisely the ones in the kernel of V. It is important to notice that if $f \in \operatorname{Ker} V$, then $fV \in \mathfrak{X}(X)$ is also a complete vector field. For this reason, we are interested in the ring of invariant functions $\mathbb{C}[X]^{(\mathbb{C},+)}$

with respect to the action of φ_V^t. If $\mathbb{C}[X]^{(\mathbb{C},+)}$ is finitely generated, its spectrum is the GIT quotient and it is affine. When this ring is not finitely generated, we will use the following result of Winkelmann as a substitute:

Theorem 2.5.5. [60] *Let k be a field, X be an irreducible, reduced, normal k-variety and $G \subset \mathrm{Aut}(X)$ be an algebraic subgroup of the group of algebraic automorphisms. Then there exist a quasi-affine k-variety Z and a rational map $\pi\colon X \to Z$ such that the following hold:*

1. *the rational map π induces an inclusion $\pi^*\colon k[Z] \subset k[X]$;*
2. *the image of the pull-back $\pi^*(k[Z])$ coincides with the ring of invariant functions $k[X]^G$;*
3. *for every affine k-variety Θ and every G-invariant morphism $f : X \to \Theta$, there exists a morphism $F\colon Z \to \Theta$ such that $F \circ \pi$ is a morphism and $f = F \circ \pi$.*

When $G = (\mathbb{C}, +)$ and the action is given by a complete vector field V, we will denote the quotient by $\pi_V : X \to Q_V$. This is a useful construction to produce functions in the kernel of such a derivation and we will use it repeatedly in Chapter 5.

Locally nilpotent derivations are important on their own and we point the interested reader to the textbook of Freudenburg [21].

3

Parametric Jet Interpolation by Automorphisms of Complex Euclidean Spaces

The present chapter aims to present the first result obtained by the author during his doctoral studies. While the main theorem already appeared in [57], we will expand on it with comments about its proof and possible applications.

3.1 Main result and proof

The main result of this chapter was already formulated in the introduction; for convenience we state it again.

Theorem 3.1.1 (same as Theorem 1.3.7)**.** [57] *Let W be a finite dimensional Stein space, $\{a_j\}_{j\in\mathbb{N}}, \{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n$ $(n>1)$ be tame sequences of points and $r_j \in \mathbb{N}_{>0}$, $j>0$ be a sequence of natural numbers. For every $j \in \mathbb{N}$, let $P_j \colon W \to J^{r_j}_{a_j,b_j}(\mathbb{C}^n)$ be a holomorphic family of non-degenerate r_j-jets such that $P_j^w(a_j) = b_j$ for all $w \in W$. Then there exists a null-homotopic holomorphic map $F \colon W \to \mathrm{Aut}(\mathbb{C}^n)$ such that*

$$F^w(z) = P_j^w(z) + O(|z-a_j|^{r_j+1}) \quad \textit{for } z \to a_j,\ j \in \mathbb{N},\ w \in W \tag{3.1}$$

if and only if the linear part map $Q_j \colon W \to GL_n(\mathbb{C})$ of P_j at the point a_j is null-homotopic for every $j \in \mathbb{N}$.

Recall that a map $F \colon W \to \mathrm{Aut}(\mathbb{C}^n)$ from a complex manifold W into the holomorphic automorphism group of $\mathbb{C}^n$ is said to be holomorphic if the evaluation map $W \times \mathbb{C}^n \to \mathbb{C}^n$ given by $F(w)(z) =: F^w(z)$ is holomorphic. A map is null-homotopic if it is homotopic to a constant map.

This kind of interpolation results often include a *relative* interpolation; in this case, this means the following. Let $W' \subset W$ be an analytic subvariety and assume that a holomorphic map $G \colon W' \to \mathrm{Aut}(\mathbb{C}^n)$ satisftying (3.1) for all $w \in W'$ is given. Then there exists a holomorphic map F as in Theorem 3.1.1, additionally satisfying

$F^w = G^w$ for all $w \in W'$. The reason this relative interpolation has not been proved is highly not-trivial and it is related to the solution of the holomorphic Vaserstein problem (Theorem 1.3.8); more details can be found in [50].

In the special case of a finite number of families of jets $P_j\colon W \to J^K_{a_j,a_j}(\mathbb{C}^n)$, $j = 1,\dots,j_0$, $K \in \mathbb{N}$, tangent to the identity map at the points $a_1,\dots,a_{j_0} \in \mathbb{C}^n$, Theorem 3.1.1 was established by Kutzschebauch and Lodin [38, Lemma A.4] who also proposed a solution for finite but arbitrary (not necessarily tangent to the identity) non-degenerate families of jets. Here we prove jet interpolation on any infinite tame set in $\mathbb{C}^n$. The special case of Theorem 3.1.1 for a finite family of jets, not necessarily tangent to the identity, is an easy consequence of our main technical tool, Proposition 3.1.2, which deals with the possibility of interpolating a family of jets at one point of $\mathbb{C}^n$ while at the same time interpolating the identity map up to a given finite order N at finitely many other points of $\mathbb{C}^n$.

Proposition 3.1.2. [57] *Let W be a finite dimensional Stein space, $n > 1$ and $r \in \mathbb{N}$ be integers, $p,q \in \mathbb{C}^n$, and $P : W \to J^r_{p,q}(\mathbb{C}^n)$ be a holomorphic family of jets at p with $P^w(p) = q$ for all $w \in W$. Let Q^w be the linear part of P^w at p, i.e. $P^w(z) = q + Q^w(z-p) + O(|z-p|^2)$ as $z \to p$ for every $w \in W$, and assume that the map $Q : W \to GL_n(\mathbb{C})$ is null-homotopic. Given finitely many points $\{a_i\}_{i=1}^{i_0} \subset \mathbb{C}^n \setminus \{p,q\}$, an integer $N \in \mathbb{N}$, a compact set $T \subset W$, a compact convex set $K \subset \mathbb{C}^n$ such that $p,q \notin K$, and a number $\varepsilon > 0$, there exists a holomorphic map $F : W \to \mathrm{Aut}(\mathbb{C}^n)$ satisfying the following conditions:*

(i) $F^w(z) = P^w(z) + O(|z-p|^{r+1})$ *for $z \to p$ and for every $w \in W$;*

(ii) $F^w(z) = z + O(|z-a_i|^N)$ *for $z \to a_i$, $1 \le i \le i_0$ and for every $w \in W$;*

(iii) $|F^w(z) - z| < \varepsilon$ *for every $w \in T$ and $z \in K$;*

(iv) *if $\{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n \setminus (K \cup \{p,q\})$ is a discrete set contained in the z_1-axis, we can also ensure that $F^w(c_j) = c_j$ for every $w \in W$ and $j \in \mathbb{N}$.*

The proof of Theorem 3.1.1 will consist in an inductive application of this proposition on larger and larger compact sets. At each step of the induction we will interpolate the given jet at one more point, eventually exhausting the sequence in the limit.

Let us comment on the four properties required in Proposition 3.1.2. Condition (i) ensures that we are interpolating the correct jet at the point p, while condition (ii) is needed to preserve the interpolation already obtained in the previous steps of the induction. It is clear that (iii) will ensure the convergence of the process, thanks to Theorem 2.4.2. The fourth condition is more subtle and it allows us to fix the points where we are yet to perform our jet interpolation. Without this requirement, we would not be able to apply Proposition 3.1.2 inductively as the points $p,q \in \mathbb{C}^n$ would not be fixed; they would in fact depend holomorphically on $w \in W$.

The reason that our topological assumption on the jet only entails its linear part is due to the fact that $\mathrm{Aut}(\mathbb{C}^n)$ retracts onto $\mathrm{GL}_n(\mathbb{C})$ [18] via the map

$$\begin{aligned}[0,1]\times \mathrm{Aut}(\mathbb{C}^n) &\to \mathrm{Aut}(\mathbb{C}^n)\\ (t,F) &\mapsto \frac{F(tz)-F(0)}{t}+tF(0).\end{aligned}$$

Before proving Proposition 3.1.2, let us show how to inductively use it to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. As both sequences a_j and b_j are tame, we can change the families of jets and assume that $a_j=b_j=(j,0,\dots,0)=:je_1,\ j\in\mathbb{N}$. Furthermore, as shown in [8], we only need to prove this result at a discrete sequence $\{c_j\}_{j\in\mathbb{N}}$ of points contained in the z_1-axis, as for any such sequence there exists an automorphism Φ of $\mathbb{C}^n$ such that

$$\Phi(z)=je_1+(z-c_j)+O(|z-c_j|^{r_j+1}),\qquad z\to c_j,\ j\in\mathbb{N}.$$

Fix an exhausting sequence of compacts $T_1\subset T_2\subset\cdots\subset\bigcup_{j=1}^{\infty}T_j=W$ and a sequence of positive numbers $\{\varepsilon_j\}_{j\in\mathbb{N}}\subset\mathbb{R}_{>0}$ such that $\sum_{j=1}^{\infty}\varepsilon_j<+\infty$. We will inductively construct the following:

(a) a discrete sequence of points $\{\alpha_j\}_{j\in\mathbb{N}}\subset\mathbb{N}\subset\mathbb{C}$;

(b) an exhausting sequence of convex compact sets $K_1\subset K_2\subset\cdots\subset\bigcup_{j=1}^{\infty}K_j=\mathbb{C}^n$ such that $\mathrm{dist}(K_{j-1},\mathbb{C}^n\setminus K_j)>\varepsilon_j$ and $\alpha_je_1\notin K_j$ for all $j>1$;

(c) a sequence of holomorphic maps $\psi_j:W\to\mathrm{Aut}(\mathbb{C}^n)$ for $j\in\mathbb{N}$

such that for $F_k^w:=\psi_k^w\circ\cdots\circ\psi_1^w\in\mathrm{Aut}(\mathbb{C}^n)$ $(w\in W,\ k\in\mathbb{N})$ we have that

(i_k) $F_k^w(z)=P_j^w(z)+O(|z-\alpha_je_1|^{r_j+1})$ for $z\to\alpha_je_1$ and each $j=1,\dots,k$;

(ii_k) $F_k^w(ie_1)=ie_1$ for every $i>\alpha_k$;

(iii_k) ψ_j^w is ε_j-close to the identity on K_j for every $w\in T_j$ and $j\le k$.

For the base of our induction, let $K_1=\mathbb{B}$, the ball of radius one in $\mathbb{C}^n$, and $\alpha_1=2$. By Proposition 3.1.2 we can pick a family of automorphisms $\psi_1^w\in\mathrm{Aut}(\mathbb{C}^n)$ $(w\in W)$ such that properties (i_1), (ii_1) and (iii_1) hold.

For the induction step, suppose we have constructed the objects in (a), (b), and (c) satisfying properties (i_j), (ii_j), and (iii_j) for all $j=1,\dots,k$. Pick a compact convex set $K_{k+1}\subset\mathbb{C}^n$ such that

$$(\alpha_k+1)\mathbb{B}\cup F_k^w((\alpha_k+1)\mathbb{B})\subset K_{k+1}\quad\text{for every } w\in T_{k+1}$$

and

$$\mathrm{dist}(K_k,\mathbb{C}^n\setminus K_{k+1})>\varepsilon_{k+1}.$$

Choose $\alpha_{k+1} \in \mathbb{N}$ such that $\alpha_{k+1} e_1 \notin K_{k+1}$. We again invoke Proposition 3.1.2 to obtain a holomorphic map $\psi_{k+1} : W \to \mathrm{Aut}(\mathbb{C}^n)$ with the following properties:

1. $\psi^w_{k+1}(z) = z + O(|z - \alpha_j e_1|^N)$ as $z \to \alpha_j e_1$ for every $j = 1, \ldots, k$, where the integer $N > m_j$ for every $j < k+1$;

2. $\psi^w_{k+1}(z) = [P^w_{k+1} \circ (F^w_k)^{-1}]^{m_{k+1}}_{\alpha_{k+1} e_1} + O(|z - \alpha_{k+1} e_1|^{m_{k+1}+1})$ as $z \to \alpha_{k+1} e_1$;

3. ψ^w_{k+1} is ε_{k+1}-close to the identity on K_{k+1} for every $w \in T_{k+1}$;

4. $\psi^w_{k+1}(je_1) = je_1$ for every $j > \alpha_{k+1}$.

We then see that the holomorphic family of automorphisms defined by

$$F^w_{k+1} = \psi^w_{k+1} \circ F^w_k \in \mathrm{Aut}(\mathbb{C}^n), \quad w \in W$$

satisfies properties (i_{k+1}), (ii_{k+1}) and (iii_{k+1}), so the induction may proceed.

The sequence of compacts $K_j \subset \mathbb{C}^n$, $j \in \mathbb{N}$ constructed in this way clearly satisfies condition (b). According to Theorem 2.4.2, the sequence $\{F_k\}_{k\in\mathbb{N}}$ converges to a holomorphic family of automorphisms $F : W \to \mathrm{Aut}(\mathbb{C}^n)$ which interpolates the given families of jets P^w_j at the points $\alpha_j e_1$ ($j \in \mathbb{N}$) thanks to property (i). □

In this proof we already see an interesting property of tame sets, namely that any of its subsets is also a tame set and therefore equivalent to the initial one via an automorphism. This is of course obvious when one uses Definition 1.4.1 to characterize tame sets in $\mathbb{C}^n$.

Before proceeding with the proof of Proposition 3.1.2, we provide a useful lemma for holomorphic functions in one variable. Its importance will be clear later.

Lemma 3.1.3. *Let $T \subset W$ be a compact set and $K \subset \mathbb{C}$ be a convex compact set such that $0 \notin K$. Let $\{a_i\}_{i=1}^{i_0} \subset \mathbb{C} \setminus \{0\}$ and $\{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C} \setminus \{0\}$ be a discrete sequence. Given a holomorphic function $\beta \in \mathscr{O}(W)$, $\varepsilon > 0$ and $r, N \in \mathbb{N}$, there exists a holomorphic $f : W \times \mathbb{C} \to \mathbb{C}$ such that*

(1) $|f|_{T\times K} < \varepsilon$;

(2) $f^w(\zeta) = \beta(w)\zeta^r + O(|\zeta|^{r+1})$ *for* $\zeta \to 0$;

(3) $f^w(\zeta) = O(|\zeta - a_j|^N)$ *for* $\zeta \to a_j$, $j = 1, \ldots, i_0$;

(4) $f^w(c_j) = 0$, $j > 0$.

Proof. By Weierstrass interpolation theorem, there exists a holomorphic function $h : \mathbb{C} \to \mathbb{C}$ which is zero exactly at the points c_j and a_j; we can further require that it is zero up to order N at the points a_j.

Since K is convex there is a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $g(0) = \frac{1}{h(0)}$ and

$$|g|_K < \frac{\varepsilon}{|\beta|_T |h|_K |\zeta^r|_K}.$$

Then $f^w(\zeta) := \beta(w)\zeta^r h(\zeta)g(\zeta)$ is the desired function. □

Shortly, we will prove Proposition 3.1.2 with the help of the above lemma. Before dwelling on the technicalities, we wish to give an intuitive presentation of the underlying idea.

For simplicity, assume that $p = q = 0$. Given a holomorphic family of jets $P : W \to J^k_{0,0}(\mathbb{C}^n)$ at the origin $0 \in \mathbb{C}^n$, we begin by finding finitely many homogeneous polynomial maps $H^w_j : \mathbb{C}^n \to \mathbb{C}^n$, $j = 1, \ldots, k$, depending holomorphically on $w \in W$, such that $P^w(z) = H^w_1(z) + \cdots + H^w_k(z)$ holds for all $w \in W$ and $z \in \mathbb{C}^n$. We will then find holomorphic maps $S_j : W \to \mathrm{Aut}(\mathbb{C}^n)$ for $j = 0, \ldots, k$ satisfying

(i) $S^w_j \circ \cdots \circ S^w_0(z) = H^w_1(z) + \cdots + H^w_j(z) + O(|z|^{j+1})$ for $z \to 0$;

(ii) $S^w_j(z) = z + O(|z - a_i|^N)$ for $z \to a_i$;

for every $w \in W$, every $1 \le j \le k$, and each prescribed point a_i, $1 \le i \le m$, where $N \in \mathbb{N}$ is an arbitrarily large integer. Our interpolating map $F : W \to \mathrm{Aut}(\mathbb{C}^n)$ will then be the composition $F^w = S^w_k \circ \cdots \circ S^w_0$ ($w \in W$) of such automorphisms. For $j \ge 2$ the existence of S_j will be a straightforward generalization of the construction needed in the proof of Theorem 1.3.2, while interpolating the linear part of the jet will not be as easy.

We can identify the linearization of P at $0 \in \mathbb{C}^n$ with a map $Q : W \to GL_n(\mathbb{C})$. We will first take care of the determinant in order to reduce the problem to the case when $Q : W \to SL_n(\mathbb{C})$; this step will already require the map Q to be null-homotopic. In the next and crucial step we use Theorem 1.3.8 to obtain a decomposition of $Q : W \to SL_n(\mathbb{C})$ into a product of unipotent matrices. The terms of this decomposition provide us with automorphisms interpolating the linear part of the jet. The details appear below.

Proof of Proposition 3.1.2. We would like to assume that $p = q = 0$. An automorphism moving the point p to the point q while satisfying (ii) and (iii) (independently of $w \in W$) is provided by [8, Theorem 1.2], where the authors construct an automorphism approximatively fixing a compact set $K \subset \mathbb{C}^n$, precisely fixing some points in it, while moving a point p ouside of K to another point q, also outside of K. Using the same result we can assume that the point $p = q$ is in the z_1-axis and positioned in such a way that the projection of K on this axis is far from p. Without loss of generality we can assume that $p = 0$.

We shall inductively construct automorphisms $S^w_j : \mathbb{C}^n \to \mathbb{C}^n$ for $j = 0, 1, \ldots k$, depending holomorphically on $w \in W$ and satisfying the following conditions for every $r \in \{1, \ldots k\}$:

(a$_r$) $P^w \circ (S_0^w)^{-1} \circ (S_1^w)^{-1} \circ \cdots \circ (S_r^w)^{-1} = z + O(|z|^r)$ for $z \to 0$;

(b$_r$) $S_r^w \circ \cdots \circ S_0^w(z) = z + O(|z - a_i|^N)$ for $z \to a_i$, $1 \le i \le i_0$;

together with (iii) for $\frac{\varepsilon}{k+1}$ and (iv) from the statement of Proposition 3.1.2. Taking

$$F^w(z) := S_k^w \circ \cdots \circ S_0^w(z), \quad w \in W,$$

we obtain a holomorphic map $F\colon W \to \mathrm{Aut}(\mathbb{C}^n)$ satisfying the required conditions.

Let $\{\lambda_j\}_{j=1}^n$ be a basis of the dual space $(\mathbb{C}^n)^*$ such that $\lambda_j(a_i) \neq 0$, $1 \le i \le i_0$, $1 \le j \le n$. We further require that their kernels are almost orthogonal to the z_1-axis. A basis with these properties exists as being a basis is a generic condition and our requirements determine an open subset of $(\mathbb{C}^n)^*$. Note that this choice implies that the image of K under any of these maps is disjoint from 0 and the image of $\{c_j\}_{j\in\mathbb{N}}$ is still a discrete sequence. Let $\{e_j\}_{j=1}^n$ be its dual basis of $\mathbb{C}^n$ such that $|e_2| = 1$.

We will first determine the maps S_0^w and S_1^w. For the linear part of the jet we have

$$P^w(z) = H_1^w(z) + O(|z|^2) = Q^w(z) + O(|z|^2), \quad z \to 0;$$

hence we require that $S_1^w \circ S_0^w(z) = Q^w(z) + O(|z|^2)$ as $z \to 0$.

Since the map $Q\colon W \to GL_n(\mathbb{C})$ is null-homotopic, so is the determinant map $\det Q\colon W \to \mathbb{C} \setminus \{0\}$. Hence by the homotopy lifting property there exists a holomorphic function $g : W \to \mathbb{C}$ such that $e^{g(w)} = \det(Q^w)$ holds for all $w \in W$. By Lemma 3.1.3, there exists $f \in \mathcal{O}(W \times \mathbb{C})$ such that

(1) $|f|_{T\times\lambda_1(K)} < \frac{\varepsilon}{k+1}$;

(2) $f^w(\zeta) = g(w) + O(|\zeta|)$ for $\zeta \to 0$;

(3) $f^w(\zeta) = O(|\zeta - \lambda_1(a_j)|^N)$ for $\zeta \to a_i$, $i = 1, \ldots, i_0$;

(4) $f^w(\lambda_1(c_j)) = 0$, $j > 0$.

We now use it to define the holomorphic family of overshears

$$S_0^w(z) = z + (e^{f^w(\lambda_1(z))} - 1)\langle z, e_2\rangle e_2 = (z_1, z_2 e^{f^w(\lambda_1(z))}, z_3 \ldots, z_n), \quad w \in W.$$

It is easily seen that its Jacobian determinant satisfies

$$J(S_0^w)(z) = e^{f^w(\lambda_1(z))} = e^{g(w)} + O(|z|) \quad \text{for } z \to 0$$

as the Jacobian matrix is triangular. Since we also have that

$$S_0^w(z) = z + O(|z - a_i|^N) \quad \text{for } z \to a_i,\ 1 \le i \le i_0,$$

we see that S_0^w fixes the point a_i for $i = 1, \ldots, i_0$ and that the linear part of $Q^w \circ (S_0^w)^{-1}$ at $0 \in \mathbb{C}^n$ belongs to $\mathrm{SL}_n(\mathbb{C})$ for all $w \in W$. Furthermore, since the linear part of the map $W \ni w \mapsto S_0^w$ is null-homotopic, so is the linear part of $w \mapsto Q^w \circ (S_0^w)^{-1}$.

Conditions (1) and (4) provide (iii) and (iv). (When $Q^w \in SL_n(\mathbb{C})$ for all $w \in W$, we may simply take $S_0^w = \mathrm{Id}$).

We can now apply Theorem 1.3.8 (the holomorphic Vaserstein problem) in order to find an integer $M \in \mathbb{N}$ and holomorphic maps $G_1, \ldots, G_M : W \to \mathbb{C}^{\frac{n(n-1)}{2}}$ such that the linear part of $Q^w \circ (S_0^w)^{-1}$ at $0 \in \mathbb{C}^n$ equals the product of lower and upper triangular matrices

$$\begin{pmatrix} 1 & 0 \\ G_1^w & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2^w \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_M^w \\ 0 & 1 \end{pmatrix}. \qquad (\star)$$

Each of these triangular matrices can be further decomposed into a product of elementary matrices $T_{j,l} = I + \alpha^w e_{j,l},\ j \neq l$, where $\alpha^w \in \mathbb{C}$ and $e_{j,l}$ is the matrix with 1 in position (j,l) and zero elsewhere. The function $\alpha : W \to \mathbb{C}$ is holomorphic since it is the term in position (j,l). For each element in this decomposition, consider the holomorphic family of shear automorphisms

$$L^w(z) = z + h^w(\lambda_j(z))e_l, \quad w \in W,\ z \in \mathbb{C}^n,$$

where $h : W \times \mathbb{C} \to \mathbb{C}$ is such that

(1) $|h|_{T \times \lambda_j(K)} < \frac{\varepsilon}{(k+1)Mn^2}$;

(2) $h^w(\zeta) = \alpha^w \zeta + O(|\zeta|^2)$ for $\zeta \to 0$;

(3) $h^w(\zeta) = O(|\zeta - \lambda_j(a_j)|^N)$ for $\zeta \to a_j$, $i = 1, \ldots, i_0$;

(4) $h^w(\lambda_j(c_i)) = 0$, $i > 0$.

Note that

$$L^w(z) = (\mathrm{Id} + \alpha^w e_{j,l})z + O(|z|^2) \quad \text{for } z \to 0,$$

and

$$L^w(z) = z + O(|z - a_i|^N) \quad \text{for } z \to a_i,\ 1 \le i \le i_0.$$

Hence, the composition S_1^w of these families of shears, ordered as they appear in the corresponding matrix decomposition $(\star)$, is a volume preserving automorphism of $\mathbb{C}^n$ depending holomorphically on $w \in W$ and satisfying conditions (a_1), (b_1), (iii) and (iv).

This concludes the construction of the maps $S_0, S_1 : W \to \mathrm{Aut}(\mathbb{C}^n)$.

In order to find maps $S_2, \ldots, S_k : W \to \mathrm{Aut}(\mathbb{C}^n)$ we proceed inductively. Assume that for some integer $r \in \{2, \ldots, k\}$ we have already found maps $S_0, S_1, \ldots, S_{r-1}$ such that conditions (a_{r-1}), (b_{r-1}), (iii) and (iv) hold. Then

$$P^w \circ (S_0^w)^{-1} \circ (S_1^w)^{-1} \circ \cdots \circ (S_{r-1}^w)^{-1}(z) = z + H_r^w(z) + O(|z|^{r+1}) \quad \text{for } z \to 0,$$

where H_r^w is a homogeneous polynomial vector field of order r on $\mathbb{C}^n$ depending holomorphically on $w \in W$.

We now use Lemma 2.3.2 as we did in the proof of Theorem 2.3.1 to obtain numbers $A, B \in \mathbb{N}$, linear maps $\{\lambda_j\}_{j=1}^A$, $\{\mu_j\}_{j=1}^B \subset (\mathbb{C}^n)^*$ and vectors $\{v_j\}_{j=1}^A$, $\{v_j'\}_{j=1}^B \subset \mathbb{C}^n$ with $\lambda_j(v_j) = 0$ and $\mu_j(v_j') = 0$ for all j such that the homogeneous polynomial maps of degree r given by

$$z \mapsto (\lambda_j(z))^r v_j, \quad j = 1, \dots A,$$

together with

$$z \mapsto (\mu_j(z))^{r-1} \langle z, v_j' \rangle v_j', \quad j = 1, \dots B$$

form a basis for the vector space of homogeneous polynomial vector fields of degree r on $\mathbb{C}^n$. Hence we can write

$$H_r^w(z) = \sum_{j=1}^{A} c_j^w (\lambda_j(z))^r v_j + \sum_{j=1}^{B} d_j^w (\mu_j(z))^{r-1} \langle z, v_j' \rangle v_j', \quad w \in W,\ z \in \mathbb{C}^n$$

for uniquely determined holomorphic functions $c_j, d_j \colon W \to \mathbb{C}$.

Thanks to Remark 2.3.3, we can furthermore ensure that $\mu_j(a_i) \neq 0$ and $\lambda_j(a_i) \neq 0$ for all $i = 1, \dots, i_0$ and all j in the appropriate range and the kernel of each of these linear forms is almost orthogonal to the z_1-axis.

For each term in the above decomposition we consider the following families of shears depending holomorphically on $w \in W$:

$$L^w(z) = z + f^w(\lambda_j(z)) v_j, \quad z \in \mathbb{C}^n,$$

where f satisfies

(1) $|f|_{T \times \lambda_j(K)} < \frac{\varepsilon}{(k+1)AB}$;

(2) $f^w(\zeta) = c_x^j \zeta^r + O(|\zeta|^{r+1})$ for $\zeta \to 0$;

(3) $f^w(\zeta) = O(|\zeta - a_j|^N)$ for $\zeta \to a_j$, $j = 1, \dots, i_0$;

(4) $f^w(c_j) = 0$, $j > 0$;

and

$$R^w(z) = z + (e^{h^w(\mu_j(z))} - 1) \langle z, v_j' \rangle v_j', \quad z \in \mathbb{C}^n,$$

where h satisfies

(1) $|h|_{T \times \lambda_j(K)} < \frac{\varepsilon}{(k+1)AB}$;

(2) $h^w(\zeta) = c_j^w \zeta^{r-1} + O(|\zeta|^r)$ for $\zeta \to 0$;

(3) $h^w(\zeta) = O(|\zeta - a_j|^N)$ for $\zeta \to a_j$, $j = 1, \dots, i_0$;

(4) $h^w(c_j) = 0$, $j > 0$.

We define S^w_r to be the composition of all mentioned L^w and R^w, the order not being relevant. Examining the behaviour of each L^w and R^w near 0 and $a_1, \ldots, a_{i_0}$ like it was done for S^w_0 and S^w_1 then proves that S^w_r satisfies conditions (a_r) and (b_r). As (iii) and (iv) are clearly satisfied as well, this closes the induction step.

After finitely many steps we find maps S^w_j for $j = 0, \ldots, k$ such that conditions (a_k) and (b_k) hold. Taking $F^w(z) := S^w_k \circ \cdots \circ S^w_0(z)$ for $w \in W$ furnishes a holomorphic map $F\colon W \to \mathrm{Aut}(\mathbb{C}^n)$ satisfying the required conditions. □

The proof we just gave implicitly contains the following corollary, which allows us to interpolate a holomorphic family of jets at just one point without any topological assumption.

Corollary 3.1.4. [57] *Let W be a Stein space and $P : W \to J^r_{0,0}(\mathbb{C}^n)$ ($r \in \mathbb{N}$, $n > 1$) be a holomorphic family of jets fixing the origin of $\mathbb{C}^n$. Then there exists a holomorphic map $F : W \to \mathrm{Aut}(\mathbb{C}^n)$ such that the following holds for any point $w \in W$:*

$$F^w(z) = P^w(z) + O(|z|^{r+1}) \quad \text{for } z \to 0.$$

Proof. Let Q^w be the linear part of P^w. Then the linear part of $(Q^w)^{-1} \circ P^w$ is constant and we can apply Proposition 3.1.2 to obtain a family of automorphisms $G\colon W \to \mathrm{Aut}(\mathbb{C}^n)$ such that

$$[G^w]^r_0 = (Q^w)^{-1} \circ P^w.$$

The automorphism $F^w := Q^w \circ G^w$ satisfies the required property. □

3.2 Parametric Jet Interpolation fixing an axis

Theorem 1.3.2 proved to be an extremely useful tool in the study of holomorphic dynamics; that is, the study of iterates of holomorphic maps. In the presence of a fixed point, the local dynamical behaviour is completely determined by the jet of the map; the result of Forstnerič provides an automorphims with the given jet, hence allowing to extend the study of the local behaviour to information about the global dynamics of the automorphism.

In [52], Rosay and Rudin proved that if an automorphism of $\mathbb{C}^n$ has an attracting fixed point (a fixed point such that the norms of the eigenvalues of its Jacobian matrix are strictly between 0 and 1), then its basin of attraction (the set of points converging to the fixed point under interation) is biholomorphic to $\mathbb{C}^n$. Note that this is also true for $n = 1$; there every basin of attraction of an automorphism with an attracting fixed point is the whole of $\mathbb{C}$. This sparked an immediate interest in the field of Several Complex Variables, as it provided a striking difference with the usual theory of one complex variable.

Definition 3.2.1. Let X be a complex manifold. A proper subset $\Omega \subset X$ such that Ω is biholomorphic to $\mathbb{C}^{\dim X}$ is a *Fatou-Bieberbach* domain of the first kind. A proper subset $\Omega \subset X$ such that Ω is biholomorphic to X is a *Fatou-Bieberbach* domain of the second kind.

For $X = \mathbb{C}^n$, the two notions coincide and we just call them Fatou-Bieberbach domains. When the *kind* of a Fatou-Bieberbach domain is not specified, we are referring to Fatou-Bieberbach domains of the first kind i.e. biholomorphic copies of $\mathbb{C}^{\dim X}$.

One can use Theorem 1.3.4 to show that any Stein manifold with the density property admits a Fatou-Bieberbach domain of the first kind. Existence of Fatou-Bieberbach domains of the second kind follows from Theorem 2.3.4. Details for both constructions can be found in [58].

Long before the density property was introduced, the following question already challenged researchers in the field; it is still not answered to this day.

Question 3.2.2. Does the complex torus $\mathbb{C}^* \times \mathbb{C}^*$ admit a Fatou-Bieberbach domain?

It is known that $\mathbb{C}^2$ minus a line admits Fatou-Bieberbach [52, Example 9.5], while $\mathbb{C}^2$ minus two non-intersecting lines does not admit such domains by Picard's Theorem.

Since $\mathbb{C}^* \times \mathbb{C}$ is a Stein manifold with the density property, Theorem 1.3.4 applies and we can easily obtain a Fatou-Bieberbach domain. Yet there is perhaps a more interesting reason for why this is the case. To present it, we need to introduce the notion of density property for geometric structures.

Definition 3.2.3. [59] Let X be a complex manifold and $\mathfrak{g} \subset \mathfrak{X}(X)$ be a Lie subalgebra of holomorphic vector fields on X (i.e. a *geometric structure*). The Lie subalgebra $\mathfrak{g}$ has the density property if the subalgebra $\mathfrak{g}_{int} \subset \mathfrak{g}$ generated by complete vector fields is dense in $\mathfrak{g}$ equipped with the uniform topology on compact sets.

In this language, a manifold has the density property if the Lie algebra $\mathfrak{X}(X)$ does.

The reason we introduced this notion is that Varolin proved a jet interpolation theorem for automorphisms produced by such Lie algebras. To formulate it, we will first need to understand what it means for a jet and for an automorphism to be *produced* by a Lie algebra of vector fields.

Definition 3.2.4. Let $\mathfrak{g} \subset \mathfrak{X}(X)$ be a Lie subalgebra of holomorphic vector fields and let $p \in X$. We write $J^k_{p,*}(X)_{\mathfrak{g}}$ for the set of k-jets at p of the form $[\varphi^{t_n}_{X_n} \circ \cdots \circ \varphi^{t_1}_{X_1}]^k_p$, where $X_1, \ldots, X_n \in \mathfrak{g}$ and $\varphi^t_{X_i}$ denotes the flow of X_i for $i = 1, \ldots, n$.

Definition 3.2.5. We denote by $\mathrm{Aut}_{\mathfrak{g}}(X)$ the subgroup of $\mathrm{Aut}(X)$ generated by time-1 maps of complete vector fields in $\mathfrak{g}_{int}$.

In his second paper on the density property [58], Varolin proved jet interpolation for Lie algebras with the density property.

Theorem 3.2.6. [58] *Let X be a complex manifold, $p \in X$ and $\mathfrak{g} \subset \mathfrak{X}(X)$ be a Lie algebra of vector fields with the density property. Then for all $P \in J^K_{p,*}(X)_{\mathfrak{g}}$, $K \in \mathbb{N}$, there exists an automorphism $F \in \mathrm{Aut}_{\mathfrak{g}}(X)$ such that*

$$[F]^K_p = P.$$

We now give some examples of geometric structures with the density property.

Proposition 3.2.7. [59] *Given $k, n \in \mathbb{N}$ with $1 \leq k < n$, let $\mathfrak{g}^{n,k}_0 \subset \mathfrak{X}(\mathbb{C}^n)$ be the Lie algebra of holomorphic vector fields vanishing on $\{z_{k+1} = \cdots = z_n = 0\} \cong \mathbb{C}^k$. Then $\mathfrak{g}^{n,k}_0$ has the density property for all $1 \leq k < n$.*

This fact, together with Theorem 3.2.6, allows us to construct automorphisms of $\mathbb{C}^n$ fixing a k-dimensional complex subspace L $(k < n)$ with prescribed jets outside of such subspace, hence proving the existence of Fatou-Bieberbach domains avoiding L. For the Lie algebras $\mathfrak{g}^{n,k}_0$ defined above, we can prove a parametric version of Theorem 3.2.6.

Theorem 3.2.8. [57] *Let W be a Stein space and $P\colon W \to J^K_{0,0}(\mathbb{C}^n)_{\mathfrak{g}^{n,k}_0}$ be a holomorphic family of K-jets pointwise fixing the linear subspace $\{z_{k+1} = \cdots = z_n = 0\}$. Then there exists a holomorphic map $F\colon W \to \mathrm{Aut}_{\mathfrak{g}^{n,k}_0}(\mathbb{C}^n)$ such that for any $w \in W$ the following hold:*

(i) $F^w(z) = P^w(z) + O(|z|^{K+1})$ *for $z \to 0$;*

(ii) $F^w(z_1, \ldots, z_k, 0, \ldots, 0) = (z_1, \ldots, z_k, 0, \ldots, 0)$ *for any $(z_1, \ldots, z_k) \in \mathbb{C}^k$.*

We point out that property (ii) is another way of saying that F^w belongs to the subgroup $\mathrm{Aut}_{\mathfrak{g}^{n,k}_0}(\mathbb{C}^n) \subset \mathrm{Aut}(\mathbb{C}^n)$ for all $w \in W$.

Before providing a proof of Theorem 3.2.8, we will establish a lemma analogous to Lemma 2.3.2. A polynomial vector field is called homogeneous of degree $r \in \mathbb{N}$ if each of its components is either zero or a homogeneous polynomial of degree r. Denote by $\mathscr{H}^r_{\mathfrak{g}^{n,k}_0}(\mathbb{C}^n)$ the vector space of homogeneous polynomial vector fields of degree $r \in \mathbb{N}$ that vanish on the set $\{z_{k+1} = \cdots = z_n = 0\} \subset \mathbb{C}^n$.

Lemma 3.2.9. *For any integer $r > 1$, there exists a basis of $\mathscr{H}^r_{\mathfrak{g}^{n,k}_0}(\mathbb{C}^n)$ such that each element is a Lie combination of complete vector fields in $\mathfrak{g}^{n,k}_0$.*

Proof. The proof of this lemma is contained in [58, Proof of 5.1.1]. Given a vector field $V = (V_1, \ldots, V_n)$ in $\mathscr{H}^r_{\mathfrak{g}^{n,k}_0}(\mathbb{C}^n)$, define a map δ by setting

$$\delta(V) := \begin{cases} \sum_{j=1}^n \frac{\partial V_j}{\partial z_j} & \text{for } k < n-1, \\ \sum_{j=1}^n \frac{\partial V_j}{\partial z_j} - \frac{V_n}{z_n} & \text{for } k = n-1. \end{cases}$$

We see that δ is linear and $\mathscr{H}^r_{\mathfrak{g}_0^{n,k}}(\mathbb{C}^n)$ splits into $\ker\delta \oplus W$, where W is a vector space isomorphic to $\delta\mathscr{H}^r_{\mathfrak{g}_0^{n,k}}(\mathbb{C}^n)$. The latter is itself isomorphic to the space of homogeneous polynomials of degree $r-1$ for $k < n-1$, while each monomial has to be divisible by z_n when $k = n-1$ [58].

A basis for $\ker\delta$ such that each element is a Lie combination of complete vector fields in $\mathfrak{g}_0^{n,k}$ is provided by [58, Proof of Lemma 5.4], while a basis for W with the same property is given by [58, Proof of Lemma 5.6, 5.7]. □

Proof of Theorem 3.2.8. We proceed as in the proof of Proposition 3.1.2. Precisely, we are looking for holomorphic maps $S_j\colon W \to \mathrm{Aut}_{\mathfrak{g}_0^{n,k}}(\mathbb{C}^n)$, $j = 1,\dots,K$ such that

$$P^w \circ (S_1^w)^{-1} \circ \cdots \circ (S_{j-1}^w)^{-1} = z + H_j^w(z) + O(|z|^{j+1}) \quad \text{for } z \to 0 \qquad (\star)$$

holds for every $j = 1,\dots,K$, where $H_j\colon W \to \mathscr{H}^j_{\mathfrak{g}_0^{n,k}}(\mathbb{C}^n)$ is a holomorphic family of homogeneous polynomial vector fields.

Choose S_1^w to be the linear part of P^w. By induction, suppose we have families satisfying $(\star)$ for $1 \le j \le r-1 < K$. We need to find $S_r\colon W \to \mathrm{Aut}_{\mathfrak{g}_0^{n,k}}(\mathbb{C}^n)$ such that

$$S_r^w(z) = z + H_r^w(z) + O(|z|^{k+1}) \quad \text{for } z \to 0.$$

Thanks to Lemma 3.2.9 we can write H_r^w as a Lie combination of families of complete vector fields in $\mathfrak{g}_0^{n,k}$, each depending holomorphically on $w \in W$. It is known [58, Proof of Lemma 2.1] that a composition of the time-1 maps of these families of complete vector fields gives a family of automorphisms having the prescribed r-jet at the origin. We set S_r equal to this composition to conclude the induction step.

The family of automorphisms $F^w = S_K^w \circ \cdots \circ S_1^w$ has the correct family of K-jets in the origin and pointwise fixes the subspace $\{z_{k+1} = \cdots = z_n = 0\}$. □

It is not known whether the Lie algebra consisting of vector fields on $\mathbb{C}^n$ vanishing on each axis has the density property. This problem is in the same spirit of determining whether $(\mathbb{C}^*)^n$ has the density property; both cases seem to boil down to this old basic question. Recall that a *holomorphic volume form* is a $(\dim X, 0)$-form which is holomorphic and nowhere zero. A self-map F is *volume preserving* if there exists $\lambda \in \mathbb{C}^*$, $|\lambda| = 1$ such that the pullback $F^*\omega = \lambda\omega$.

Question 3.2.10. Does there exist an automorphism $F \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ which is not volume preserving for the holomorphic volume form $\omega := \frac{dz\wedge dw}{zw}$?

It is easy to produce a large class of automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$ using overshears, but all of them preserve the form ω. For this reason, we know that $(\mathbb{C}^*)^n$ has the *volume density property* [59].

Definition 3.2.11. Let X be a manifold with a holomorphic volume form ω. X has the volume density property if the Lie algebra of volume-preserving vector fields $\{V \in \mathfrak{X}(X)\colon 0 = \mathrm{div}_\omega V := d(V \rfloor \omega)\}$ has the density property, where $V \rfloor \omega$ denotes a contraction.

Here are a few examples of affine manifolds which are known to enjoy the volume density property:

(i) $\mathbb{C}^n$ with the volume form $dz_1 \wedge \cdots \wedge dz_n$ [3];

(ii) $(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$ for $1 \leq k \leq n$ with volume form $\frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \ldots z_k}$ [59];

(iii) the Koras-Russell cubic threefold $\{x^2 y + x + z^2 + w^3 = 0\} \subset \mathbb{C}^4$ with volume form $\frac{dx}{x^2} \wedge dy \wedge dz \wedge dw$ [42];

(iv) every linear algebraic Lie group G with a left-invariant volume form and its affine homogeneous spaces admitting a left-invariant volume form for the action of G [33].

4

Parametric Jet Interpolation by Automorphisms of Stein Manifolds with the Density Property

Soon after Forstnerič published his Theorem 1.3.2, Varolin realized that the possibility to interpolate jets by automorphisms was not exclusive to complex Euclidean spaces. In 2000, he published Theorem 1.3.4, providing jet interpolation by automorphisms in any Stein manifold with the density property.

In this chapter, we present a parametric jet interpolation theorem by automorphisms for Stein manifolds with the density property obtained by the author together with Ramos-Peon [50]. Before stating the result, we will give a precise definition of the space of jets that we wish to consider.

4.1 The Jet Space

Let X be a complex manifold of dimension n and fix $r \in \mathbb{N}$. Recall the definition of a jet given in the introduction.

Definition 4.1.1. A pair of holomorphic functions $F, G : U \to X$ in a neighbourhood U of a point $p \in X$ have the same r-jet at p if $|F(z) - G(z)| = O(|z|^{r+1})$, $z \to 0$ in some (hence in any) local chart centered at p. This defines an equivalence relation for germs and its equivalence classes are called r-jets. We define a jet to be non-degenerate if its linear part (the Jacobian matrix of any representative) has non-zero determinant. We denote by $[F]_p^r$ the equivalence class of F at p.

Thus in a local chart, an r-jet can be uniquely represented by a polynomial mapping of total degree (i.e. the maximal degree of its n components) at most r. We will often drop the index r and write $[F]_p$ when the degree is clear from context.

The set of all non-degenerate r-jets at a point $p \in X$ will be denoted by $J_{p,*}^r(X)$. Again, we will often drop r and just write $J_{p,*}(X)$. Note that this is a complex manifold. Since a jet $\gamma \in J_{p,*}(X)$ is an equivalence class of germs, the value $\gamma(p) \in X$

of the jet is well-defined and we call this the *image* of γ at p. Furthermore, the map

$$\begin{aligned}\pi : J_{p,*}(X) &\to X \\ \gamma &\mapsto \gamma(p)\end{aligned}$$

is surjective and holomorphic.

For $q \in X$ we can think of $\pi^{-1}(q) =: J_{p,q}(X)$ as the set of non-degenerate r-jets at p whose image is q. For convenience and unless noted otherwise we will use the word *jet* for *non-degenerate r-jet*.

As we are interested in jet interpolation by automorphisms at more than one point, let us define the relevant spaces. Fix $N \in \mathbb{N}$ distinct points $\{\hat{x}_i\}_{i=1}^N \subset X$. We interpret this N-tuple as a point $\hat{x} \in X^N \setminus \Delta$ in the product manifold, where

$$\Delta = \bigcup_{1 \le i < j \le N} \left\{ (z_1, \ldots, z_N) \in X^N : z_i = z_j \right\}.$$

Applying the projection π coordinatewise, we obtain a map (which we still denote by π) from $J_{\hat{x}_1,*}(X) \times \cdots \times J_{\hat{x}_N,*}(X)$ to X^N. The manifold

$$Y := J_{\hat{x}_1,*}(X) \times \cdots \times J_{\hat{x}_N,*}(X) \setminus \pi^{-1}(\Delta). \tag{4.1}$$

is the complex manifold representing all possible configurations of non-degenerate r-jets at the points $\{\hat{x}_i\}_{i=1}^N$ such that their respective images are distinct.

If U is an open set containing all of the points $\{\hat{x}_i\}_{i=1}^N$ and $F : U \to X$ is an injective holomorphic map, we denote by $[F]_{\hat{x}}$ the N-tuple of jets $([F]_{\hat{x}_1}, \ldots, [F]_{\hat{x}_N}) \in Y$.

Given a jet of the form $\gamma = [F]_{\hat{x}} \in Y$, we denote by $(\gamma)^{-1}$ the jet $[F^{-1}]_{F(\hat{x})}$ of the inverse map and observe that $(\gamma)^{-1} \in Y$ if $\gamma(\hat{x}) = \hat{x}$.

In order to compare jets, we fix a distance function on $J_{p,*}(X)$; it naturally induces a distance function on X, hence on $X^N \setminus \Delta$ by taking the maximum distance of each coordinate projection. Similarly, we also have a distance function on Y. It follows from the Cauchy estimates that uniform convergence on compacts of X implies convergence in Y with respect to this distance. In order to avoid using more subscripts than necessary, we will use the abbreviation *dist* to refer to any of them; the correct domain will be clear from context.

Let W be a complex manifold and $\gamma : W \to Y$ be a holomorphic map. We are interested in finding a *holomorphic* map $F : W \to \mathrm{Aut}(X)$ such that $[F^w]_{\hat{x}} = \gamma^w$ for all $w \in W$. It is clear that a necessary condition for the existence of such a map is that $\mathrm{Aut}(X)$ is *large*; more precisely we will require X to be a Stein manifold with the density property.

Let us return to our setting. Given $\gamma = (\gamma_1, \ldots, \gamma_N) \in Y$ and any holomorphic vector field $V \in \mathfrak{X}(X)$, it defines a flow φ_V^t in a neighbourhood of $\{\pi(\gamma_i)\}_{i=1}^N$ for small enough values of t. Hence the jet $[\varphi_V^t \circ \gamma]_{\hat{x}} \in Y$ is well defined for small t. Differentiating with respect to t, we obtain a holomorphic vector field $\tilde{V} \in \mathfrak{X}(Y)$ such that $\varphi_{\tilde{V}}^t(\gamma) = [\varphi_V^t \circ \gamma]_{\hat{x}}$ for all $\gamma \in Y$; we call $\tilde{V}$ *the lift* of V. Note that if V is complete, then so is $\tilde{V}$. We denote by $\mathrm{CVF}(X)$ the set of complete holomorphic vector fields on X. The set $\widetilde{\mathfrak{X}(X)} = \{\tilde{V} \in \mathfrak{X}(Y) : V \in \mathfrak{X}(X)\}$ is a Lie subalgebra of $\mathfrak{X}(Y)$ and we note that if X has the density property, so does $\widetilde{\mathfrak{X}(X)}$ [58]. Similarly, an automorphism α of X lifts to an automorphism $\tilde{\alpha}$ of Y.

We now prove that complete vector fields on X can be lifted to span the tangent space of Y.

Lemma 4.1.2. *Let X be a Stein manifold with the density property and let $\gamma \in Y$. Then there exist $M = \dim Y \in \mathbb{N}$ complete vector fields $\{\theta_i\}_{i=1}^M \subset \mathrm{CVF}(X)$ such that $\{\tilde{\theta}_i(\gamma)\}_{i=1}^M$ is a basis for the tangent space of Y at the point γ. In particular the map*

$$\begin{aligned} \mathbb{C}^M &\to Y \\ (t_1, \dots, t_M) &\mapsto \varphi_{\tilde{\theta}_1}^{t_1} \circ \cdots \circ \varphi_{\tilde{\theta}_M}^{t_M}(\gamma) \end{aligned}$$

is a biholomorphism from a neighbourhood of 0 to a neighbourhood of γ.

Proof. It is sufficient to prove the lemma for $\gamma = [\mathrm{Id}]_{\hat{x}}$ since, thanks to Theorem 1.3.4, there exists $F \in \mathrm{Aut}(X)$ such that $[F \circ \gamma]_{\hat{x}} = [\mathrm{Id}]_{\hat{x}}$. If the vector fields $\{\tilde{\theta}_i\}_{i=1}^M$ span the tangent space of Y at $[\mathrm{Id}]_{\hat{x}}$, then $\{\widetilde{F^*\theta_i}\}_{i=1}^M$ span $T_\gamma Y$.

We first claim that the conclusion is true for $N = 1$ and $X = \mathbb{C}^n$, $n > 1$. To see this, suppose $\hat{x} = (0, \dots, 0)$ and consider the vector fields $V_{I,j} = z^I \frac{\partial}{\partial z_j}$ where I runs through the multi-indexes of degree less or equal than r and $j = 1, \dots, n$. Not all of them are complete, but they do span the tangent space of Y at $[\mathrm{Id}]_{\hat{x}}$. As X has the density property, we can approximate each $V_{I,j}$ by a sum of complete vector fields. For a good enough approximation, this new collection must also span the tangent space of Y at $[\mathrm{Id}]_{\hat{x}}$. The claim is proved by picking out a basis from this generating set.

A similar proof works for an arbitrary Stein manifold X with the density property. Let U be a Runge coordinate neighbourhood of $\hat{x}$ such that $\hat{x}$ corresponds to $0 \in \mathbb{C}^n$ under the chart map. On U we consider the pullback of the vector fields $V_{I,j}$ above. As U is Runge and X has the density property, we conclude as above by approximating these pulled-back fields by sums of complete vector fields.

Let now X be as above and $N > 1$. We choose coordinate neighbourhoods U_i of $\hat{x}_i$, $i = 1, \dots N$ such that $U = U_1 \cup \cdots \cup U_N$ is Runge in X and each $\hat{x}_j$ is mapped to zero under the appropriate coordinate chart. For each $i = 1, \dots, N$, we pull back $V_{I,j}$ on U_i and extend it to the zero field on the other coordinate neighbourhoods in order to obtain a generating set for Y at $[\mathrm{Id}]_{\hat{x}}$, and proceed as above to obtain complete vector fields. □

The above proof is just one step short of showing that Y is an *elliptic manifold*.

Definition 4.1.3. A manifold Y is elliptic if it admits a *dominating holomorphic spray*, which is a triple (E, π, s) consisting of

(i) a holomorphic vector bundle $\pi \colon E \to Y$;

(ii) a holomorphic map $s\colon E \to Y$ such that for each $y \in Y$ we have $s(0_y) = y$ and its differential restricted to the *vertical space* $ds_{|\operatorname{Ker} d\pi}(0_y)\colon \operatorname{Ker} d\pi_{0_y} \subset T_{0_y}E \to T_yY$ is surjective.

Any elliptic manifold Y is also an Oka-Forstnerič manifold (see [20] for both the theorem and the definition), hence it satisfies the following h-principle: given a continuous homotopy $f^t : W \to Y$ ($t \in [0,1]$), where W is Stein and f^0, f^1 are holomorphic, there exists a new homotopy $g^t : W \to Y$ which is smooth in $t \in [0,1]$, holomorphic in $w \in W$ for all $t \in [0,1]$ and $f^0 = g^0, f^1 = g^1$. We call such a special homotopy a *smooth isotopy of holomorphic maps* (or jets, in the case that Y is as defined previously in this section).

Proposition 4.1.4. *Let Y be a Stein manifold such that* $\operatorname{Aut}(Y)$ *acts transitively on Y. Assume there exists a collection $\{V_1, \ldots, V_N\} \subset \operatorname{CVF}(Y) \subset \mathfrak{X}(Y)$ of complete holomorphic vector fields such that $\{V_1(p), \ldots, V_N(p)\} \subset T_pY$ span the tangent space at a point $p \in Y$. Then Y is elliptic.*

Proof. This proof is the same as the one given by Kaliman and Kutzschebauch in [31, Theorem 4], we include it here for completeness.

We begin by pointing out that $\operatorname{Aut}(Y)$ equipped with the topology of uniform convergence on compacts together with uniform convergence on compacts of the inverses is Hausdorff, regular, and second-countable, hence metrizable by Urysohn's metrization theorem and therefore a Baire space.

As the vector fields $\{V_1, \ldots, V_N\}$ span the tangent space at p, they must span it everywhere on Y besides at the points of an analytic set $A \subset Y$. We want to prove the existence of $F \in \operatorname{Aut}(Y)$ such that the collection $\{dF(V_1), \ldots, dF(V_N)\} \subset \operatorname{CVF}(Y)$ spans the tangent space at a generic point of A. By induction on the dimension of this exceptional set, we will finally produce finitely many vector fields $\{W_1, \ldots, W_M\} \subset \operatorname{CVF}(Y)$, $M \in \mathbb{N}$ spanning the tangent space at every point of Y. This proves the proposition as we can then produce the following dominating holomorphic spray:

$$\begin{aligned} s\colon \mathbb{C}^M \times Y &\to Y \\ (t_1, \ldots, t_M, y) &\mapsto \varphi_{W_M}^{t_M} \circ \cdots \circ \varphi_{W_1}^{t_1}(y). \end{aligned}$$

Let $A = \bigcup_i A_i$ be the decomposition of A in its (at most countably many) irreducible components. Let $B_i \subset \operatorname{Aut}(Y)$ be the set of automorphisms $F \in \operatorname{Aut}(Y)$ such that $\{dF(V_1), \ldots, dF(V_N)\}$ span the tangent space of Y at a generic point of A_i. It is clear that B_i is a non-empty open set. We will show that it is also dense, hence by Baire's Theorem there exists an automorphism in $\bigcap_{i \in \mathbb{N}} B_i$, which is precisely what we are looking for.

To show that B_i is dense in $\operatorname{Aut}(Y)$, consider $G \notin B_i$ and a complete vector field $V \in \mathfrak{X}(Y)$ which is not tangent to A_i at a point $p \in A_i$. The latter exists as $\operatorname{Aut}(Y)$ acts transitively and we already have a generating set in a point. For any small value of $t > 0$, the automorphism $\varphi_V^t \circ G$ is then in B_i, hence B_i is dense in $\operatorname{Aut}(Y)$ and the proof is concluded. □

4.2 Main result and proof

Now that we carefully defined our configuration space of jets Y, we can formulate the main result of this chapter. As we will deal with parametrized families of jets and automorphisms, let us fix the following notation: for $F: W \to \mathrm{Aut}(X)$ denote by $F^w \in \mathrm{Aut}(X)$ the automorphism obtained by evaluating F at $w \in W$.

Theorem 4.2.1. [50] *Let W be a Stein manifold, X a Stein manifold with the density property, $r, N \in \mathbb{N}$, $(\hat{x}_1, \dots, \hat{x}_N)$ an N-tuple of distinct points in X, and let Y be as defined in (4.1) above. Given a holomorphic map $\gamma: W \to Y$, there exists a null-homotopic holomorphic map $F: W \to \mathrm{Aut}(X)$ such that $[F^w]_{\hat{x}} = \gamma^w$ if and only if γ is null-homotopic.*

The proof of Theorem 4.2.1 is rather technical and it follows the same ideas portrayed in [39], where Kutzschebauch and Ramos-Peon prove Theorem 4.2.1 for $r = 0$ i.e. pointwise interpolation, without any condition on the derivatives [39, Theorem 1.1]; we will now give a brief outline of it.

As our family of jets is null-homotopic, denote by γ^t, $t \in [0,1]$ the homotopy such that $\gamma^1 = \gamma$ and γ^0 is a constant map, which we can assume to be $[\mathrm{Id}]_{\hat{x}} \in Y$. Thanks to [39, Theorem 1.1] we can further assume that $\gamma^{t,w}(\hat{x}) = \hat{x}$ for all $(t,w) \in [0,1] \times W$. After fixing a compact set $L_0 \subset W$, we will be able to produce a map $f^t: L_0 \times U \subset W \times X \to X$ depending smoothly on $t \in [0,1]$ and holomorphically on $(w,z) \in L_0 \times U$, where $U \subset X$ is an open set containing each $\hat{x}_i$, $i = 1, \dots, N$, such that $[f^{t,w}]_{\hat{x}} = \gamma^{t,w}$ for $(t,w) \in [0,1] \times L_0$. We then apply Theorem 2.3.7 to obtain a parametrized family of automorphisms $F_0^t: W \to \mathrm{Aut}(X)$ such that $[F_0^{t,w}]_{\hat{x}}^{-1} \circ \gamma^{t,w}$ is close to the jet of the identity $[\mathrm{Id}]_{\hat{x}}$, hence obtaining an approximate solution for $w \in L_0$. This first step is formalized in Proposition 4.2.2.

We now wish to repeat the same process inductively on larger and larger compact sets $L_0 \subset L_1 \subset \cdots \subset \bigcup_{j \geq 0} L_j = W$ exhausting W. This can be done easily and we do obtain a sequence of automorphisms whose jet at $\hat{x}$ converges to γ, but we do not know whether such sequence converges to an automorphism as well. In order to solve this problem, we will require our next automorphism $F_1^t: W \to \mathrm{Aut}(X)$ to be close to the identity on a compact set $K_1 \subset X$ for $w \in L_0$, while still approximatively interpolating the correct jet for $w \in L_1$. After exhausting X by compact sets $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j \geq 0} K_j = X$, an inductive procedure will provide the correct sequence of automorphisms.

The main difficulty in our induction step is combining the jet approximation on L_j together with the approximation of the identity map on $L_{j-1} \times K_j$. In fact, the approximation of the identity map can only be performed if our jet approximation on L_{j-1} obtained in the previous step is good enough. This is the content of Proposition 4.2.3.

Conversely, if F in Theorem 4.2.1 is null-homotopic, then clearly so is γ.

We now go into details, starting with the proposition that will allow us to start the induction. From now on, Y will denote the jet configuration space in (4.1).

Proposition 4.2.2. *Let W be a Stein manifold and X be a Stein manifold with the density property. Let $\gamma^1 \colon W \to Y$ be holomorphic and null-homotopic, with γ^t denoting the homotopy from γ^1 to the constant jet $\gamma^0 = [\mathrm{Id}]_{\hat{x}}$. Given a number $\varepsilon > 0$ and a holomorphically convex compact set $L = \hat{L} \subset W$, there exists a family of parametrized automorphisms $F \colon [0,1] \times W \to \mathrm{Aut}(X)$ with $F^0 = \mathrm{Id}$ such that*

$$d([F^{t,w} \circ \gamma^{t,w}]_{\hat{x}}, [\mathrm{Id}]_{\hat{x}}) < \varepsilon \quad \text{for all } (t,w) \in [0,1] \times L.$$

Proof. Since both γ^0 and γ^1 are holomorphic, W is Stein and Y is elliptic by Proposition 4.1.4, the h-principle applies; we can therefore assume that $\gamma^t \colon W \to Y$ is in fact holomorphic and its dependence on t is smooth.

Define $x^t := \pi(\gamma^t)$, i.e. the image of the jet γ^t at $\hat{x}$. As x^t takes values in $X^N \setminus \Delta$, we may apply [39, Theorem 1.1], hence assume without any loss of generality that for all $(t,w) \in [0,1] \times W$ the image of $\gamma^{t,w}$ (hence also the one of $(\gamma^{t,w})^{-1}$) is the fixed N-tuple $\hat{x}$.

This allows us to uniquely represent the jets $(\gamma^{t,w})^{-1}$ by parametrized polynomial mappings of total degree at most r fixing $0 \in \mathbb{C}^n$. Indeed, let $U \subset X$ be a disjoint union of coordinate neighbourhoods U_j of the fixed points $\hat{x}_j$ and $\phi_j : U_j \to \phi_j(U_j) \subset \mathbb{C}^n$ be charts centered at $\hat{x}_j$. For each $j = 1, \ldots, N$ and $(t,w) \in [0,1] \times W$ there exists a uniquely determined polynomial mapping $Q_j^{t,w}$ of total degree at most r such that

$$\left[\phi_j \circ (\gamma^{t,w})^{-1} \circ \phi_j^{-1}\right]_0 = \left[Q_j^{t,w}\right]_0.$$

By uniqueness, these polynomials mappings which fix $0 \in \mathbb{C}^n$ depend smoothly on $t \in [0,1]$ and holomorphically on $w \in W$. For fixed $w \in W$ and for each $j = 1, \ldots, N$, since non-degenerate mappings are locally invertible, there exists a neighbourhood V_j of 0 in $\mathbb{C}^n$ such that for all $t \in [0,1]$, $Q_j^{t,w}|_{V_j}$ is injective and $Q_j^{t,w}(V_j) \subset \phi_j(U_j)$. Given a compact set $L' \supset L$ such that $\mathring{L}' \supset L$, by compactness we can shrink V_j such that the above holds for all $(t,w) \in [0,1] \times L'$. Let V be the disjoint union of $\phi_j^{-1}(V_j)$, and define the injective holomorphic map P^t on $\mathring{L}' \times V$ by setting

$$P^t(w,z) := (w, \phi_j^{-1} \circ Q_j^{t,w} \circ \phi_j(z)) : \mathring{L}' \times V \to W \times X.$$

Note that the union of the "graphs of the N fixed points" $K = \bigcup_{j=1}^N \{(w, \hat{x}_j) : w \in L\} \subset W \times X$ is a $\mathcal{O}(W \times X)$-convex set which is fixed by P^t. Since P^0 is the identity, we can apply Theorem 2.3.7 and obtain a family of parametrized automorphisms $F^{t,w}$ which approximates the second component of $P^{t,w}$ uniformly on compacts in a neighbourhood of K. By the Cauchy estimates, this implies the approximation of the jet. □

The following proposition is similar to [39, Proposition 4.4] but with jet approximation instead of just pointwise approximation. It will provide us with the inductive step.

Proposition 4.2.3. *Let $L_1, L_2 \subset W$ be $\mathscr{O}(W)$-convex compact sets such that $L_1 \subset \mathring{L}_2$ and let $K, C \subset X$ be $\mathscr{O}(X)$-convex compact sets with $K \subset \mathring{C}$. For every $\varepsilon > 0$, there exists $\delta = \delta(K, L_1, \varepsilon) > 0$ such that if $\gamma\colon [0,1] \times W \to Y$ is a smooth isotopy of holomorphic jets satisfying $\gamma^{0,w} = [\mathrm{Id}]_{\hat{x}}$ for all $w \in W$ and*

$$\mathrm{dist}(\gamma^{t,w}, [\mathrm{Id}]_{\hat{x}}) < \delta \quad \forall (t,w) \in [0,1] \times L_1, \tag{4.2}$$

then

(i) there exists $\psi\colon [0,1] \times L_1 \to \mathrm{Aut}(X)$ holomorphic in a neighbourhood of L_1, with $\psi^{0,w} = \mathrm{Id}$ for all $w \in L_1$ and such that

$$\mathrm{dist}(\psi^{t,w}(z), z) < \varepsilon \quad \textit{for } (t,w,z) \in [0,1] \times L_1 \times K, \tag{4.3}$$

$$[\psi^{t,w}]_{\hat{x}} = \gamma^{t,w} \quad \textit{for } (t,w) \in [0,1] \times L_1. \tag{4.4}$$

(ii) For every $\alpha > 0$, there exists $F\colon [0,1] \times W \to \mathrm{Aut}(X)$ holomorphic with $F^{0,w} = \mathrm{Id}$ for all $w \in W$ and such that

$$\mathrm{dist}([F^{1-t,w} \circ \gamma^{1,w}]_{\hat{x}}, \gamma^{t,w}) < \alpha \quad \textit{for } (t,w) \in [0,1] \times L_2, \tag{4.5}$$

$$\mathrm{dist}(F^{1-t,w}(z), \psi^{t,w}(z)) < \varepsilon \quad \textit{for } (t,w,z) \in [0,1] \times L_1 \times C. \tag{4.6}$$

Proof. Let us examine the nature of the local biholomorphism near $[\mathrm{Id}]_{\hat{x}} \in Y$ given by Lemma 4.1.2. Note in particular that given a compact $K \subset X$, for $(t_1, \dots, t_M) \in U \subset \mathbb{C}^M$ small enough, the automorphism $\varphi^{t_1}_{\theta_1} \circ \cdots \circ \varphi^{t_M}_{\theta_M} \in \mathrm{Aut}(X)$ is going to be arbitrarily close to the identity on K. We can then pick δ such that

$$\{\gamma^{t,w} : (t,w) \in [0,1] \times L_1\} \subset \{\varphi^{t_1}_{\tilde{\theta}_1} \circ \cdots \circ \varphi^{t_M}_{\tilde{\theta}_M}([\mathrm{Id}]_{\hat{x}}) : (t_1, \dots, t_M) \in U\} \subset Y.$$

Apply Lemma 4.1.2 to obtain a family of parametrized automorphisms $\psi\colon [0,1] \times L_1 \to \mathrm{Aut}(X)$ such that $\psi^0 = \mathrm{Id}$, $\mathrm{dist}(\psi^{t,w}(z), z) < \varepsilon/2$ and $[\psi^{t,w}]_{\hat{x}} = \gamma^{t,w}$ for $(t,w,z) \in [0,1] \times L_1 \times K$. This proves (i).

Consider the non-autonomous parametrized vector field on X

$$\Theta^{t,w}(z) = \frac{d}{ds}\bigg|_{s=t} \psi^{1-s,w}((\psi^{1-t,w})^{-1}(z))$$

with flow $\varphi^{t,0,w}_{\Theta} = \psi^{1-t} \circ (\psi^1)^{-1}$ defined for $(t,w,z) \in [0,1] \times L_1 \times X$ and note that the lift of Θ to Y satisfies

$$\tilde{\Theta}^{t,w}(\gamma^{1-t,w}) = \frac{d}{ds}\bigg|_{s=t} \gamma^{1-s,w}.$$

Therefore Θ is well defined on $[0,1] \times L_1 \times X \cup [0,1] \times W \times \{\hat{x}_1, \dots, \hat{x}_N\}$.

By Lemma 2.3.8 there is a Runge neighbourhood $\Omega \subset W \times X$ of $L_1 \times K \cup L_2 \times \{\hat{x}_1, \dots, \hat{x}_N\}$ such that the flow $f^t\colon \Omega \to W \times X$ of Θ^t is injective and $f^t(\Omega)$ is Runge for every $t \in [0,1]$. Using Theorem 2.3.7, we obtain the desired $F^t\colon W \to \mathrm{Aut}(X)$. Observe that condition (4.4) only depends on this last step, hence we can approximate arbitrarily well and choose α only when invoking (ii). □

To prove the main theorem, we will construct families of holomorphic automorphisms $F\colon [0,1]\times W \to \mathrm{Aut}(X)$ using the above result inductively on a growing sequence of compacts of $W \times X$. In order to apply Proposition 4.2.3 again (on a larger compact, in views of exhausting the parameter space W), we need a smooth isotopy of parametrized jets that are close to the identity for all $t \in [0,1]$ over the compact L_2. This homotopy does not come for free during the induction step for the following reason.

Let γ^t be as above starting at the constant jet $\gamma^0 = [\mathrm{Id}]_{\hat{x}}$ and apply Proposition 4.2.3 to obtain a family of parametrized automorphisms F^t. We now have a new homotopy of jets

$$h : [0,1] \times W \to Y$$

given by

$$h^{t,w} = \begin{cases} \gamma^{2t,w} & \text{for } 0 \le t \le \frac{1}{2}, \\ [F^{2t-1,w} \circ \gamma^{1,w}]_{\hat{x}} & \text{for } \frac{1}{2} \le t \le 1, \end{cases}$$

connecting $h^{0,w} = [\mathrm{Id}]_{\hat{x}}$ to $h^{1,w} \approx [\mathrm{Id}]_{\hat{x}}$ for $w \in L_2$. We cannot immediately use Proposition 4.2.3 for $h^{t,w}$ over the larger L_2, as the smallness condition (4.2) required for all $t \in [0,1]$ is only satisfied at the end point $t = 1$.

However, this issue can be handled as follows:

Lemma 4.2.4. [39, Lemma 4.2] *Let $L \subset W$ be a $\mathcal{O}(W)$-convex compact set and $h^t : W \to Y$ be a smooth homotopy between the constant $h^0 = [\mathrm{Id}]_{\hat{x}}$ and a holomorphic map h^1. Then there exists $\varepsilon = \varepsilon(h, L) > 0$ such that for every $0 < \alpha < \varepsilon$, every smooth $F\colon [0,1]\times W \to Y$ with $F^t = h^{2t}$ for $t \le \frac{1}{2}$ satisfying*

$$\mathrm{dist}(F^{t,w}, F^{1-t,w}) < \alpha/2 \quad \textit{for } (t,w) \in [0,1] \times L,$$

and every $\mathcal{O}(W)$-convex compact set $L' \subset \mathring{L}$, there exists an analytic homotopy H : $[0,1] \times W \to Y$ between $[\mathrm{Id}]_{\hat{x}}$ and h^1 such that

$$\mathrm{dist}(H^{t,w}, [\mathrm{Id}]_{\hat{x}}) < \alpha \quad \textit{for } (t,w) \in [0,1] \times L'.$$

Note that [39, Lemma 4.2] is *stated* for the manifold Y which stands for $X^N \setminus \Delta$. However, the proof uses only general topological constructions, as well as the Oka property which holds for any elliptic manifold Y. The idea is to introduce a new parameter $s \in [0,1]$ and use it to connect each point $F^{t,w}$ for $t \le \frac{1}{2}$ to its *across neighbour* $F^{1-t,w}$. This gives a map G from the square $[0,1]_s \times [0,1]_t$ to the space of mappings Y^W, satisfying $G(0,t) = F^{t,\cdot}$ and $G(1,t) = F^{1-t,\cdot}$ for all $t \le \frac{1}{2}$. We then retract on the bottom interval $[0,1]_s \times \{0\}$ while keeping the bottom left corner $(0,0)$ and the bottom right corner $(1,0)$ fixed thanks to the Oka property. The details can be found in [39].

We now proceed with the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $L_j \subset W$, $K_j \subset X$, $j \in \mathbb{N}$ be exhaustions by compact holomorphically convex compact sets such that $L_j \subset \mathring{L}_{j+1}$ and $K_j \subset \mathring{K}_{j+1}$. Assume that

$L_0 = \emptyset$ and $(\hat{x}_1, \dots, \hat{x}_N) \in \mathring{K}_0$. Fix a sequence $\varepsilon_j > 0$ such that $\varepsilon_j < \operatorname{dist}(K_{j-1}, X \setminus K_j)$ and $\sum \varepsilon_j < +\infty$.

Since $\gamma : W \to Y$ is null-homotopic, there exists an isotopy on holomorphic maps $\gamma^t : W \to Y$ such that $\gamma^0 = [\mathrm{Id}]_{\hat{x}}$ and $\gamma^1 = \gamma$. We can immediately apply Proposition 4.2.2 to obtain $\varphi_0 : [0,1] \times W \to \operatorname{Aut}(X)$ such that

$$\operatorname{dist}([\varphi_0^{t,w} \circ \gamma^{t,w}]_{\hat{x}}, [\mathrm{Id}]_{\hat{x}}) < \min(\varepsilon_0, \delta(K_0, L_1, \varepsilon_1/2)) \quad \text{for } (t,w) \in [0,1] \times L_1,$$

where δ is as in Proposition 4.2.3.

Conclusion (i) in the latter gives $\psi : [0,1] \times L_1 \to \operatorname{Aut}(X)$ such that

$$\begin{aligned} \operatorname{dist}(\psi^{t,w}(z), z) &< \varepsilon_1/2 \quad \text{for } (t,w,z) \in [0,1] \times L_1 \times K_0, \\ [\psi^{t,w}]_{\hat{x}} &= [\varphi_0^{t,w} \circ \gamma^{t,w}]_{\hat{x}} \quad \text{for } (t,w) \in [0,1] \times L_1. \end{aligned}$$

Let $C_1 \subset X$ be a $\mathscr{O}(X)$-convex compact containing the $\varepsilon_1/2$-envelope of $K_1 \cup \psi^{[0,1],L_1}(K_1)$ i.e. all points at distance at most $\varepsilon_1/2$ from $K_1 \cup \psi^{[0,1],L_1}(K_1)$. Let

$$\alpha_1 := \min(\varepsilon_1, \delta(C_1, L_2, \varepsilon_2/2), \varepsilon([\varphi_0^{t,w} \circ \gamma^{t,w}]_{\hat{x}}, L_2))/2,$$

where $\varepsilon([\varphi_0^{t,w} \circ \gamma^{t,w}]_{\hat{x}}, L_2)$ is the one arising from Lemma 4.2.4.

By (ii) in Proposition 4.2.3 there exists $\varphi_1 : [0,1] \times W \to \operatorname{Aut}(X)$ such that

$$\operatorname{dist}([\varphi_1^{1-t,w} \circ \varphi_0^{1,w} \circ \gamma^{1,w}]_{\hat{x}}, [\varphi_0^{t,w} \circ \gamma^{t,w}]_{\hat{x}}) < \alpha_1$$

for $(t,w) \in [0,1] \times L_2$ and

$$\operatorname{dist}(\varphi_1^{1-t,w}(z), \psi^{t,w}(z)) < \varepsilon_1/2 \quad \text{for } (t,w,z) \in [0,1] \times L_1 \times C_1.$$

We just proved the initial step of the following induction.

Suppose for $j = 1, \dots, k$ we have $C_j \subset X$ holomorphically convex sets and $\varphi_j : [0,1] \times W \to \operatorname{Aut}(X)$ such that

(a) $\varphi_k^{0,w} = \mathrm{Id} \in \operatorname{Aut}(X)$ for all $w \in W$;

(b) $\operatorname{dist}([\varphi_k^{1-t,w} \circ \varphi_{k-1}^{1,w} \circ \cdots \circ \varphi_0^{1,w} \circ \gamma^{1,w}]_{\hat{x}}, h^{t,w}) < \alpha_k$ for $(t,w) \in [0,1] \times L_{k+1}$;

(c) $\mathring{C}_k \supset K_k \cup F_k^{t,w}((F_{j-1}^{t,w})^{-1}(K_k))$ for each $j = 1, \dots, k$, $t \in [0,1]$ and $w \in L_j \setminus L_{j-1}$;

(d) for each $j = 1, \dots, k$, if $w \in L_j \setminus L_{j-1}$ then $\operatorname{dist}(\varphi_k^{t,w}(z), z) < \varepsilon_k$ for each $t \in [0,1]$ and $z \in K_k \cup F_{k-1}^{t,w}((F_{j-1}^{t,w})^{-1}(K_k))$.

where $h : [0,1] \times W \to Y$ is a homotopy between $[\mathrm{Id}]_{\hat{x}}$ and $[\varphi_{k-1}^{1,w} \circ \cdots \circ \varphi_0^{1,w} \circ \gamma^{1,w}]_{\hat{x}}$,

$$F_k^{t,w}(z) = \varphi_k^{t,w} \circ \varphi_{k-1}^{t,w} \circ \cdots \circ \varphi_0^{t,w}(z),$$

and

$$\alpha_k = \min(\varepsilon_k, \delta(C_k, L_{k+1}, \varepsilon_{k+1}/2), \varepsilon(h, L_{k+1}))/2.$$

We first explain how the induction provides a family of parametrized automorphisms $G\colon [0,1]\times W \to \mathrm{Aut}(X)$ such that $G^{0,w} = \mathrm{Id}$ and $G^{1,w}$ satisfies the conclusion of the theorem. Thanks to (c) and (d), Theorem 2.4.2 ensures that the sequence $F_k\colon [0,1]\times W \to \mathrm{Aut}(X)$ converges to $F\colon [0,1]\times W \to \mathrm{Aut}(X)$ uniformly on compacts, while condition (b) evaluated at $t = 0$ shows that the inverse of $F^{1,w}$ provides the required parametrized automorphism $G^{1,w}$. The fact that such inverse is null-homotopic is guaranteed by (a).

Let us now assume that we have the required objects for $j = 1,\dots,k$, we begin by considering the homotopy $\tilde{H}\colon [0,1]\times W \to \mathrm{Aut}(X)$ given by

$$\tilde{H}^{t,w} = \begin{cases} h^{2t,w} & \text{for } 0 \le t \le \frac{1}{2}, \\ [\varphi_k^{2t-1,w} \circ \varphi_{k-1}^{1,w} \circ \cdots \circ \varphi_0^{1,w} \circ \gamma^{1,w}]_{\hat{x}} & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that condition (b) and the definition of α_k ensure we can apply Lemma 4.2.4, hence there exists an homotopy $H\colon [0,1]\times W \to \mathrm{Aut}(X)$ such that $H^{0,w} = \mathrm{Id}$, $H^{1,w} = [\varphi_k^{1,w} \circ \varphi_{k-1}^{1,w} \circ \cdots \circ \varphi_0^{1,w} \circ \gamma^{1,w}]_{\hat{x}}$ and

$$\mathrm{dist}(H^{t,w}, [\mathrm{Id}]_{\hat{x}}) < \delta(C_k, L_{k+1}, \varepsilon_{k+1}/2) \quad \text{for } (t,w) \in [0,1]\times L_k.$$

Part (i) of Proposition 4.2.3 provides the existence of $\psi\colon [0,1]\times L_k \to \mathrm{Aut}(X)$ such that

$$\mathrm{dist}(\psi^{t,w}(z), z) < \varepsilon_{k+1}/2 \quad \text{for } (t,w,z) \in [0,1]\times L_k \times C_k, \tag{4.7}$$

$$[\psi^{t,w}]_{\hat{x}} = H^{t,w} \quad \text{for } (t,w) \in [0,1]\times L_k. \tag{4.8}$$

Let $C_{k+1} \subset X$ be a holomorphically convex compact set containing the $\varepsilon_{k+1}/2$-envelope of

$$C_k \cup K_{k+1} \cup \psi^{[0,1],L_k}(K_{k+1}) \cup \psi^{[0,1],L_{j-1}}(K_{k+1}) \tag{$*$}$$

and of

$$\psi^{t,w}(F_k^{t,w}((F_{j-1}^{t,w})^{-1}(K_{k+1})) \tag{$*$}$$

for each $j = 1,\dots,k$ and $(t,w) \in [0,1]\times L_{j-1}$.

Part (ii) of Proposition 4.2.3 gives $\varphi_{k+1}\colon [0,1]\times W \to \mathrm{Aut}(X)$ with $\varphi_{k+1}^{0,w} = \mathrm{Id}$ for all $w \in W$ and such that

$$\mathrm{dist}([\varphi_{k+1}^{1-t,w} \circ H^{1,w}]_{\hat{x}}, H^{t,w}) < \alpha_{k+1} \quad \text{for } (t,w) \in [0,1]\times L_{k+2}, \tag{4.9}$$

$$\mathrm{dist}(\varphi_{k+1}^{1-t,w}(z), \psi^{t,w}(z)) < \varepsilon_{k+1}/2 \quad \text{for } (t,w,z) \in [0,1]\times L_k \times C_{k+1}, \tag{4.10}$$

with $\alpha_{k+1} = \min(\varepsilon_{k+1}, \delta(C_{k+1}, L_{k+2}, \varepsilon_{k+2}/2), \varepsilon(H, L_{k+2}))/2$.

We now explain the reason these choices provide the $k+1$-th step. Condition (a) is clearly satisfied. Equation (4.9) and the fact that $H^{1,w} = [\varphi_k^{1,w} \circ \cdots \circ \varphi_0^{1,w} \circ \gamma^{1,w}]_{\hat{x}}$ provide condition (b). Equation (4.10) tells that the image of φ_{k+1} is ε_{k+1}-close to the one of ψ, hence C_{k+1} also contains the sets in ($*$) if we substitute ψ with φ_{k+1}, thus condition (c) is fulfilled. Condition (d) is obtained as a combination of (4.7) and (4.10).

□

5

Strongly Tame Sets

The main resource for this chapter is the joint work [7] with Rafael Andrist. Let us recall the definition of tame and strongly tame set:

Definition 5.0.1. [52] Let e_1 be the first standard basis vector of $\mathbb{C}^n$. A closed infinite discrete sequence $\{a_j\}_{j\geq 1} \subset \mathbb{C}^n$ is called *tame* (in the sense of Rosay and Rudin) if there exists $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that $F(a_j) = j \cdot e_1$ for all $j \geq 1$.

When referring to the set $\mathbb{N}e_1 \subset \mathbb{C}^n$, we will often just write $\mathbb{N} \subset \mathbb{C}^n$.

Definition 5.0.2. [7] Let X be a complex manifold and let $\mathrm{Aut}(X)$ be its group of holomorphic automorphisms. A closed discrete infinite set $A \subset X$ is a *strongly tame set* if for every injective mapping $f : A \to A$ there exists a holomorphic automorphism $F \in \mathrm{Aut}(X)$ such that $F_{|_A} = f$.

We begin with some basic properties of strongly tame sets, then focus on the problem of existence of such sets. We first consider the special linear group $\mathrm{SL}_2(\mathbb{C})$, where we can prove the existence of tame sets using automorphisms coming from the explicit description of the corresponding Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. In Section 5.3, we use this knowledge to construct tame sequences in any linear algebraic Lie group of dimension at least 2. There, we also show that existence is guaranteed by the presence of two commuting vector fields which kernels are not contained one into the other (Theorem 5.3.7).

5.1 Basic Properties

The first use of the word *tame* in this context is due to Rosay and Rudin, who introduced Definition 5.0.1 in 1988. The first lemma we wish to prove shows why Definition 5.0.2 coincides with the standard definition of tameness for $X = \mathbb{C}^n$.

Lemma 5.1.1. *Let $A \subset \mathbb{C}^n$ be a closed discrete infinite subset. Then A is strongly tame in the sense of Definition 5.0.2 if and only if it is tame in sense of Definition 5.0.1.*

Proof. Assume that A is a strongly tame set. Consider the projections of A to the coordinates axes in $\mathbb{C}^n$. Since A must be an unbounded set, at least one projection $\pi\colon \mathbb{C}^n \to L$ to a certain coordinate axis must be such that $\pi(A)$ is an unbounded set. Pick an infinite discrete closed subset $B := \{b_j\}_{j\in\mathbb{N}} \subset \pi(A)$. Without loss of generality we may assume that L coincides with the first coordinates axis. By the Mittag-Leffler Theorem we find a holomorphic automorphism of the form $F(z_1, z_2, \dots, z_n) = (z_1, z_2 + f_2(z_1), \dots, z_n + f_n(z_1))$ such that $F(B) \subseteq A$. Since A is strongly tame, there exists an automorphism $G \in \mathrm{Aut}(\mathbb{C}^n)$ such that $G(F(B)) = A$. We are reduced to the situation where $A \subset L$. According to [52, Proposition 3.1, Remark 3.4] an infinite closed discrete set that is contained in a complex line $L \subset \mathbb{C}^n$ is tame.

Assume now that $A \subset \mathbb{C}^n$ is a tame set in the sense of Rosay and Rudin. Then every infinite subset of $B \subset A$ is tame as well, again by [52, Proposition 3.1, Remark 3.4]. Let $G, H \in \mathrm{Aut}(\mathbb{C}^n)$ be such that $G(A) = \mathbb{N}$ and $H(B) = \mathbb{N}$. Then $F = H^{-1} \circ G \in \mathrm{Aut}(\mathbb{C}^n)$ satisfies $F(A) = B$. Moreover we can achieve that for any given injective map $f : A \to B$ the automorphism F coincides with f on A, see [52, Remark 3.2]. □

Now that this is established, we will use the word *tame* to refer to strongly tame sets in any manifold.

The original Definition 5.0.1 of Rosay and Rudin clearly implies that any two tame sequences can be mapped bijectively onto one another by an automorphism of $\mathbb{C}^n$, as they can both be mapped to $\mathbb{N} \subset \mathbb{C}^n$. It is not immediate from Definition 5.0.2 that any two strongly tame sets are equivalent, in the sense that they can be mapped bijectively into one another by an automorphism of X. We show that this is the case when X is a Stein manifold with the density property.

Proposition 5.1.2. *Let X be a Stein manifold with the density property and let $A, B \subset X$ be tame sets. Then there exists $F \in \mathrm{Aut}(X)$ such that $F(A) = B$.*

Proof. Write $A = \{a_i\}_{i\in\mathbb{N}}$ and $B = \{b_j\}_{j\in\mathbb{N}}$. By the definition of tame set, it is enough to find subsequences $\{a_{i_k}\}_{k\in\mathbb{N}} \subset A$, $\{b_{j_k}\}_{k\in\mathbb{N}} \subset B$ and $F \in \mathrm{Aut}(X)$ such that $F(a_{i_k}) = b_{j_k}$ for all $k \in \mathbb{N}$. Fix a sequence $\varepsilon_k > 0$ such that $\sum_{k\geq 1} \varepsilon_k < \infty$ and an exhaustion K_k of X by compact sets. We will obtain F as the limit of a composition of automorphisms $F_k = \phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1$. These automorphisms will be constructed together with an exhaustion C_k of X by holomorphically convex compact sets satisfying the following properties:

(i) $F_k(a_{i_r}) = b_{j_r}$ for all $r \leq k$;

(ii) ϕ_k is ε_k-close to the identity on C_{k-1};

(iii) $K_k \cup F_k(K_k) \subset C_k$;

(iv) $C_{k-1} \subset \mathring{C}_k$ and $dist(X \setminus C_k, C_{k-1}) > \varepsilon_k$.

Suppose we have such a sequence. Then it converges (uniformly on compacts) to $F \in \mathrm{Aut}(X)$ thanks to Theorem 2.4.1 and it has the required property because of (i).

We will construct the automorphisms ϕ_k by induction. For $k=1$, let $j_k = 1$ and $C_0 = \emptyset$. Then ϕ_1 is any automorphism such that $\phi_1(a_1) = b_1$; it exists because X has the density property.

Suppose we have chosen j_r and constructed ϕ_r and C_{r-1} as above for all $r \leq k$. First of all pick a holomorphically convex compact set C_k satisfying (iii) and (iv) above and such that $b_{j_r} \in C_k$ for all $r \leq k$; this is possible because X is Stein. Choose j_{k+1} such that $F_k(a_{j_{k+1}}), b_{j_{k+1}} \in X \setminus C_k$; this can be done as $F_k(A)$ and B are necessarily unbounded. By means of Theorem 2.3.1, we can then obtain an automorphism ϕ_{k+1} which is ε_{k+1}-close to the identity on C_k and is such that $\phi_{k+1}(b_{j_r}) = b_{j_r}$ for $r \leq k$ and $\phi_{k+1}(F_k(a_{j_{k+1}})) = b_{j_{k+1}}$. The composition $F_{k+1} = \phi_{k+1} \circ \phi_k \circ \cdots \circ \phi_1$ then satisfies (i) and we can proceed with the induction. □

5.2 Special Linear Group

We will now explore the special linear group $\mathrm{SL}_2(\mathbb{C})$ in search for tame sets. As $\mathrm{SL}_2(\mathbb{C})$ is a Stein manifold with the density property [55], Proposition 5.1.2 applies and all tame sets on this manifold are equivalent under the action of the automorphisms group. We begin by showing our first non-trivial example of a tame set.

Lemma 5.2.1. *The set*

$$\left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{N} \right\} \subset \mathrm{SL}_2(\mathbb{C})$$

is tame.

Before we start with the actual proof of the lemma, let us recall some facts about the special linear group $\mathrm{SL}_2(\mathbb{C})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ in the adjoint representation.

In the following we will fix the names of the entries for

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

There are three linear and hence $\mathbb{C}$-complete vector fields that generate $\mathfrak{sl}_2(\mathbb{C})$ and correspond to conjugations (see for instance [34]):

$$\begin{aligned} V &= c\frac{\partial}{\partial a} + (d-a)\frac{\partial}{\partial b} - c\frac{\partial}{\partial d}, \\ W &= -b\frac{\partial}{\partial a} + (a-d)\frac{\partial}{\partial c} + b\frac{\partial}{\partial d}, \\ H &= -2b\frac{\partial}{\partial b} + 2c\frac{\partial}{\partial c}, \end{aligned}$$

satisfying $[V,W]=H$, $[H,V]=2V$ and $[H,W]=2W$, with their respective flows

$$\varphi_V^t\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} 1 & -t\\ 0 & 1\end{pmatrix}=\begin{pmatrix} ct+a & t(d-ct)-at+b\\ c & d-ct\end{pmatrix},$$
$$\varphi_W^t\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 1 & 0\\ t & 1\end{pmatrix}\cdot\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} 1 & 0\\ -t & 1\end{pmatrix}=\begin{pmatrix} a-bt & b\\ t(a-bt)-dt+c & bt+d\end{pmatrix},$$
$$\varphi_H^t\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} e^{-t} & 0\\ 0 & e^{t}\end{pmatrix}\cdot\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix} e^{t} & 0\\ 0 & e^{-t}\end{pmatrix}.$$

We also have the following relations for the kernels:

$$\ker V\supseteq\mathbb{C}[c,a+d],$$
$$\ker W\supseteq\mathbb{C}[b,a+d].$$

Given a holomorphic function f and a vector field V, $V(f)$ is itself a function. Hence, we can apply V again; denote this operation by $V^2(f)=V(V(f))$. For these second kernels we have:

$$\ker V^2\supseteq\mathbb{C}[a,c,d],$$
$$\ker W^2\supseteq\mathbb{C}[a,b,d].$$

Note that $ad-bc=1$ is trivially contained in all the kernels.

Proof. We are given an injective map $\ell:\mathbb{N}\to\mathbb{N}$ that prescribes the injection

$$\begin{pmatrix} 1 & k\\ 0 & 1\end{pmatrix}\mapsto\begin{pmatrix} 1 & \ell(k)\\ 0 & 1\end{pmatrix}.$$

Next, we construct an interpolating holomorphic automorphism of $\mathfrak{sl}_2(\mathbb{C})$ as a composition of suitable time-1 maps of complete vector fields corresponding to conjugations.

The desired automorphism is given by

$$\varphi_W^G\circ\varphi_V^F\circ\varphi_W^1$$

where F and G are holomorphic functions given as follows:

1. $$F\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=f(-c).$$

   By the Mittag-Leffler interpolation theorem we find a holomorphic function $f:\mathbb{C}\to\mathbb{C}$ such that $f(k)=\sqrt{\frac{\ell(k)}{k}}-1$ for all $k\in\mathbb{N}$. The root can be chosen arbitrarily. Note that $F\in\ker V$, hence $F\cdot V$ is $\mathbb{C}$-complete and its time-1 map φ_V^F is a holomorphic automorphism.

2.
$$G\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = g(b).$$

Note that $\ell \colon \mathbb{N} \to \mathbb{N}$ is injective and has in particular a closed discrete image. Again by the Mittag-Leffler interpolation theorem we find a holomorphic function $g \colon \mathbb{C} \to \mathbb{C}$ such that $g(\ell(k)) = -\frac{k(1+f(k))}{\ell(k)}$ for all $k \in \mathbb{N}$. Note that $G \in \ker W$, hence $G \cdot W$ is $\mathbb{C}$-complete and φ_W^G is a holomorphic automorphism.

We now check that we indeed interpolate correctly:

$$\begin{aligned}
\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} &\overset{\varphi_W^1}{\mapsto} \begin{pmatrix} 1-k & k \\ -k & k+1 \end{pmatrix} \\
&\overset{\varphi_V^F}{\mapsto} \begin{pmatrix} -kf(k)-k+1 & k\cdot(1+f(k))^2 \\ -k & k\cdot f(k)+k+1 \end{pmatrix} \\
&\overset{\varphi_W^G}{\mapsto} \begin{pmatrix} 1 & k\cdot(1+f(k))^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ell(k) \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$
□

Here is a second example of a tame set in $\mathrm{SL}_2(\mathbb{C})$.

Lemma 5.2.2. *The set*
$$\left\{\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix} : k \in \mathbb{N}\right\} \subset \mathrm{SL}_2(\mathbb{C})$$
is tame.

To prove this fact we will use a combination of left and right multiplication. Consider the complete vector fields

$$\begin{aligned}
A &= c\frac{\partial}{\partial a} + d\frac{\partial}{\partial b}, \\
B &= a\frac{\partial}{\partial c} + b\frac{\partial}{\partial d}, \\
C &= a\frac{\partial}{\partial b} + c\frac{\partial}{\partial d},
\end{aligned}$$

and the respective flows

$$\begin{aligned}
\varphi_A^t\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
\varphi_B^t\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
\varphi_C^t\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

In this case we have the following:

$$\begin{aligned}\ker A &\supseteq \mathbb{C}[c,d],\\ \ker B &\supseteq \mathbb{C}[a,b],\\ \ker C &\supseteq \mathbb{C}[a,c].\end{aligned}$$

Proof. Denote the given injective map by

$$\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k}\end{pmatrix} \mapsto \begin{pmatrix} \ell(k) & 0 \\ 0 & \frac{1}{\ell(k)}\end{pmatrix}.$$

As before, we will obtain the interpolating automorphism as a composition of time-1 maps of suitably defined vector fields. Choose holomorphic functions $f = f(c) \in \ker A$, $g = g(a) \in \ker B$, and $h = h(a) \in \ker C$ such that

$$\begin{aligned} f\begin{pmatrix} k & 0 \\ k & \frac{1}{k}\end{pmatrix} &= \frac{\ell(k)}{k} - 1,\\ g\begin{pmatrix} \ell(k) & \frac{\ell(k)}{k^2} - \frac{1}{k} \\ k & \frac{1}{k}\end{pmatrix} &= -\frac{k}{\ell(k)},\\ h\begin{pmatrix} \ell(k) & \frac{\ell(k)}{k^2} - \frac{1}{k} \\ 0 & \frac{1}{\ell(k)}\end{pmatrix} &= \frac{1}{k\ell(k)} - \frac{1}{k^2}.\end{aligned}$$

Let $F, G, H \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ be the time-1 maps of fA, gB and hC respectively. Then

$$H \circ G \circ F\left(\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k}\end{pmatrix}\right) = \begin{pmatrix} \ell(k) & 0 \\ 0 & \frac{1}{\ell(k)}\end{pmatrix}.$$

□

As mentioned before, any two tame sets must be equivalent, hence the following.

Corollary 5.2.3. *There exists* $F \in \mathrm{Aut}(\mathrm{SL}_2(\mathbb{C}))$ *such that*

$$F\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k}\end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1\end{pmatrix}$$

for all $k \in \mathbb{N}$.

Proof. Lemmas 5.2.1 and 5.2.2 give that the two sequences are tame. Since $\mathrm{SL}_2(\mathbb{C})$ has the density property [55], the result is a consequence of Proposition 5.1.2. □

5.3 Existence

The results obtained in the previous section are not just important on their own, but they provide a blueprint for the discussion of linear *semi-simple* Lie groups.

Definition 5.3.1. A Lie algebra is *simple* if it is not one-dimensional and it contains only trivial ideals. A Lie algebra is semi-simple if it is a direct sum of simple Lie algebras. A Lie group is called semi-simple if its Lie algebra is semi-simple.

We recall that every semi-simple Lie group G admits an algebraic immersive morphism $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow G$; this follows for instance from the classification of semi-simple Lie algebras, we point to the textbook [27] by Humphreys for a complete picture. We will use this fact together with our previous discussion to produce tame sets in any semi-simple Lie group. In the main theorem of this section, we take one more step and prove the following theorem. Recall that a complex algebraic Lie group is *linear* if it is (algebraically isomorphic to) a subgroup of a general linear group. Note that every such group is also a Lie group.

Theorem 5.3.2. *Every connected linear algebraic Lie group G different from the complex line $\mathbb{C}$ or the punctured complex line $\mathbb{C}^*$ contains a tame set.*

In order to prove Theorem 5.3.2, we need to take a step back and establish some basic properties.

Lemma 5.3.3. *Let X and Y be Stein manifolds and let V resp. W be a complete holomorphic vector field on X resp. Y with at least one unbounded orbit. Then there exists a tame set in $X \times Y$.*

Proof. Choose $x_0 \in X$ resp. $y_0 \in Y$ such that its orbit under the flow φ_V^t resp. φ_W^t is unbounded. We define the candidate tame set A to be $A := \{\varphi_V^{a_n}(x_0) : n \in \mathbb{N}\} \times \{y_0\} \subset X \times Y$ for a sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that A is closed and discrete; this is possible as $t \mapsto \varphi_V^t(x_0)$ is unbounded.

We need to show that for any injective map $\ell \colon \mathbb{N} \to \mathbb{N}$ we can find a holomorphic automorphism F of $X \times Y$ such that $F(\varphi_V^{a_n}(x_0), y_0) = (\varphi_V^{a_{\ell(n)}}(x_0), y_0)$.

We construct this automorphism as a composition $F = F_3 \circ F_2 \circ F_1$, where

$$\begin{aligned} F_1(x,y) &= (x, \varphi_W^{h(x)}(y)),\\ F_2(x,y) &= (\varphi_V^{f(y)}(x), y)),\\ F_3(x,y) &= (x, \varphi_W^{g(x)}(y)), \end{aligned}$$

and we chose holomorphic functions $h \colon X \to \mathbb{C}$, $f : Y \to \mathbb{C}$, and $g \colon X \to \mathbb{C}$ as follows:

$$\begin{aligned} h(\varphi_V^{a_n}(x_0)) &= b_n,\\ f(\varphi_W^{b_n}(y_0)) &= a_{\ell(n)} - a_n,\\ g(\varphi_V^{a_{\ell(n)}}(x_0)) &= -b_n, \end{aligned}$$

where $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ is such that $\{x_0\} \times \{\varphi_W^{b_n}(y_0) : n \in \mathbb{N}\} \subset X \times Y$ is unbounded. Such functions exist, since we can prescribe the values of a holomorphic function on a closed discrete set in a Stein manifold. □

Corollary 5.3.4. *For $m, n \in \mathbb{N}$ with $m + n \geq 2$, the manifold $\mathbb{C}^m \times (\mathbb{C}^*)^n$ contains a tame set.*

Proof. Note that $z\frac{\partial}{\partial z}$ is a complete holomorphic vector field on $\mathbb{C}^*$ resp. $\frac{\partial}{\partial z}$ is a complete holomorphic vector field on $\mathbb{C}$ such that the assumptions of the preceding lemma are satisfied. □

It is now time to give what we deem to be the most difficult result in this chapter.

Proposition 5.3.5. *Let G be a linear algebraic group and assume there exists an algebraic Lie group homomorphism $i\colon \mathrm{SL}_2(\mathbb{C}) \to G$ that is an immersion. Then G contains a tame set.*

Proof. Let V and W be the vector fields on G with flows given by

$$\begin{aligned} &\mathbb{C} \times G \to G \\ &(t, x) \to i\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot x \cdot i\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{C} \times G \to G \\ &(t, x) \to i\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \cdot x \cdot i\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \end{aligned}$$

respectively. These are algebraic flows of complete vector fields as they give an algebraic action of the additive group $(\mathbb{C}, +)$.

Denote by $\pi_V : G \to Q_V$ and $\pi_W : G \to Q_W$ the quasi-affine quotients for these actions, as they appear in Theorem 2.5.5. Recall that regular functions on these quotients correspond to functions in the respective kernels of the vector fields.

We consider the following polynomial maps:

$$\begin{aligned} \beta_V : \mathbb{C} \to G, &\quad t \mapsto \varphi_W^1 \circ i\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = i\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}, \\ \beta_W : \mathbb{C} \to G, &\quad t \mapsto i\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \\ \alpha_V : \mathbb{C} \to Q_V, &\quad \alpha_V = \pi_V \circ \beta_V, \\ \alpha_W : \mathbb{C} \to Q_W, &\quad \alpha_W = \pi_W \circ \beta_W. \end{aligned}$$

Since the $(\mathbb{C}, +)$-actions of V and W arise from the action of the reductive group $\mathrm{SL}_2(\mathbb{C})$ on G which is an affine variety, their ring of invariant functions is actually finitely generated, hence Q_V and Q_W are affine, see e.g. [21, Proposition 6.2].

We wish to choose a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ such that its image under both α_V and α_W is closed, discrete and unbounded. This is possible if $\alpha_V(\mathbb{C})$ and $\alpha_W(\mathbb{C})$ are unbounded as maps to complex-affine space. Let us consider the differentials γ_V and γ_W of β_V and β_W at $0 \in \mathbb{C}$ to obtain maps into the Lie algebra $i_*(\mathrm{SL}_2(\mathbb{C})) \subset \mathfrak{g}$. The differentials $d\pi_V$ and $d\pi_W$ at $\mathrm{Id} \in G$ are maps from $\mathfrak{g}$ into the tangent spaces of Q_V and Q_W respectively. By the third isomorphism theorem [48] $T_{\pi_V(\mathrm{Id})}Q_V$ is isomorphic to the Lie algebra $\mathfrak{g}$ modulo the kernel of the projection $d_{\mathrm{Id}}\pi_V$. Restricting our attention to $i_*(\mathrm{SL}_2(\mathbb{C})) \subset \mathfrak{g}$, we have that

$$K_V := i_*\mathfrak{sl}_2(\mathbb{C}) \cap \ker d_{\mathrm{Id}}\pi_V = \{i_*\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) : b \in \mathbb{C}\}.$$

For W we have that

$$K_W := i_*\mathfrak{sl}_2(\mathbb{C}) \cap \ker d_{\mathrm{Id}}\pi_W = \{i_*\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right) : c \in \mathbb{C}\}.$$

The differentials

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\gamma_V} & i_*\mathfrak{sl}_2(\mathbb{C}) & \xrightarrow{d\pi_V} & i_*\mathfrak{sl}_2(\mathbb{C})/K_V \subset T_{\pi_V(\mathrm{Id})}Q_V \\ t & \longmapsto & i_*\left(t \cdot \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right) & \longmapsto & \left[t \cdot \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right] \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\gamma_W} & i_*\mathfrak{sl}_2(\mathbb{C}) & \xrightarrow{d\pi_W} & i_*\mathfrak{sl}_2(\mathbb{C})/K_W \subset T_{\pi_W(\mathrm{Id})}Q_W \\ t & \longmapsto & i_*\left(t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) & \longmapsto & \left[t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \end{array}$$

are not constant, hence α_V and α_W are not constant.

We can now make our choice of $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that its image under both α_V and α_W is closed, discrete and unbounded. We claim that $\left\{i\begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix}\right\}_{n\in\mathbb{N}} \subset G$ is a tame sequence.

Let $\ell : \mathbb{N} \to \mathbb{N}$ denote the injective self-map we need to interpolate. As $\{\alpha_V(t_n)\}_{n\in\mathbb{N}} \subset Q_V$ is closed and discrete, there exists $f \in \ker V$ such that

$$f\left(\varphi_W^1\left(i\begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix}\right)\right) = \sqrt{\frac{t_{\ell(n)}}{t_n}} - 1.$$

For the same reason there exists $g \in \ker W$ such that

$$g\left(i\begin{pmatrix} 1 & t_{\ell(n)} \\ 0 & 1 \end{pmatrix}\right) = \sqrt{\frac{t_n}{t_{\ell(n)}}}.$$

Let $F = \varphi_V^f \in \mathrm{Aut}(G)$ and $H = \varphi_W^g \in \mathrm{Aut}(G)$, then the same computation from Lemma 5.2.1 shows that $H^{-1} \circ F \circ \varphi_W^1$ is the interpolating automorphism. □

Corollary 5.3.6. *A complex semi-simple Lie group always admits a tame set.*

Proof. As G is semi-simple, there exists an immersion $SL_2(\mathbb{C}) \hookrightarrow G$. This follows from the proof of the classification of simple Lie algebras (see e.g. [27]).

Let i be the induced immersion $SL_2(\mathbb{C}) \hookrightarrow G$ and apply Proposition 5.3.5. □

Now that we know that semi-simple Lie groups and certain products of Stein manifolds admit tame sets, it is time to prove Theorem 5.3.2. We will use Mostow's decomposition Theorem [43] to reduce to the case of a product, then apply Lemma 5.3.3 if the factors are not trivial. If one of the factors in said product is trivial, we will further reduce ourselves to the case of a simple Lie group.

Proof of Theorem 5.3.2. By Mostow's Theorem [43], the group G is isomorphic to a semi-direct product $N \rtimes M$ where N is the connected normal subgroup consisting of the unipotent elements of G and M is a maximal fully reductive subgroup of G. Hence, G is isomorphic as affine variety to the direct product $N \times M$. If both M and N are non-trivial complex Lie groups, we can apply Lemma 5.3.3 directly. Since N is a unipotent group, it is also nilpotent and hence an affine variety isomorphic to $\mathbb{C}^n$. If M is trivial, then by assumption we have $n \geq 2$ and we know that tame sets exist.

It remains to consider the case where N is trivial and M is non-trivial. We will reduce this case to the situation where M is a simple complex Lie group. Assume first that M is indeed simple. Then we could apply Proposition 5.3.5 to M and obtain two complete algebraic vector fields V resp. W with algebraic flows. Our goal is to understand the behaviour of these vector fields under the process of reducing M to a simple Lie group.

Let $Z \cong (\mathbb{C}^*)^n$ denote the identity component of the center of M and assume it is non-trivial. If M decomposes as a product $M' \times Z$ we may again apply Lemma 5.3.3. In the general case, $M \cong (M' \times Z)/\Gamma$ for a central normal subgroup Γ. The flows of the complete vector fields will commute with Γ and induce complete vector fields on M with the same properties.

Next we may assume that the identity component of the center Z is trivial. If M is not simply connected, we pass to its universal cover $\widetilde{M}$. If a semi-simple Lie group M is simply connected, then it decomposes as a product of simple Lie groups and we are done by applying Proposition 5.3.5 to one factor or just Lemma 5.3.3 in case there are at least two factors.

If $\widetilde{M} \to M$ is the universal cover, then $M \cong \widetilde{M}/H$ for a discrete (actually finite) subgroup H of its center, hence as above, this induces complete vector fields on M with the same properties except that the inclusion $i\colon SL_2(\mathbb{C}) \to \widetilde{M}$ might now just be an immersion in M. □

In Proposition 5.3.5, we use the action of $SL_2(\mathbb{C})$ to produce tame sets. We are able to do so thanks to the discussion in Section 5.2. Another group which is known to admit tame sets in simply $\mathbb{C}^2$; let us see how we can use its action to produce tame sets.

Theorem 5.3.7. *Let X be an affine algebraic complex manifold and let V, W be complete algebraic vector fields whose flows are algebraic. If $[V, W] = 0$ and their kernels are not contained one into the other, then there exists a tame sequence in X.*

Proof. As the kernels are not contained one into the other, there exist $g, h \in \mathbb{C}[X]$ such that

$$V(g) = W(h) = 0,$$
$$V(h), W(g) \neq 0.$$

Denote by π_V and π_W the rational maps given by Theorem 2.5.5. Consider the following properties for a point $x \in X$:

(i) $V(h)(x) \neq 0$;

(ii) $W(x) \neq 0$;

(iii) $\pi_V^{-1}(\pi_V(x)) = \{\varphi_V^t(x) : t \in \mathbb{C}\}$;

(iv) x is not in the singular set of π_V;

and the analogous ones

(a) $W(g)(x) \neq 0$;

(b) $V(x) \neq 0$;

(c) $\pi_W^{-1}(\pi_W(x)) = \{\varphi_W^t(x) : t \in \mathbb{C}\}$;

(d) x is not in the singular set of π_W.

Conditions (i),(ii),(iv),(a),(b) and (d) are clearly generically satisfied. This is true also for conditions (iii) and (c) by a theorem of Rosenlicht [53].

We now prove that if $\hat{x} \in X$ satisfies (i), (ii), (iii) and (iv) then the rational map

$$\mathbb{C} \to Q_V$$
$$t \mapsto \pi_V(\varphi_W^t(\hat{x}))$$

is not constant. By (i), the map

$$s \mapsto h(\varphi_V^s(\hat{x}))$$

is not constant. Assume that $\pi_V(\varphi_W^t(\hat{x})) = \pi_V(\hat{x})$ for all $t \in \mathbb{C}$. Then by (iii) we have that $\varphi_W^t(\hat{x}) \in \{\varphi_V^s(x) : s \in \mathbb{C}\}$ for all $t \in \mathbb{C}$. Since $W(\hat{x}) \neq 0$ by condition (ii), the flow of W starting at $\hat{x}$ is not constant, hence there must be $\hat{t} \neq 0$ and $\hat{s} \neq 0$ such that $\varphi_W^{\hat{t}}(\hat{x}) = \varphi_V^{\hat{s}}(\hat{x})$ and $h(\varphi_V^{\hat{s}}(\hat{x})) \neq h(\hat{x})$. We get a contradiction because $h(\varphi_W^{\hat{t}}(\hat{x})) = h(\hat{x})$, since $h \in \ker W$.

If $\hat{x}$ also satisfies (a), (b), (c) and (d), we obtain that the rational images $\{\pi_V(\varphi_W^t(\hat{x})) : t \in \mathbb{C}\} \subset Q_V$ and $\{\pi_W(\varphi_V^t(\hat{x})) : t \in \mathbb{C}\} \subset Q_W$ are unbounded.

We claim that conditions (a),(b), (c) and (d) are generically true (with respect to $t \in \mathbb{C}$) for points in $\{\varphi_W^t(\hat{x}) : t \in \mathbb{C}\} \subset X$. This is true for conditions (a) and (b) as

$$\begin{aligned} \mathbb{C} &\mapsto \mathbb{C} \\ t &\mapsto W(g)(\varphi_W^t(\hat{x})) \end{aligned}$$

and

$$\begin{aligned} \mathbb{C} &\to TX \\ t &\mapsto V(\varphi_W^t(\hat{x})) \end{aligned}$$

are not constant, since $\hat{x}$ satisfies (a) and (b). Conditions (c) and (d) are always true because $\{\varphi_W^s(\varphi_W^t(\hat{x})) : s \in \mathbb{C}\} = \{\varphi_W^s(\hat{x}) : s \in \mathbb{C}\} = \pi_W^{-1}(\pi_W(\hat{x}))$, where the last equality is condition (c) for $\hat{x}$.

We obtain our tame sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ by setting $x_n = \varphi_W^{t_n}(\hat{x})$, for a sequence $\{t_n\} \subset \mathbb{C}$ such that $\{\pi_V(x_n)\} \subset Q_V$ is discrete and each x_n satisfies (a), (b), (c) and (d). It exists because $\{\pi_V(\varphi_W^t(\hat{x})) : t \in \mathbb{C}\} \subset Q_V$ is unbounded and (a), (b), (c) and (d) are generic with respect to t. We will show that such a sequence is tame.

An injective map $f : \{x_n\} \to \{x_n\}$ is nothing but a relabeling (possibly with omissions) of the points, in the sense that for every $n \in \mathbb{N}$ there is $k_n \in \mathbb{N}$ such that $f(x_n) = x_{k_n}$ and $k_n \neq k_m$ for $n \neq m$.

The quasi-affine variety Q_V is equal to $\Omega \setminus A$, where Ω is an affine variety and A is an algebraic subset. Since $\{\pi_V(x_n)\} \subset Q_V \subset \Omega$ is unbounded and discrete in Q_V we may assume that, after passing to a subsequence, it is also discrete in Ω. Since Ω is affine, there exist $\tilde{f}_1, \tilde{f}_3 \in \mathcal{O}(\Omega) \subset \mathcal{O}(Q_V)$ such that

$$\begin{aligned} \tilde{f}_1(\pi_V(x_n)) &= a_n, \\ \tilde{f}_3(\pi_V(x_{k_n})) &= a_n, \end{aligned}$$

for any sequence $\{a_n\} \subset \mathbb{C}$. We postpone the choice of this sequence.

We claim that $f_1 := \pi_V^* \tilde{f}_1$, $f_3 := \pi_V^* \tilde{f}_3 \in \mathcal{O}(X)$ are well-defined and in the kernel of V. Since Ω is affine there exist sequences of regular maps on Q_V converging to $\tilde{f}_1$ and $\tilde{f}_3$ respectively. We define the pullbacks of $\tilde{f}_1$ and $\tilde{f}_3$ as the limit of the pullbacks of the respective sequences. They are in the kernel of V because they are obtained as limits of invariant maps.

This gives complete vector fields $f_1 V, f_3 V$ whose flows are determined by

$$\begin{aligned} \mathbb{C} \times X &\to X \\ (t, x) &\mapsto \varphi_V^{f_i(x)t}(x) \end{aligned}$$

for $i = 1, 3$. The time-1 maps are automorphisms F_1, F_3 of X such that

$$\begin{aligned} F_1(x_n) &= \varphi_V^{a_n}(x_n) = \varphi_V^{a_n}(\varphi_W^{t_n}(\hat{x})); \\ F_3(x_{k_n}) &= \varphi_V^{a_n}(x_{k_n}) = \varphi_V^{a_n}(\varphi_W^{t_{k_n}}(\hat{x})). \end{aligned}$$

As $[V,W]=0$ the respective flows commute, hence we are looking for $f_2\in\ker W$ such that $f_2(\varphi_V^{a_n}(x_n))=t_{k_n}-t_n$. We could then set F_2 to be the time-1 map of f_2W and obtain the interpolating automorphism $F:=(F_3)^{-1}\circ F_2\circ F_1$. To obtain f_2 we turn our attention to Q_W, in particular to the sequence $\{\pi_W(\varphi_V^{a_n}(x_n))\}\subset Q_W$. If it is discrete and without repetition then we can find $\tilde f_2\in\mathcal{O}(Q_W)$ such that $f_2:=\pi_W^*\tilde f_2$ has the required property. Since we chose x_n such that (a), (b), (c) and (d) hold, the rational map

$$\begin{aligned}&\mathbb{C}\to Q_W\\ &t\mapsto \pi_W(\varphi_V^t(x_n))=\pi_W(\varphi_V^t(\varphi_W^{t_n}(\hat x))=\pi_W(\phi_V^t(\hat x))\end{aligned}$$

is not constant, hence unbounded. We conclude the proof by choosing the sequence $\{a_n\}$ in such a way that $\{\pi_W(\varphi_V^{a_n}(x_n))\}\subset Q_W$ is discrete and without repetition.

□

Note that even if we start from algebraic vector fields with algebraic flows, it is not true in general that the interpolating automorphism will be algebraic.

Corollary 5.3.8. *Let G and H be non-trivial affine complex Lie groups whose connected components are not biholomorphic to $(\mathbb{C}^*)^n$. Then $G\times H$ contains a tame sequence.*

Corollary 5.3.9. *Each variety of the family $X_{a,b}=\{x^2y=a(z)+xb(z)\}\subset\mathbb{C}^{n+3}$ for $a,b\in\mathcal{O}(\mathbb{C}^{n+1})$, $n\geq 1$ contains a tame sequence. In particular, the Koras-Russell cubic threefold $C=\{x^2y+x+z^2+w^3=0\}\subset\mathbb{C}^4$ contains a tame sequence.*

This class of complex varieties was recently considered by Leuenberger [42]. He proved that under suitable conditions on the holomorphic functions a and b, the space $X_{a,b}$ is a complex manifold with the density property. Here we include the case where the variety is singular, as we will use vector fields that vanish on the singular locus.

Proof. The vector fields

$$V=\left(\frac{\partial a}{\partial z_0}+x\frac{\partial b}{\partial z_0}\right)\frac{\partial}{\partial y}+x^2\frac{\partial}{\partial z_0}$$

and

$$W=\left(\frac{\partial a}{\partial z_1}+x\frac{\partial b}{\partial z_1}\right)\frac{\partial}{\partial y}+x^2\frac{\partial}{\partial z_1}$$

satisfy the hypothesis of Theorem 5.3.7. □

6

Discussion

This last chapter will describe a few of the still unanswered questions that arose while working on tame sets and parametric jet interpolation. In Section 6.1 we also announce two possible answers to these questions, but do not give any detail as they will appear elsewhere.

Problems concerning tame sets and parametric interpolation by automorphisms will be considered simultaneously in Section 6.2, in particular we wonder whether small perturbations of tame sets are still tame and if we can find a parametric family of automorphisms following the (parametric) perturbation. Particular interest will be devoted to the action of $\mathrm{Aut}(\mathbb{C}^n)$ on the space of all closed discrete sequences.

The last section will consider a variation of the Vaserstein problem portrayed in Theorem 1.3.8, following ideas from Geometric Control Theory that were already put to use by Varolin in his proof of Theorem 3.2.6.

6.1 Work in Progress

When presented with the definition of strongly tame set (Definition 5.0.2), two problems naturally arise: it is not clear whether tame sets exist and it is not clear whether they all lie in the same equivalence class under the action of $\mathrm{Aut}(X)$. In Chapter 5, we gave partial answers to these questions with Theorem 5.3.2, showing the existence of tame sets in any linear algebraic Lie group, and Proposition 5.1.2, giving the equivalence of two tame sets in any Stein manifold with the density property.

Together with Andrist, we are able to provide a better understanding of these qualities.

Proposition 6.1.1. *The spectral ball* $\{A \in \mathrm{Mat}(n \times n, \mathbb{C}) : \mathrm{Spec} A \subset \mathbb{D}\}$ *for* $n > 1$ *is a complex manifold admitting non-equivalent tame sets.*

Here $\mathrm{Spec} A$ denotes the spectrum of the matrix A, that is its set of eigenvalues. We only mention that this result is possible thanks to the work of Andrist and Kutzschebauch [6], who determined the automorphisms group of the spectral ball.

A much harder result is the following.

Claim 6.1.2. *Let G be a complex linear algebraic Lie group and $H \subset G$ a closed algebraic subgroup such that the quotient $X := G/H$ is an affine variety of dimension greater than* 1*. Then X admits a tame set.*

We point out that all homogeneous manifolds of this sort are known to have the density property [11]. This was established by using *compatible pairs* of vector fields, that is complete vector fields which behave similarly to two of the generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We plan to use these compatible pairs to obtain an action of $\mathrm{SL}_2(\mathbb{C})$ on the manifold X, then perform computations similar to the ones presented in Proposition 5.3.5. The main difficulty will be the study of the kernel of the induced vector fields, for this reason we might need to weaken Claim 6.1.2 to include only *reductive* linear Lie groups.

6.2 Topology of Tame Sets

A natural question which has never been considered in the literature is the following.

Question 6.2.1. Let $A \subset \mathbb{C}^n$ be a tame set and denote by A_ε a *perturbation* of A. Is A tame?

The word *perturbation* is emphasized because we have to define its meaning.

It is important to notice that the choice of an *allowed perturbation* implies the choice of a topology on the space of all sequences with values in $\mathbb{C}^n$. Recall that for a sequence to be tame, we require that it is without repetion and has a closed and discrete image. For this reason, we will only consider the space X of such sequences instead of the space of all sequences. It is also interesting to study the action of the group $\mathrm{Aut}(\mathbb{C}^n)$ on such space, later we will give a short overview of the possible choices for a topology and examine how they behave with respect to this action.

To define X as a set we will begin with the space of all sequences $Y := (\mathbb{C}^n)^{\mathbb{N}}$ and remove first the ones that admit an accumulation point and then the ones with repetitions.

As mentioned before, there are three standard choices for a topology on Y. Perhaps the most natural one from a categorical point of view is the product topology.

Definition 6.2.2. Let X_j be topological spaces indexed by a set J. The product topology on the product $M := \Pi_{j\in J} X_j$ is the coarsest topology which makes all the projection maps $\pi_j : M \to X_j,\ j \in J$ continuous.

Equivalently, it is the topology generated by open sets of the form $\Pi_{j\in J} U_j$ where $U_j \subset X_j$ are open sets such that $U_j \neq X_j$ for only finitely many $j \in J$.

For $J = \mathbb{N}$ the product topology is intuitively not concerned with the behaviour of sequences at infinity, this poses a great limit to its utility in our discussion. More concrete evidence will be provided shortly.

Definition 6.2.3. Let X_j be metric spaces indexed by a set J. The uniform topology on the product $M := \Pi_{j\in J} X_j$ is the topology generated by open sets of the form $\Pi_{j\in J}\mathbb{B}(p_j,\delta)$ where $p_j \in X_j$, $0<\delta\leq+\infty$ and $\mathbb{B}(p_j,\delta)$ is the ball of radius δ and center p_j for the metric on X_j.

Definition 6.2.4. Let X_j be topological spaces indexed by a set J. The box topology on the product $M := \Pi_{j\in J} X_j$ is the topology generated by open sets of the form $\Pi_{j\in J} U_j$ where $U_j \subset X_j$ are open sets.

When J is finite, the three topologies coincide. In general, the box topology is finer than the uniform one which in turn is finer than the product topology.

Let us define the sets we wish to remove. For $p\in\mathbb{C}^n$ and $r>0$ denote by $\mathbb{B}(p,r)$ the ball of center p and radius r, then define

$$
\begin{aligned}
V_{p,k} :=& \{x=(x_j)_{j\in\mathbb{N}}\in Y : |\mathbb{B}(p,1/k)\cap\{x_j\}_{j\in\mathbb{N}}|=\infty\},\\
V_p :=& \bigcap_{k\in\mathbb{N}} V_{p,k},\\
V :=& \bigcup_{p\in\mathbb{C}^n} V_p.
\end{aligned}
$$

where $|A|$ denotes the cardinality of the set A.

Note that V is the set of all sequences which accumulate at least at one point.

We now wish to remove diagonals to ensure there are no repeating sequences. For $a,b\in\mathbb{N},\ a<b$ let

$$
\begin{aligned}
\Delta_{a,b} :=& \{x=(x_j)_{j\in\mathbb{N}}\in Y : x_a=x_b\}\\
\Delta :=& \bigcup_{a<b}\Delta_{a,b}
\end{aligned}
$$

We define $X := Y\setminus(\Delta\cup V)$ to be the set of all closed discrete sequences without repetition. The first result gives evidence that the product topology is not suitable for our discussion.

Proposition 6.2.5. *The space $X\subset Y$ is open in the box topology but not in the product topology.*

Proof. As every open set in the product topology admits sequences with repetition, the second claim is trivial.

Let $x=(x_j)_{j\in\mathbb{N}}\in X$ be a closed discrete sequence without repetition, then there exist a sequence $\delta_j>0,\ j\in\mathbb{N}$ such that $\mathbb{B}(x_a,\delta_a)\cap\mathbb{B}(x_b,\delta_b)=\emptyset$ for all $a\neq b$. The open set in the box topology $\Pi_{j\in\mathbb{N}}\mathbb{B}(x_j,\delta_j)\subset Y$ is then contained in X, hence X is open in the box topology. □

Both the uniform and the box topology are somewhat pathological. The former has many (path) connected components, the latter is not even locally path connected.

Proposition 6.2.6. [45, Exercise 25.2.b] *Two sequences $x, y \in Y$ equipped with the uniform topology belong to the same path connected component if and only if the sequence $(x_j - y_j)_{j\in\mathbb{N}}$ is bounded. Furthermore Y is locally path connected, hence the path connected components coincide with the connected components.*

Proposition 6.2.7. [45, Exercise 25.2.c] *Two sequences $x, y \in Y$ equipped with the box topology belong to the same path connected component if and only if the sequence $(x_j - y_j)_{j\in\mathbb{N}}$ is eventually zero. Furthermore Y is not locally path connected, nevertheless the path connected components coincide with the connected components.*

It is still not clear what would be a suitable choice for a topology on X. Since we are interested in the action of Aut($\mathbb{C}^n$) on this space, let us see how it behaves under the different topologies.

Given any holomorphic self-map $f : \mathbb{C}^n \to \mathbb{C}^n$, we can consider its *diagonal extension* $\tilde{f} : Y \to Y$ such that $\tilde{f}(x_j)_{j\in\mathbb{N}} = (f(x_j))_{j\in\mathbb{N}}$. If we restrict our attention to holomorphic automorphisms of $\mathbb{C}^n$ we obtain an action of Aut($\mathbb{C}^n$) on X and we are interested in the study of the equivalence classes (i.e. orbits) generated by this action. Note that the set of tame sequences $\mathscr{T}$ is exactly the equivalence class of $\mathbb{N} \times \{0\}^{n-1}$.

This action is clearly continuous in the box and product topology, unfortunately this in not true for the uniform topology.

Proposition 6.2.8. *There exists $F \in$ Aut($\mathbb{C}^n$) which is not uniformly continuous.*

Proof. Consider the sequence $x = (x_j)_{j\in\mathbb{N}}$ defined by

$$x_j = (\sum_{k=1}^{j} \frac{1}{k}, 0, \ldots, 0) \in \mathbb{C}^n.$$

It is tame as it is a discrete set in the first axis [52]. Let $F \in$ Aut($\mathbb{C}^n$) be such that $F(j, 0, \ldots, 0) = x_j$, then the image of the uniform neighbourhood $U = \Pi_j \mathbb{B}((j, 0, \ldots, 0), 1/2) \subset X$ cannot contain a uniform neighbourhood of $x \in X$ as $|x_j - x_{j-1}| = 1/j$. Hence, F^{-1} is not uniformly continuous. □

The following proposition is again another reason to not focus on the product topology.

Proposition 6.2.9. *Any orbit* Aut($\mathbb{C}^n$)x *for $x \in X$ is dense in X equipped with the product topology.*

Proof. Let $a \in X$. For each $N \in \mathbb{N}$ there exists an automorphisms $F_N \in$ Aut($\mathbb{C}^n$) such that $b_j^N := F_N(x_j) = a_j$ for all $j \leq N$. Notice that $(b^N)_{N\in\mathbb{N}} \subset X$ converges to $a \in X$ in the product topology, hence $a \in \overline{\mathrm{Aut}(\mathbb{C}^n)x}$ since each b^N is in Aut($\mathbb{C}^n$)x. □

The results above make clear that the most suitable topology to discuss our problem is the box topology, an opinion which is reinforced by the following proposition.

Proposition 6.2.10. *The set of tame sequences $\mathscr{T}$ is open in X equipped with the box topology.*

This result is a consequence of [61, Theorem 1]:

Theorem 6.2.11. [61, Theorem 1] *Let $x = (x_j)_{j\in\mathbb{N}} \in X$ be such that*

$$\sum_j \frac{1}{||x_j||^{2n-2}} < +\infty. \tag{6.1}$$

Then $x \in \mathscr{T}$.

Proof of Proposition 6.2.10. Consider the standard tame sequence $x_j = (j, 0, \ldots, 0)$. There exists a number $\varepsilon > 0$ such that if $||z_j - x_j|| < \varepsilon$ for all $j \in \mathbb{N}$, then the sequence $z \in X$ satisfies Equation 6.1. Therefore the open set $U := \Pi_{j\in\mathbb{N}}\mathbb{B}(x_j, \varepsilon) \subset X$ is an open neighbourhood of x contained in $\mathscr{T}$. Since for any other tame set $y \in \mathscr{T}$ there exists $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that $\tilde{F}(x) = y$, we have that $\tilde{F}(U)$ is an open neighbourhood of y contained in $\mathscr{T}$. □

Note that the neighbourhood of the standard tame sequence is indeed a uniform neighbourhood and not just a box neighbourhood.

Let us recall that there are infinitely many equivalence classes of closed discrete sequences [52]. The most important example of non-tame sequences is given by *rigid* sequences.

Theorem 6.2.12. [52, Theorem 5.1] *There is a closed discrete set $D \subset \mathbb{C}^n$ such that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic, its Jacobian is not identically zero and $F(\mathbb{C}^n \setminus D) \subset \mathbb{C}^n \setminus D$, then $F = \mathrm{Id}$. Such sets are called* rigid.

We will now use these sets to construct a map $\mathbb{C} \to X$ which is in some sense transversal to the orbits.

Proposition 6.2.13. *There are infinitely many rigid sets whose orbits are open neither in X nor in $X \setminus \mathscr{T}$ equipped with any of the three mentioned topologies.*

Proof. We will recall a construction used by Rosay and Rudin [52] to show the existence of infinitely many non-equivalent closed discrete sequences and use it to show the existence of a continuous map $f : \mathbb{C} \to X \setminus \mathscr{T}$ which is in some sense transversal to the orbits. To be more precise $f(t)$ will be a rigid set for all $t \in \mathbb{C}$ and for $t \neq s$ it will be not equivalent to $f(s)$, hence the preimage of an orbit will consist of only one point and the orbit will not be open.

Let $D = (d_j)_{j\in\mathbb{N}}$ be a rigid set and let $d_0 : \mathbb{C} \to \mathbb{C}^n \setminus D$ be an injective continuous function. Define $f : \mathbb{C} \to X$ such that $f(t) = (d_0(t), d_1, d_2, \ldots)$. It is clear that $f(t)$ is still rigid and $f(t)$ is not equivalent to $f(s)$ for all $t \neq s$. The function f is continuous (in any topology) because the first coordinate is continuous and the rest are constants. □

The rigid sets above belong to the same connected component of X in the box topology, a very different situation compared to the orbit $\mathcal{T}$ which is a union of connected components.

Proposition 6.2.13 defies the idea that when the *isotropy group* is small, the corresponding orbit should be large.

Definition 6.2.14. Let G be a group acting on a set S. For $s \in S$ define the isotropy group of s as

$$G_s := \{g \in G : gs = s\}$$

It is clear from the definitions that the isotropy group of a rigid set is trival, while the isotropy group of a tame sequence is very large because of Theorem 1.3.6. Nevertheless, $\mathcal{T}$ is an open set which is the union of connected components while Proposition 6.2.13 tells that orbits of rigid sets are much smaller, at least locally.

We conclude this section with a general problem.

Problem 6.2.15. Understand the orbits of the action of $\mathrm{Aut}(\mathbb{C}^n)$ on X.

6.3 Generalized Vaserstein Problem

Our last topic of discussion will be a generalization of the holomorphic Vaserstein problem solved by Ivarsson and Kutzschebauch [28](see Theorem 1.3.8).

As noted at the beginning of the present chapter, the ideas portrayed in this section come from the use of tools from Geometric Control Theory made by Varolin in his proof of Theorem 3.2.6.

Earlier in Definition 3.2.4, we introduced the manifold $J^k_{p,*}(X)_{\mathfrak{g}}$ of jets generated by a Lie algebra of vector fields $\mathfrak{g}$. The nature of the elements in $J^k_{p,*}(X)_{\mathfrak{g}}$ is strictly related to the Lie algebra $\mathfrak{g}$. For $k = 0$ this is clear by the following definition.

Definition 6.3.1. Given $p \in X$ and a Lie algebra $\mathfrak{g} \subset \mathfrak{X}(X)$, we define the orbit of $\mathfrak{g}$ through p as

$$R_{\mathfrak{g}}(p) := \{\varphi^{t_n}_{X_n} \circ \cdots \circ \varphi^{t_1}_{X_1}(p) : X_1, \ldots, X_n \in \mathfrak{g}\}.$$

It is known that $R_{\mathfrak{g}}(p)$ is a complex manifold and that for any $q \in R_{\mathfrak{g}}(p)$ there exist holomorphic vector fields $X_1, \ldots, X_n \in \mathfrak{g}$ such that the holomorphic map $(t_1, \ldots t_n) \mapsto \varphi^{t_n}_{X_n} \circ \cdots \circ \varphi^{t_1}_{X_1}(q)$ gives holomorphic local coordinates in a neighbourhood of q (see for instance the textbook [29]).

If $\mathfrak{g}$ has the density property, it is natural to expect a behaviour similar to the one showed in Chapter 3 for holomorphic families of jets in $J^k_{p,*}(X)_{\mathfrak{g}}$. For this reason, we were led to consider the following generalization of the Vaserstein problem:

Claim 6.3.2. *Let W be a Stein manifold, X be a complex manifold and $\mathfrak{g} \subset \mathfrak{X}(X)$ be a Lie algebra with the density property. Given $p \in X$ and a null-homotopic holomorphic* *function $F : W \to R_{\mathfrak{g}}(p) \subset X$, then there exist complete vector fields $X_1, \ldots X_N \in \mathfrak{g}_{int}$ and holomorphic functions $f_1, \ldots, f_N : W \to \mathbb{C}$ such that*

$$F(w) = \varphi_{X_N}^{f_N(w)} \circ \cdots \circ \varphi_{X_1}^{f_1(w)}(p).$$

Theorem 1.3.8 solves the *generalized Vaserstein problem* in the following special situation. The finite dimensional Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of left invariant vector fields has the density property, as it consists of complete vector fields only. The solution of Theorem 1.3.8 appears to be more restrictive since the authors prove that the representation uses only the special vector fields corresponding to the one parameter subgroups $t \to \exp(te_{i,j})$, $i \neq j$, where $e_{i,j}$ is the matrix with entry 1 in place (i, j) and 0 elsewhere. However it is only a matter of simple linear algebra to show that any representation using left invariant vector fields can be reduced to a representation using these special fields only. The generalized Vaserstein problem becomes much less restrictive if one enlarges the Lie algebra from $\mathfrak{sl}_n(\mathbb{C})$ to the algebra of all holomorphic vector fields on $X = SL_n(\mathbb{C})$ (which has the density property by the work of Toth and Varolin [55]). We wonder whether the solution is easier in this case than that of the original Vaserstein problem.

In our opinion, the ability to solve the generalized Vaserstein problem could involve an answer to the following question, which has its own independent charm.

Question 6.3.3. Let X be a complex manifold. Given an open set $U \subset X$, a vector field $V \in \mathfrak{X}(U)$ and a Lie algebra $\mathfrak{g} \subset \mathfrak{X}(X)$, what conditions do we need to approximate V by elements of $\mathfrak{g}$ on compact subsets of U?

We know that for a Stein manifold X with the density property we only need U to be Runge in X (setting $\mathfrak{g} = \mathfrak{X}(X)$). This is one of the key ingredients of Theorem 2.3.1. We expect an answer to this question to give a new version of this result, which would allow to obtain elements of $\mathrm{Aut}_{\mathfrak{g}}(X)$ with specific local properties.

An instance of such a result has been proved by Kutzschebauch, Leuenberger and Liendo in [37, Theorem 6.3], where they deal with singular varieties and vector fields vanishing on a subvariety containing the singular locus. Another particular solution to Claim 6.3.2 was given in Section 3.2, while Lemma 4.1.2 settles the problem in a local setting.

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Razširjeni povzetek

Uvod

Disertacija vsebuje prikaz avtorjevih originalnih rezultatov na področju analize funkcij več kompleksnih spremenljivk, doseženih tekom doktorskega študija na FMF Univerze v Ljubljani. Poudarek je na eliptični kompleksni geometriji in predvsem na študiju kompleksnih mnogoterosti z veliko grupo holomorfnih avtomorfizmov. Modelni primer so kompleksni evklidski prostori $\mathbb{C}^n$ dimenzije $n > 1$. Disertacija uvodoma vsebuje podrobnejši prikaz nekaterih znanih rezultatov teorije Andersén-Lempert, ki predstavljajo bistveno orodje pri naših raziskavah, ter osnove teorije Steinovih mnogoterosti, to so kompleksne mnogoterosti z bogato algebro holomorfnih funkcij. Natančneje, kompleksna mnogoterost X je Steinova, če jo lahko predstavimo kot zaprto kompleksno podmnogoterost v nekem kompleksnem evklidskem prostoru $\mathbb{C}^N$. Splošneje, Steinov prostor končne dimenzije je biholomorfno ekvivalenten zaprti kompleksno analitični množici s singularnostmi v nekem kompleksnem evklidskem prostoru $\mathbb{C}^N$.

Eliptična kompleksna geometrija je sorazmerno novo področje kompleksne geometrije, ki se ukvarja s širokim spektrom holomorfnih fleksibilnostnih lastnosti kompleksnih mnogoterosti. To so lastnosti, diametralno nasprotne holomorfni rigidnosti. Med najbolj tipičnimi rigidnostnimi lastnostmi so Kobayashijeva hiperboličnost, ki pomeni neobstoj nekonstantnih holomorfnih preslikav $\mathbb{C} \to X$ kompleksne ravnine $\mathbb{C}$ v dano mnogoterost X, ter sorazmerna majhnost grupe $\mathrm{Aut}(X)$ holomorfnih avtomorfizmov mnogoterosti X. Njim nasprotne so fleksibilnostne lastnosti, kot npr. obstoj velike družine holomorfnih preslikav $S \to X$ poljubne Steinove mnogoterosti S v mnogoterost X (poseben primer je obstoj nekonstantnih holomorfnih premic $\mathbb{C} \to X$ ter, nekoliko splošneje, holomorfnih preslikav $\mathbb{C}^N \to X$ maksimalnega ranga za $N \geq \dim X$) ter obstoj neskončno razsežne grupe holomorfnih avtomorfizmov $\mathrm{Aut}(X)$. Tipični rezultati na tem področju zagotavljajo obstoj rešitev široke družine aproksimacijskih ter interpolacijskih problemov, pogosto za parametrizirane družine preslikav v mnogoterost ali avtomorfizmov mnogoterosti.

Teorija holomorfnih avtomorfizmov kompleksnih evklidskih prostorov $\mathbb{C}^n$ dimenzije $n > 1$ se je razvijala tekom 20. stoletja preko del mnogih avtorjev kot so Fatou, Bieberbach, Jung, Henon, Dixon in Esterle, Friedland in Milnor in drugi. Eden od vrhuncev klasične teorije predstavlja vplivno delo J.-P. Rosaya in W. Rudina [52] iz leta 1988, ki pa še vedno uporablja adhoc pristop k reševanju tovrstnih problemov. Teorija je doživela nesluten razvoj s pionirskim delom E. Anderséna in L. Lemperta iz obdobja 1990-92. Njun bistveni prispevek je nov pristop k obravnavi izotopij holomorfnih avtomorfizmov evklidskih prostorov $\mathbb{C}^n$ preko identifikacije neskončno razsežne Liejeve algebre (neskončno razsežne) grupe Aut($\mathbb{C}^n$). Pomemben prispevek v konceptualnem razumevanju in uporabnosti teorije je tudi delo F. Forstneriča in J.-P. Rosaya [19] iz leta1993, v katerem sta avtorja obravnavo prenesla iz neskončno razsežne Liejeve algebre na konkretnejšo obravnavo tokov holomorfnih vektorskih polj na $\mathbb{C}^n$. S tem sta omogočila efektivno aproksimacijo izotopij biholomorfnih preslikav med Rungejevimi domenami v $\mathbb{C}^n$ z izotopijami holomorfnih avtomorfizmov prostora $\mathbb{C}^n$. Bistveno vlogo v teoriji igra rezultat Anderséna in Lemperta, da je vsako polinomsko vektorsko polje na $\mathbb{C}^n$ za $n > 1$ vsota končno mnogo kompletnih polinomskih vektorskih polj. Holomorfno vektorsko polje na kompleksni mnogoterosti X se imenuje kompletno, če njegov tok $\phi_t(x)$ obstaja za poljubno začetno točko $x \in X$ dane mnogoterosti in za vsako kompleksno vrednost $t \in \mathbb{C}$ časovne spremenljivke. Vsako tako vektorsko polje generira kompleksno enoparametrično grupo $\{\phi_t\}_{t\in\mathbb{C}} \subset \mathrm{Aut}(X)$ holomorfnih avtomorfizmov mnogoterosti X.

Vsak holomorfen avtomorfizem kompleksne ravnine $\mathbb{C}$ je oblike $z \mapsto az+b$ za nek par števil $a, b \in \mathbb{C}$, $a \neq 0$. V naprotju se tem je grupa holomorfnih avtomorfizmov $\mathbb{C}^n$ za $n > 1$ zelo velika. Očitno je vsaka holomorfna preslikava oblike

$$z = (z_1, \ldots, z_n) \mapsto (z_1 + f(z_2, \ldots, z_n), z_2, \ldots, z_n),$$
$$z \mapsto (z_1 e^{f(z_2,\ldots,z_n)}, z_2, \ldots, z_n),$$

kjer je f cela holomorfna funkcija $n-1$ kompleksnih spremenljivk, holomorfen avtomorfizem prostora $\mathbb{C}^n$. Avtomorfizmi te oblike ter njihove $GL_n(\mathbb{C})$-konjugiranke se imenujejo *strigi*. Eno od najpomembnejših osnov teorije predstavlja rezultat Anderséna in Lemperta [3], da je grupa generirana s strigi gosta v grupi vseh holomorfnih avtomorfizmov $\mathbb{C}^n$ v kompaktno-odprti topologiji, ni pa ji enaka.

Pomembno konceptualno posološitev in nadaljnji razvoj teorije Andersén-Lempert na razredu splošnejših Steinovih mnogoterostih je omogočil D. Varolin [58, 59] v obdobju 2000-2001 z uvedbo in obravnavo t.i. holomorfne *lastnosti gostote*, ki jo podaja naslednja definicija.

Definicija 1.2.1. [59] *Kompleksna mnogoterost X ima lastnost gostote, če lahko vsako holomorfno vektorsko polje na X aproksimiramo enakomerno na kompaktih z Liejevimi kombinacijami (to je, vsotami in Liejevimi oklepaji oz. komutatorji) kompletnih holomorfnih vektorskih polj.*

Ta lastnost sama na sebi ni zelo signifikantna na poljubni mnogoterosti. Kot primer naj omenimo, da ima vsaka kompaktna kompleksna mnogoterost lastnost gostote, saj je vsako vektorsko polje na njej kompletno. (Večina kompaktnih kompleksnih mnogoterosti sploh nima netrivialnih holomorfnih vektorskih polj.) Po drugi strani pa je lastnost gostote bistveno bolj zanimiva in netrivialna na Steinovih mnogoterostih, ki imajo zelo veliko algebro holomorfnih vektorskih polj. Interakcija med holomorfnimi funkcijami in lastnostjo gostote na Steinovi mnogoterosti porodi neskončno razsežno grupo holomorfnih avtomorfizmov, podobno kot na evklidskih prostorih $\mathbb{C}^n$ za $n > 1$.

Naslednji rezultat se običajno imenuje izrek Andersén-Lempert. Prvo eksplicitno fomulacijo tega izreka na $\mathbb{C}^n$ lahko najdemo v delu Forstneriča in Rosaya [19, Theorem 1.1]; dokaz na poljubni Steinovi mnogoterosti z lastnostjo gostote je zelo podoben.

Izrek 2.3.4. *Naj bo X Steinova mnogoterost z lastnostjo gostote, $\Omega \subset X$ odprta množica in $\varphi^t : \Omega \to \Omega_t := \varphi^t(\Omega) \subset X$ $(t \in [0,1])$ izotopija holomorfnih preslikav, ki zadošča naslednjim pogojem:*

(i) *Ω_t je Rungejeva domena v X za vsak $t \in [0,1]$;*

(ii) *$\varphi^0 = \mathrm{Id}_\Omega$;*

(iii) *preslikava $\varphi^t(x)$ je razreda $\mathscr{C}^1$ v spremenljivki $t \in [0,1]$ in za vsak fiksen t je injektivna holomorfna v spremenljivki $x \in \Omega$.*

Potem lahko preslikavo φ^1 aproksimiramo enakomerno na kompaktih v Ω s holomorfnimi avtomorfizmi mnogoterosti X.

Ob tem naj spomnimo, da je injektivna holomorfna preslikava med dvema n-razsežnima kompleksnima mnogoterostima lokalno obrnljiva, torej je biholomorfna na svojo sliko. Navedeni izrek je izjemno uporaben v številnih konstrukcijah. Podroben dokaz in pregled njegove dosedanje uporabe lahko bralec najde v monografiji [20].

Parametrična interpolacija brstičev s holomorfnimi avtomorfizmi kompleksnih evklidskih prostorov

Prvi originalni rezultat disertacije, ki ga bomo predstavili, se dotika grupe holomorfnih avtomorfizmov evklidskega prostora $\mathbb{C}^n$ za poljuben $n > 1$. Začeli bomo z definicijami relevantnih pojmov.

Definicija 1.3.3. *Naj bo X kompleksna mnogoterost, $p \in X$ točka in $r \in \mathbb{N}$ naravno število. Par holomorfnih preslikav $F, G : U \to X$ na okolici $U \subset X$ točke p določa isti* brstič *reda r v p, če velja $|F(z) - G(z)| = O(|z|^{r+1})$ pri $z \to 0$ v nekem (in zato v poljubnem) paru lokalnih holomorfnih kart na X s središčem v točki p oziroma $F(p) = G(p)$.*

Očitno dobimo s tem ekvivalenčno relacijo na množici zarodkov holomorfnih preslikav na okolicah točke p v mnogoterost X; ekvivalenčne razrede te relacije imenujemo r-brstiče. Za dan izbor lokalnih holomorfnih kart na X v točkah p

in $q = F(p)$ je r-brstič preslikave F natanko določen z njenim (holomorfnim) Taylorjevim polinomom stopnje r. Brstič se imenuje *neizrojen*, če je njegov linearni del neizrojen; ekvivalentno, Jacobijeva matrika preslikave v poljubnem paru lokalnih kart ima neničelno determinanto. Ekvivalenčni razred (r-brstič) preslikave F v točki p bomo označili z $[F]_p^r$, množico vseh neizrojenih r-brstičev z vrednostjo $F(p) = q \in X$ pa z $J_{p,q}^r(X)$.

Druga pomembna definicija se tiče *pohlevnih zaporedij*, ki jih bomo podrobneje obravnavali v naslednjem razdelku.

Definicija 1.3.5. [52] *Naj bo $e_1 = (1, 0, \ldots, 0)$ prvi standardni bazni vektor prostora $\mathbb{C}^n$. Zaprto diskretno zaporedje $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n$ brez ponovitev se imenuje* pohlevno *(v smislu Rosaya in Rudina [52]), če obstaja holomorfni avtomorfizem $F \in \mathrm{Aut}(\mathbb{C}^n)$, tako da je $F(a_j) = j \cdot e_1$ za vse $j \in \mathbb{N}$.*

Lahko je videti, da grupa holomorfnih avtomorfizmov prostora $\mathbb{C}^n$ deluje tranzitivno na množici $\mathbb{N} \cdot e_1 = \{(j, 0, \ldots, 0) : j \in \mathbb{N}\}$ (glej [52]), zato smemo govoriti o pohlevnih (zaprtih diskretnih) množicah namesto o pohlevnih zaporedjih.

Naslednji izrek je posplošitev rezultata G. Buzzarda in F. Forstneriča [8] iz leta 2000. Njun izrek se tiče posameznih avtomorfizmov brez odvisnosti od parametra, to je, prostor W v naslednjem izreku je ena sama točka.

Preslikava topoloških prostorov, ki je homotopna konstantni preslikavi, se imenuje *nulhomotopna*.

Izrek 1.3.7. [57] *Naj bo W končno razsežni Steinov prostor, $\{a_j\}_{j\in\mathbb{N}}$, $\{b_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n$ ($n > 1$) pohlevni zaporedji brez ponovitev in $r_j \in \mathbb{N}$ ($j \in \mathbb{N}$) zaporedje naravnih števil. Za vsak $j \in \mathbb{N}$ naj bo $P_j : W \to J_{a_j,b_j}^{r_j}(\mathbb{C}^n)$ družina neizrojenih r_j-brstičev, ki je holomorfno odvisna od parametra $w \in W$. Potem obstaja nulhomotopna holomorfna preslikava $F : W \to \mathrm{Aut}(\mathbb{C}^n)$, ki zadošča pogoju*

$$F^w(z) = P_j^w(z) + O(|z - a_j|^{r_j+1}) \quad za\ z \to a_j,\ j \in \mathbb{N},\ w \in W,$$

natanko tedaj, ko je linearni del $Q_j : W \to GL_n(\mathbb{C})$ preslikave P_j v točki a_j nulhomotopna preslikava za vsak $j \in \mathbb{N}$.

Grupa avtomorfizmov $\mathrm{Aut}(\mathbb{C}^n)$ ni kompleksna mnogoterost, saj ima neskončno dimenzijo. Kljub temu lahko definiramo pojem holomorfne preslikave $F : W \to \mathrm{Aut}(\mathbb{C}^n)$: to je preslikava, za katero je pridružena evalvacijska preslikava

$$\begin{aligned} W \times \mathbb{C}^n &\to \mathbb{C}^n \\ (w, z) &\mapsto F^w(z) \end{aligned}$$

holomorfna v običajnem smislu.

V dokazu izreka 1.3.7 bomo uporabili rešitev holomorfnega Vasersteinovega problema, ki sta jo našla B. Ivarsson in F. Kutzschebauch [28] leta 2012 kot spektakularen primer uporabe moderne teorije Oka.

Izrek 1.3.8. [28] *Naj bo W končno razsežni Steinov prostor in $f : W \to SL_n(\mathbb{C})$ nulhomotopna holomorfna preslikava. Potem obstaja število $K \in \mathbb{N}$ in holomorfne preslikave*

$$G_1, \dots, G_K : W \to \mathbb{C}^{n(n-1)/2},$$

tako da je f produkt zgornje in spodnje trikotnih unipotentnih matričnih funkcij spremenljivke $w \in W$, prirejenih preslikavam G_i:

$$f(w) = \begin{pmatrix} 1 & 0 \\ G_1(w) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(w) \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & G_K(w) \\ 0 & 1 \end{pmatrix}.$$

Topološka predpostavka na brstiče v izreku 1.3.7 se tiče zgolj njihovega linearnega dela, saj obstaja retrakcija grupe $\mathrm{Aut}(\mathbb{C}^n)$ na njeno linearno podgrupo $\mathrm{GL}_n(\mathbb{C})$ (glej Forstnerič in Larusson [18]), podana z

$$\begin{aligned} [0,1] \times \mathrm{Aut}(\mathbb{C}^n) &\to \mathrm{Aut}(\mathbb{C}^n) \\ (t, F) &\mapsto \frac{F(tz) - F(0)}{t} + tF(0). \end{aligned}$$

Z uporabo izreka 1.3.8 bomo poenostavili linearni del brstiča, ki mora biti homotopen konstantni preslikavi.

V dokazu izreka 1.3.7 bomo induktivno uporabili naslednjo trditev.

Trditev 3.1.2. [57] *Naj bo W končno razsežen Steinov prostor, $n > 1$ in $r \in \mathbb{N}$ naravni števili, $p, q \in \mathbb{C}^n$ in $P : W \to J^r_{p,q}(\mathbb{C}^n)$ holomorfna družina brstičev v točki p, ki zadošča pogoju $P^w(p) = q$ za vse $w \in W$. Naj bo Q^w linearni del brstiča P^w v točki p, torej je $P^w(z) = q + Q^w(z-p) + O(|z-p|^2)$ pri $z \to p$ za vsak $w \in W$. Predpostavimo, da je preslikava $Q : W \to GL_n(\mathbb{C})$ nulhomotopna. Tedaj za poljubno končno množico $\{a_i\}_{i=1}^{i_0} \subset \mathbb{C}^n \setminus \{p, q\}$, naravno število $N \in \mathbb{N}$, kompaktno množico $T \subset W$, kompaktno konveksno množico $K \subset \mathbb{C}^n \setminus \{p, q\}$ ter število $\varepsilon > 0$ obstaja holomorfna preslikava $F : W \to \mathrm{Aut}(\mathbb{C}^n)$, ki zadošča naslednjim pogojem:*

(i) *$F^w(z) = P^w(z) + O(|z-p|^{r+1})$ pri $z \to p$ za vse $w \in W$;*

(ii) *$F^w(z) = z + O(|z-a_i|^N)$ pri $z \to a_i$ za vse $i \in \{1, \dots, i_0\}$ in $w \in W$;*

(iii) *$|F^w(z) - z| < \varepsilon$ za vse $w \in T$ in $z \in K$;*

(iv) *če je $\{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C}^n \setminus (K \cup \{p, q\})$ zaprta diskretna množica v koordinatni z_1-premici, lahko dodatno zagotovimo pogoj $F^w(c_j) = c_j$ za vse $w \in W$ in $j \in \mathbb{N}$.*

Na vsakem koraku v induktivnem postopku bomo z uporabo te trditve interpolirali dani brstič v eni dodatni točki ter hkrati ohranili že doseženo vrednost brstičev v prejšnjih točkah. V limiti bomo na ta način dobili predpisani brstič v vsaki točki dane diskretne množice.

Želimo komentirati pomen pogojev v trditvi 3.1.2. Pogoj (i) zagotovi, da dobimo pravi brstič v točki p, medtem ko pogoj (ii) zagotavlja ohranitev že dobljenih brstičev iz prejšnjih korakov indukcije. Pogoj (iii) zagotavlja konvergenco procesa (kot v članku Kuztschebauch in Ramos-Peon [39]). Pogoj (iv) je subtilnejše narave in nam omogoči ohranjanje točk, v katerih bomo interpolirali v bodočih korakih induktivnega procesa. Brez tega pogoja ne bi mogli induktivno uporabiti trditve 3.1.2, saj točki $p, q \in \mathbb{C}^n$ ne bi mirovali, pač pa bi se njuna lokacija holomorfno spreminjala v odvisnosti od parametra $w \in W$.

Sedaj bomo podali skico dokaza trditve 3.1.2. Zaradi enostavnosti si oglejmo primer $p = q = 0$, ki ga zagotovimo brez izgube splošnosti. Naj bo dana holomorfna družina brstičev $P : W \to J^r_{0,0}(\mathbb{C}^n)$ v izhodišču $0 \in \mathbb{C}^n$. Najprej poiščemo končno mnogo homogenih polinomskih preslikav $H^w_j : \mathbb{C}^n \to \mathbb{C}^n$, $j = 1, \ldots, k$, holomorfno odvisnih od parametra $w \in W$, tako da velja

$$P^w(z) = H^w_1(z) + \cdots + H^w_k(z) \quad \text{za vse } w \in W \text{ in } z \in \mathbb{C}^n.$$

V naslednjem koraku konstruiramo holomorfne preslikave $S_j : W \to \mathrm{Aut}(\mathbb{C}^n)$ za $j = 0, 1, \ldots, k$, ki zadoščajo naslednjim pogojem za vse točke $w \in W$, indekse $j \in \{1, \ldots, k\}$ ter za poljubno izbrano naravno število $N \in \mathbb{N}$:

(i) $S^w_j \circ \cdots \circ S^w_0(z) = H^w_1(z) + \cdots + H^w_j(z) + O(|z|^{j+1})$ za $z \to 0$;

(ii) $S^w_j(z) = z + O(|z - a_i|^N)$ za $z \to a_i$, $i \in \{1, \ldots, i_0\}$.

Iskano interpolacijsko preslikavo $F : W \to \mathrm{Aut}(\mathbb{C}^n)$ tedaj dobimo kot kompozicijo $F^w = S^w_k \circ \cdots \circ S^w_0$ $(w \in W)$.

Preslikave S_j za $j \geq 2$, ki izpolnjujejo zgoraj navedene pogoje, dobimo s preprosto parametrično posplošitvijo konstrukcije v članku Buzzarda in Forstneriča [8]. Interpolacija linearnega dela predpisanih brstičev pa je v parametričnem primeru precej zahtevnejši problem in na tem mestu bomo potrebovali izrek 1.3.8 (rešitev holomorfnega Vasersteinovega problema).

Linearizacijo preslikave P v točki $0 \in \mathbb{C}^n$ lahko predstavimo s preslikavo $Q : W \to GL_n(\mathbb{C})$. Najprej bomo prevedli problem na primer, ko je $Q : W \to SL_n(\mathbb{C})$ preslikava v grupo kompleksnih $n \times n$ matrik z determinanto 1; že v tem koraku bomo uporabili predpostavko, da je Q nulhomotopna. V naslednjem bistvenem koraku bomo uporabili izrek 1.3.8 in s tem dobili predstavitev preslikave $Q : W \to SL_n(\mathbb{C})$ v obliki produkta končnega števila unipotentnih holomorfnih matričnih funkcij $W \to SL_n(\mathbb{C})$. Elementi tega produkta nam bodo dali družine avtomorfizmov prostora $\mathbb{C}^n$, ki interpolirajo linearni del brstiča P^w za vsak $w \in W$.

Parametrična interpolacija brstičev z avtomorfizmi na Steinovih mnogoterostih z lastnostjo gostote

Drugi glavni rezultat disertacije je analog izreka 1.3.7 za Steinove mnogoterosti z Varolinovo lastnostjo gostote. Ta rezultat je plod sodelovanja avtorja z Ramos-Peonom [50].

Naj bo X kompleksna mnogoterost dimenzije n. Izberimo naravno število $r \in \mathbb{N}$. Spomnimo se (glej definicijo 1.3.3), da je r-brstič $[F]_p^r$ holomorfne preslikave $F : U \to X$ v točki $p \in X$ podan (v poljubnem paru lokalnih kart na X v točkah p in $F(p)$) s polinomsko preslikavo stopnje največ r, ki je maksimum stopenj komponent. Zaradi enostavnosti oznak bomo pogosto izpustili indeks r in pisali $[F]_p$, kadar bo stopnja očitna iz konteksta.

Množico vseh neizrojenih r-brstičev v točki $p \in X$ označimo z $J^r_{p,*}(X)$, oz. $J_{p,*}(X)$ če je r razviden iz konteksta. Lahko se prepričamo, da je $J^r_{p,*}(X)$ kompleksna mnogoterost. Vrednost brstiča $\gamma = [F]_p$ v točki p je natančno določena in jo označimo z $\gamma(p)$. Preslikava

$$\begin{aligned} \pi : J_{p,*}(X) &\to X \\ \gamma &\mapsto \gamma(p) \end{aligned}$$

je holomorfna in surjektivna. Za vsako točko $q \in X$ je praslika $\pi^{-1}(q) =: J_{p,q}(X)$ ravno množica vseh neizrojenih r-brstičev v p z vrednostjo q. V bodoče bomo zaradi preprostosti uporabljali besedo brstič za neizrojene r-brstiče, v kolikor ne bo drugače rečeno.

Izberimo število $N \in \mathbb{N}$ in različne točke $\{\hat{x}_i\}_{i=1}^N \subset X$. Tako N-terico lahko razumemo kot točko $\hat{x} \in X^N \setminus \Delta$, kjer je X^N kartezični produkt N kopij mnogoterosti X ter je

$$\Delta = \bigcup_{1 \le i < j \le N} \left\{(z_1, \ldots, z_N) \in X^N : z_i = z_j\right\}$$

njegova posplošena diagonala. Z uporabo zgoraj definirane projekcije $\pi : J_{p,*}(X) \to X$ na vsaki komponenti dobimo preslikavo $\pi : J_{\hat{x}_1,*}(X) \times \cdots \times J_{\hat{x}_N,*}(X) \to X^N$. Prostor

$$Y := J_{\hat{x}_1,*}(X) \times \cdots \times J_{\hat{x}_N,*}(X) \setminus \pi^{-1}(\Delta) \qquad (*)$$

je kompleksna mnogoterost, katere elementi so vse možne konfiguracije neizrojenih r-brstičev v točkah $\{\hat{x}_i\}_{i=1}^N$ s paroma različnimi vrednostmi.

Če je $U \subset X$ odprta množica v X, ki vsebuje vse točke $\{\hat{x}_i\}_{i=1}^N$ in je $F : U \to X$ injektivna holomofna preslikava, označimo z $[F]_{\hat{x}} \in Y$ N-terico brstičev

$$[F]_{\hat{x}} = ([F]_{\hat{x}_1}, \ldots, [F]_{\hat{x}_N}) \in Y$$

preslikave F v točkah iz $\hat{x} = \{\hat{x}_i\}_{i=1}^N$.

Za poljuben brstič $\gamma = [F]_{\hat{x}} \in Y$ označimo z $(\gamma)^{-1}$ brstič $[F^{-1}]_{F(\hat{x})}$ inverzne preslikave v točki $F(\hat{x})$. če je $\gamma(\hat{x}) = \hat{x}$, očitno velja $(\gamma)^{-1} \in Y$.

Za preslikavo $F : W \to \mathrm{Aut}(X)$ označimo z $F^w = F(w, \cdot) \in \mathrm{Aut}(X)$ avtomorfizem, dobljen z evalvacijo F v točki $w \in W$.

Sedaj lahko formuliramo naslednji glavni rezultat tega razdelka.

Izrek 4.2.1. [50] *Naj bo W Steinova mnogoterost, X Steinova mnogoterost z lastnostjo gostote, $r, N \in \mathbb{N}$, $(\hat{x}_1, \ldots, \hat{x}_N)$ N-terica različnih točk v X, in naj bo Y kot v $(*)$. Za dano holomorfno preslikavo $\gamma : W \to Y$ obstaja nulhomotopna holomorfna preslikava $F : W \to \mathrm{Aut}(X)$, ki zadošča pogoju $[F^w]_{\hat{x}} = \gamma^w$, natanko tedaj, ko je preslikava γ nulhomotopna.*

Dokaz izreka 4.2.1 je tehnično precej zahteven. Na tem mestu bomo podali le skico dokaza.

Če je preslikava F v izreku 4.2.1 nulhomotopna, je očitno tudi γ nulhomotopna, saj homotopija preslikav inducira homotopijo njenih brstičev v danih točkah.

Predpostavimo sedaj, da je družina brstičev $\gamma\colon W \to Y$ nulhomotopna. Naj bo $\gamma^t : W \to Y$ homotopija, gladko odvisna od parametra $t \in [0,1]$, za katero je $\gamma^1 = \gamma$ in je γ^0 konstantna preslikava. Lahko vzamemo, da je $\gamma^0 = [\mathrm{Id}]_{\hat{x}} \in Y$. Z uporabo izreka [39, Theorem 1.1] lahko nadalje predpostavimo, da je $\gamma^{t,w}(\hat{x}) = \hat{x}$ za vse $(t,w) \in [0,1] \times W$.

Izberimo neko kompaktno množico $L_0 \subset W$. Naj bo $U \subset X$ odprta množica, ki vsebuje vse točke $\hat{x}_i,\ i = 1,\dots,N$. Najprej konstruiramo homotopijo $f^t\colon L_0 \times U \subset W \times X \to X$, ki je gladko odvisna od parametra $t \in [0,1]$ in je holomorfna na množici $(w,z) \in L_0 \times U$, tako da velja $[f^{t,w}]_{\hat{x}} = \gamma^{t,w}$ za $(t,w) \in [0,1] \times L_0$. Z uporabo izreka 2.3.7 dobimo parametrizirano družino avtomorfizmov $F_0^t\colon W \to \mathrm{Aut}(X)$, holomorfnih v spremenljivki $w \in W$ in gladko odvisnih od parametra $t \in [0,1]$, tako da brstič $[F_0^{t,w}]_{\hat{x}}^{-1} \circ \gamma^{t,w}$ aproksimira brstič identične preslikave $[\mathrm{Id}]_{\hat{x}}$. S tem dobimo približno rešitev problema na množici $w \in L_0$.

Sedaj ponavljamo ta postopek na vse večjih kompaktnih množicah $L_0 \subset L_1 \subset \cdots \subset \bigcup_{j\geq 0} L_j = W$, ki izčrpajo mnogoterost W. S tem dobimo zaporedje avtomorfizmov $W \to \mathrm{Aut}(X)$, katerih brstiči v $\hat{x}$ konvergirajo h γ. Problem je sedaj v tem, da ne vemo, ali tako dobljeno zaporedje konvergira k družini avtomorfizmov $W \to \mathrm{Aut}(X)$. Da bi rešili ta problem, bomo dodatno zahtevali, da je naslednji avtomorfizem $F_1^t\colon W \to \mathrm{Aut}(X)$ v induktivno konstruiranem zaporedju blizu identiteti na neki kompaktni množici $K_1 \subset X$ za vse $w \in L_0$, pri čemer ohranimo prej navedene interpolacijske pogoje za vse vrednosti $w \in L_1$. Če X izčrpamo s kompaktnimi množicami $K_1 \subset K_2 \subset \cdots \subset \bigcup_{j\geq 0} K_j = X$ in obenem izčrpamo W s kompakti L_j kot zgoraj, bomo s primernim induktivnim postopkom našli zaporedje preslikav $L_j \to \mathrm{Aut}(X)$, ki konvergira k rešitvi $W \to \mathrm{Aut}(X)$ danega problema.

Strogo pohlevne množice

V tem razdelku bomo uvedli in študirali nov razred pohlevnih diskretnih množic. Predstavljeni rezultati so bili dobljeni v sodelovanju z Rafaelom Andristom v članku [7].

Definicija 1.4.1. [7] *Naj bo X kompleksna mnogoterost. Zaprta diskretna neskončna množica $A \subset X$ je* strogo pohlevna množica*, če za vsako injektivno preslikavo $f : A \to A$ obstaja holomorfen avtomorfizem $F \in \mathrm{Aut}(X)$, tako da je $F|_A = f$.*

Za $X = \mathbb{C}^n$ je zgornja definicija (netrivialno) ekvivalentna definiciji Rosaya and Rudina [52] (glej definicijo 1.3.5). Njuna definicija očitno implicira, da lahko poljubni dve pohlevni zaporedji brez ponovitev preslikamo eno na drugo z avtomorfizmom prostora $\mathbb{C}^n$, saj lahko obe preslikamo na standardno zaporedje $\mathbb{N}e_1 \subset \mathbb{C}^n$

v prvi koordinatni osi. Iz definicije 1.4.1 pa ni očitno, da sta poljubni dve strogo pohlevni zaporedji v mnogoterosti X ekvivalentni v smislu avtomorfizmov. Naslednja trditev nam pove, da to velja v vsaki Steinovi mnogoterosti z lastnostjo gostote.

Trditev 5.1.2. *Naj bo X Steinova mnogoterost z lastnostjo gostote in naj bosta $A, B \subset X$ strogo pohlevni množici. Tedaj obstaja holomorfen avtomorfizem $F \in \mathrm{Aut}(X)$, tako da je $F(A) = B$.*

V taki mnogoterosti bomo strogo pohlevne množice preprosteje imenovali pohlevne, torej bomo ispustili v tem primeru nepotreben pridevnik *strogo*.

Prva mnogoterost, ki jo bomo podrobneje obravnavali v smislu vsebovanja in klasifikacije pohlevnih množic, je posebna linearna grupa $\mathrm{SL}_2(\mathbb{C})$; to je Steinova mnogoterost z lastnostjo gostote [55]. Našli smo naslednja eksplicitna primera (strogo) pohlevnih množic.

Lema 5.2.1. *Množica*

$$\left\{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{N}\right\} \subset \mathrm{SL}_2(\mathbb{C})$$

je pohlevna.

Lema 5.2.2. *Množica*

$$\left\{\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix} : k \in \mathbb{N}\right\} \subset \mathrm{SL}_2(\mathbb{C})$$

je pohlevna.

Ti rezultati niso pomembni le sami zase, ampak predstavljajo vzorčni primer za študij semi-enostavnih kompleksnih Liejevih grup. Vsaka taka Liejeva grupa G ima algebraičen imerzivni morfizem $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow G$, kar sledi iz klasifikacije semi-enostavnih Liejevih algeber (glej npr. Humphreys [27]). Nadalje lahko reduciramo študij pohlevnih množic v poljubni povezani linearni algebraični Liejevi grupi na primer semi-enostavnih Liejevih grup.

Izrek 5.3.2. *Vsaka povezana linearna algebraična Liejeva grupa z izjemo kompleksne premice $\mathbb{C}$ in punktirane kompleksne premice $\mathbb{C}^*$ vsebuje pohlevno množico.*

Strategija dokaza je naslednja. Najprej pokažemo, da vsak produkt vsaj dveh Liejevih grup vsebuje pohlevno množico. To dokažemo z uporabo vektorskih polj, ki delujejo le na posameznem faktorju v produktu, podobno kot v primeru strigov na $\mathbb{C}^2$. Zatem uporabimo splošno teorijo Liejevih grup za dekompozicijo dane grupe G kot produkt Liejevih grup s specifičnimi lastnostmi. Ker produkt vsebuje pohlevno množico, lahko reduciramo študij na primer, ko je eden od faktorjev trivialen. Z večkratno uporabo tega postopka reduciramo problem na primer semi-enostavne Liejeve grupe, ki ga bomo diskutirali v nadaljevanju.

Kot že omenjeno, vsaka semi-enostavna Liejeva grupa G dopušča algebraičen imerzivni morfizem $i : \mathrm{SL}_2(\mathbb{C}) \hookrightarrow G$. Če je V levo invariantno vektorsko polje $\mathrm{SL}_2(\mathbb{C})$, lahko njegov tok ϕ_V^t predstavimo z levim množenjem z matriko; v nadaljevanju bo simbol ϕ_V^t označeval to matriko. To porodi naslednje delovanje grupe $(\mathbb{C},+)$ na G:

$$\begin{aligned} \mathbb{C} \times G &\to G \\ (t,A) &\to i(\phi_V^t) \cdot A. \end{aligned}$$

Za vsak fiksen $t \in \mathbb{C}$ je to holomorfen avtomorfizem grupe G. Podoben postopek lahko uporabimo za poljubno holomorfno vektorsko polje na $\mathrm{SL}_2(\mathbb{C})$. Na ta način dobimo veliko grupo avtomorfizmov grupe G, ki jo lahko uporabimo za konstrukcijo pohlevnih množic.