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NODAL SOLUTIONS FOR NONLINEAR NONHOMOGENEOUS ROBIN PROBLEMS

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Abstract. We consider the nonlinear Robin problem driven by a nonhomogeneous differential operator plus an indefinite potential. The reaction term is a Carathéodory function satisfying certain conditions only near zero. Using suitable truncation, comparison, and cut-off techniques, we show that the problem has a sequence of nodal solutions converging to zero in the $C^1(\overline{\Omega})$-norm.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear nonhomogeneous Robin problem:

$$(1) \qquad \left\{ \begin{array}{l} -\mathrm{div}\, a(Du(z)) + \xi(z)|u(z)|^{p-2}u(z) = f(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

In this problem, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone map (thus also maximal monotone), which satisfies certain regularity and growth conditions listed in hypotheses $H(a)$ below. These conditions are general and they incorporate in our framework many differential operators of interest, such as the p-Laplacian and the (p,q)-Laplacian. We stress that $a(\cdot)$ is not homogeneous and this is a source of difficulties in the study of problem (1). The potential function $\xi \in L^\infty(\Omega)$ is indefinite (that is, sign changing). The reaction term (the right-hand side of (1)) is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the function $z \mapsto f(z,x)$ is measurable, and for almost all $z \in \Omega$, the function $x \mapsto f(z,x)$) is continuous. We impose conditions on $f(z,\cdot)$ only near zero. In the boundary condition, $\dfrac{\partial u}{\partial n_a}$ denotes the conormal derivative corresponding to the differential operator $u \mapsto \mathrm{div}\, a(Du)$ and is defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \mapsto (a(Du), n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We are looking for nodal (that is, sign-changing) solutions for problem (1). Employing a symmetry condition on $f(z,\cdot)$ near zero and using truncation, perturbation, comparison, and cut-off techniques, and a result of Kajikiya [7], we generate a whole sequence $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ of distinct nodal solutions such that $u_n \to 0$ in $C^1(\overline{\Omega})$.

The first result in this direction was produced by Wang [27], who used cut-off techniques to produce an infinity of solutions converging to zero in $H_0^1(\Omega)$. In

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Wang [27] the problem is semilinear driven by the Dirichlet Laplacian. There is no potential term (that is, $\xi \equiv 0$). The sequence produced by Wang [27] does not consist of nodal solutions. More recently, Li & Wang [8] produced a sequence of nodal solutions for semilinear Schrödinger equations. For nonlinear equations we mention the recent works of He, Huang, Liang & Lei [5], and Papageorgiou & Rădulescu [19]. In He *et al.* [5], the problem is Neumann (that is, $\beta \equiv 0$) and the differential operator is the p-Laplacian (that is, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$, with $1 < p < \infty$). In Papageorgiou & Rădulescu [19], the differential operator is the same as in the present paper, but $\xi \equiv 0$. Also, the hypotheses on $f(z, \cdot)$ near zero are more restrictive. In the present paper we extend the results of all aforementioned works.

2. Preliminaries and Hypotheses

In the study of problem (1) we will use the following spaces: the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$, and the boundary Lebesgue spaces $L^r(\partial\Omega)$, $1 \leqslant r \leqslant \infty$.

We denote by $||\cdot||$ the norm on the Sobolev space $W^{1,p}(\Omega)$ defined by

$$||u|| = \left[||u||_p^p + ||Du||_p^p\right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space, with positive (order) cone

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior which contains the open set

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact, D_+ is the interior of C_+ when furnished with the relative $C(\overline{\Omega})$-norm topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the Lebesgue spaces $L^r(\partial\Omega), 1 \leqslant r \leqslant \infty$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns "boundary values" to all Sobolev functions. We know that the trace map is compact into $L^r(\partial\Omega)$ for all $1 \leqslant r < \dfrac{(N-1)p}{N-p}$ if $p < N$, and into $L^r(\partial\Omega)$ for all $1 \leqslant r < \infty$ if $p \geqslant N$. Furthermore, we have that

$$\ker \gamma_0 = W_0^{1,p}(\Omega) \text{ and } \operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

In what follows, for the sake of notational simplicity, we will drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the "Palais-Smale condition" (the "PS-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that
$\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence."

We shall need the following result of Kajikya [7].

Theorem 1. *Assume that X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the PS-condition, φ is even and bounded below, $\varphi(0)=0$, and for every $n \in \mathbb{N}$, there exists an n-dimensional subspace V_n of X and $\rho_n > 0$ such that*

$$\sup\{\varphi(u) : u \in V_n \cap \partial B_{\rho_n}\} < 0,$$

where $\partial B_{\rho_n} = \{u \in X : ||u||_X = \rho_n\}$. Then there exists a sequence $\{u_n\}_{n\geqslant 1} \subseteq X\backslash\{0\}$ such that

- *(i)* $\varphi'(u_n) = 0$ *for all* $n \in \mathbb{N}$ *(that is, each u_n is a critical point of φ);*
- *(ii)* $\varphi(u_n) \leqslant 0$ *for all* $n \in \mathbb{N}$; *and*
- *(iii)* $u_n \to 0$ *in X as* $n \to \infty$.

In the sequel, for any $\varphi \in C^1(X,\mathbb{R})$, we denote by K_φ the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

For $X \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for any $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Let $\vartheta \in C^1(0,\infty)$ be such that $\vartheta(t) > 0$ for all $t > 0$ and

$$0 < \hat{c} \leqslant \frac{\vartheta'(t)t}{\vartheta(t)} \leqslant c_0 \text{ and } c_1 t^{p-1} \leqslant \vartheta(t) \leqslant c_2(t^{\tau-1} + t^{p-1}) \quad \text{for all } t > 0, \text{ with } c_1, c_2 > 0, 1 \leqslant \tau < p. \tag{2}$$

Then the hypotheses on the map $a(\cdot)$ are the following:

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0,\infty)$, $a_0(t)t \to 0^+$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) $|\nabla a(y)| \leqslant c_3 \dfrac{\vartheta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N\backslash\{0\}$, and some $c_3 > 0$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \dfrac{\vartheta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N\backslash\{0\}$, $\xi \in \mathbb{R}^N$; and

(iv) If $G_0(t) = \displaystyle\int_0^t a_0(s)sds$ for all $t > 0$, then there exists $q \in (1,p]$ such that

$$t \mapsto G_0(t^{1/q}) \text{ is convex and } \limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} < +\infty.$$

Remark 1. *Hypotheses $H(a)(i),(ii),(iii)$ are dictated by the nonlinear global regularity theory of Lieberman [10] and the nonlinear maximum principle of Pucci & Serrin [24]. Hypothesis $H(a)(iv)$ reflects the particular requirements of our problem. However, $H(a)(iv)$ is not restrictive as the examples below illustrate.*

Hypotheses $H(a)$ imply that $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Evidently, $G(\cdot)$ is convex and $G(0) = 0$. Also, we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N\backslash\{0\},\ \nabla G(0) = 0.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$. Moreover, the convexity of $G(\cdot)$ implies that

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{3}$$

The next lemma summarizes the main properties of the map $a(\cdot)$ and it is an easy consequence of hypotheses $H(a)$ and condition (2) above.

Lemma 2. *If hypotheses $H(a)(i), (ii), (iii)$ hold, then*

(a) *$a(\cdot)$ is continuous, strictly monotone, hence maximal monotone, too;*
(b) *$|a(y)| \leqslant c_4(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$, and some $c_4 > 0$; and*
(c) *$(a(y), y)_{\mathbb{R}_N} \geqslant \dfrac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

This lemma and (3) lead to the following growth conditions on $G(\cdot)$.

Corollary 3. *If hypotheses $H(a)(i), (ii), (iii)$ hold, then $\dfrac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1 + |y|^p)$ for all $y \in \mathbb{R}^N$, and some $c_5 > 0$.*

Example 1. *The following maps $a(y)$ satisfy hypotheses $H(a)$:*

(a) *$a(y) = |y|^{p-2}y, 1 < p < \infty$.*
This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

(b) *$a(y) = |y|^{p-2}y + |y|^{q-2}y, 1 < q < p < \infty$. This map corresponds to the (p, q)-Laplace differential operator defined by*

$$\Delta_p u + \Delta_q u \text{ for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics. Recently (p, q)-equations have been studied by Bobkov & Tanaka [1], *Li & Zhang* [9], *Marano & Mosconi* [11], *Marano, Mosconi & Papageorgiou* [12, 13], *Mugnai & Papageorgiou* [16], *Papageorgiou & Rădulescu* [17], *Sun, Zhang & Su* [25], *and Tanaka* [26].

(c) *$a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y, 1 < p < \infty$. This map corresponds to the generalized p-mean curvature differential operator defined by*

$$\operatorname{div}((1 + |Du|^2)^{\frac{p-2}{2}}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

(d) *$a(y) = |y|^{p-2}y(1 + \dfrac{1}{1 + |y|^p}), 1 < p < \infty$.*

We denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair

$$(W^{1,p}(\Omega)^*,\ W^{1,p}(\Omega)).$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h \rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,p}(\Omega).$$

From Gasinski & Papageorgiou [3], we have:

Proposition 4. *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is bounded (maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone, too), and of type $(S)_+$, that is,*

$$\text{“}u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } \limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \Rightarrow u_n \to u\text{”}.$$

The hypotheses on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$ are the following:

$H(\xi)$: $\xi \in L^{\infty}(\Omega)$.

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Remark 2. *If $\beta \equiv 0$, then we recover the Neumann problem.*

Finally, we introduce our conditions on the reaction term $f(z,x)$:

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exists $\eta > 0$ such that for almost all $z \in \Omega, f(z,\cdot)|_{[-\eta,\eta]}$ is odd;
(ii) $|f(z,x)| \leqslant a_\eta(z)$ for almost all $z \in \Omega$, $x \in [-\eta,\eta]$, with $a_\eta \in L^{\infty}(\Omega)$;
(iii) with $q \in (1,p]$ as in hypothesis $H(a)(iv)$, we have
$$\lim_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} = +\infty \text{ uniformly for almost all } z \in \Omega; \text{ and}$$
(iv) there exists $\hat{\xi} > 0$ such that for almost all $z \in \Omega$
$$x \to f(z,x) + \hat{\xi}|x|^{p-2}x$$
is nondecreasing on $[-\eta,\eta]$

Remark 3. *We point out that all the above hypotheses concern the behaviour of $f(z,\cdot)$ only near zero.*

Finally, we mention that nonlinear problems with an indefinite potential have recently been studied in the context of equations driven by the Neumann p-Laplacian by Gasinski & Papageorgiou [4] (resonant problems) and Fragnelli, Mugnai & Papageorgiou [2] (superlinear problems). Also, nodal solutions for nonlinear Robin problems with no potential term, were obtained by Papageorgiou & Rădulescu [21].

3. Nodal solutions

Let $\varepsilon \in (0,\eta)$ and consider an even function $\gamma \in C^1(\mathbb{R})$ such that $0 \leqslant \gamma \leqslant 1$, $\gamma|_{[-\varepsilon,\varepsilon]} = 1$ and $\operatorname{supp}\gamma \subseteq [-\eta,\eta]$.

We set
$$\hat{f}(z,x) = \gamma(x)f(z,x) + (1-\gamma(x))\xi(z)|x|^{p-2}x.$$

Evidently, $\hat{f}(z,x)$ is a Carathéodory function which is odd in $x \in \mathbb{R}$ and has the following two additional properties:

$$\hat{f}(z,\cdot)|_{[-\varepsilon,\varepsilon]} = f(z,\cdot)|_{[-\varepsilon,\varepsilon]} \text{ for all } z \in \Omega; \tag{4}$$

$$\hat{f}(z,x) = \xi(z)|x|^{p-2}x \text{ for all } z \in \Omega,\ |x| \geqslant \eta. \tag{5}$$

It follows from (5) that

$$\hat{f}(z,\eta) - \xi(z)\eta^{p-1} = 0 \text{ for almost all } z \in \Omega. \tag{6}$$

Since $\hat{f}(z,\cdot)$ is odd, we have

$$\hat{f}(z,-\eta) + \xi(z)\eta^{p-1} = 0 \text{ for almost all } z \in \Omega. \tag{7}$$

On account of hypothesis $H(f)(iii)$, given any $\mu > 0$, we can find $\delta = \delta(\mu) \in (0,\varepsilon)$ such that

$$f(z,x)x = \hat{f}(z,x) \geqslant \mu|x|^q \text{ for almost all } z \in \Omega, \text{ and all } |x| \leqslant \delta \text{ (see (4))}. \tag{8}$$

Then (8) combined with hypothesis $H(f)(ii)$ implies that given $r > p$ we can find $c_6 > 0$ such that

$$\hat{f}(z,x)x \geqslant \mu|x|^q - c_6|x|^r \text{ for almost all } z \in \Omega, \text{ and all } x \in \mathbb{R}. \tag{9}$$

We introduce the following function

$$k(z,x) = \mu|x|^{q-2}x - c_6|x|^{r-2}x. \tag{10}$$

This is a Carathéodory function which is odd in $x \in \mathbb{R}$.

We consider the following auxiliary nonlinear Robin problem:

$$\left\{ \begin{array}{l} -\text{div}\, a(Du(z)) + |\xi(z)||u(z)|^{p-2}u(z) = k(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{11}$$

Proposition 5. *If hypotheses $H(a)$, $H(\xi)$, $H(\beta)$ hold, then problem (11) admits a unique positive solution*

$$u^* \in D_+$$

and since $k(z,\cdot)$ is odd, $v^* = -u^* \in D_+$ is the unique negative solution of (11).

Proof. We consider the Carathéodory function $\hat{k}(z,x)$ defined by

$$\hat{k}(z,x) = \left\{ \begin{array}{ll} k(z,-\eta) - \eta^{p-1} & \text{if } x < -\eta \\ k(z,x) + |x|^{p-2}x & \text{if } -\eta \leqslant x \leqslant \eta \\ k(z,\eta) + \eta^{p-1} & \text{if } \eta < x. \end{array} \right. \tag{12}$$

We set $\hat{K}(z,x) = \displaystyle\int_0^x \hat{k}(z,s)ds$ and consider the C^1-functional $\hat{\varphi_+} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \hat{\varphi_+}(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [|\xi(z)| + 1]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma \\ - \int_\Omega \hat{K}(z,u^+)dz \quad \text{for all } u \in W^{1,p}(\Omega). \end{aligned}$$

From (12) and Corollary 3 it is clear that

$$\hat{\varphi}_+(\cdot) \text{ is coercive.}$$

Also, from the Sobolev embedding theorem and the compactness of the trace map, we deduce that

$$\hat{\varphi}_+(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, by the Weierstrass-Tonelli theorem, we can find $u^* \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(u^*) = \inf\left\{\hat{\varphi}_+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{13}$$

On account of hypothesis $H(a)(iv)$, we can find $c_7 > 0$ such that

$$G(y) \leqslant \frac{c_7}{q}|y|^q \text{ for all } |y| \leqslant \delta, \tag{14}$$

with $\delta > 0$ as in (8). Let $u \in D_+$. Then we can find $t \in (0,1)$ small such that

$$tu(z) \in (0,\delta] \text{ and } |D(tu)(z)| \leqslant \delta \text{ for all } z \in \overline{\Omega}. \tag{15}$$

Using (10), (12), (14) and (15), we obtain

$$\begin{array}{ll}\hat{\varphi}_+(tu) & \leqslant \frac{t^q c_7}{q}||Du||_q^q + \frac{t^q}{q}\int_\Omega |\xi(z)||u|^q dz + \frac{t^q}{q}\int_{\partial\Omega}\beta(z)|u|^q d\sigma \\ +\frac{t^r}{r}||u||_r^r - \frac{t^q}{q}\mu||u||_q^q & \\ & (\text{since } t\in(0,1),\ q\leqslant p<r) \\ & \leqslant [c_8-\mu c_9]\, t^q \text{ for some } c_8, c_9>0 \text{ depending on } u.\end{array}$$

Choosing $\mu > \frac{c_8}{c_9}$, we infer that

$$\begin{array}{ll} & \hat{\varphi}_+(tu) < 0, \\ \Rightarrow & \hat{\varphi}_+(u^*) < 0 = \hat{\varphi}_+(0) \text{ (see (13))}, \\ \Rightarrow & u^* \neq 0.\end{array}$$

From (13) we have

$$\begin{array}{ll} & \hat{\varphi}'_+(u^*) = 0, \\ \Rightarrow & \langle A(u^*), h\rangle + \int_\Omega [|\xi(z)|+1]|u^*|^{p-2}u^* h dz + \int_{\partial\Omega}\beta(z)|u^*|^{p-2}u^* h dz = \\ & \qquad \int_\Omega \hat{k}(z,(u^*)^+) h dz \\ & \text{for all } h\in W^{1,p}(\Omega).\end{array} \tag{16}$$

In (16) we choose $h = -(u^*)^- \in W^{1,p}(\Omega)$. Using Lemma 2(c), we obtain

$$\begin{array}{ll} & \frac{c_1}{p-1}||D(u^*)^-||_p^p + ||(u^*)^-||_p^p \leqslant 0 \text{ (see hypothesis } H(B)), \\ \Rightarrow & u^* \geqslant 0,\ u^*\neq 0.\end{array}$$

In (16) we choose $h = (u^*-\eta)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{array}{ll} & \langle A(u^*), (u^*-\eta)^+\rangle + \int_\Omega [|\xi(z)|+1](u^*)^{p-1}(u^*-\eta)^+ dz \\ + & \int_{\partial\Omega}\beta(z)(u^*)^{p-1}(u^*-\eta)^+ d\sigma \\ = & \int_\Omega \left[\mu\eta^{q-1} - c_6\eta^{r-1} + \eta^{p-1}\right](u^*-\eta)^+ dz \text{ (see (12) and (10))} \\ \leqslant & \int_\Omega \left[\hat{f}(z,\eta) + \eta^{p-1}\right](u^*-\eta)^+ dz \text{ (see (9))} \\ = & \int_\Omega [\xi(z)+1]\,\eta^{p-1}(u^*-\eta)^+ dz \text{ (see (6))} \\ \leqslant & \langle A(\eta), (u^*-\eta)^+\rangle + \int_\Omega [|\xi(z)|+1]\,\eta^{p-1}(u^*-\eta)^+ dz + \int_{\partial\Omega}\beta(z)\eta^{p-1}(u^*-\eta)^+ d\sigma \\ & \text{(note that } A(\eta)=0 \text{ and see hypothesis } H(\beta)), \\ \Rightarrow & \langle A(u^*) - A(\eta), (u^*-\eta)^+\rangle + \int_\Omega [|\xi(z)|+1]\,((u^*)^{p-1} - \eta^{p-1})(u^*-\eta)^+ dz \leqslant 0 \\ & \text{(see hypotesis } H(\beta)), \\ \Rightarrow & u^* \leqslant \eta.\end{array}$$

So, we have proved that

$$u^* \in [0,\eta] = \left\{u\in W^{1,p}(\Omega) : 0\leqslant u(z)\leqslant \eta \text{ for almost all } z\in\Omega\right\}. \tag{17}$$

From (10), (12), (16) and (17), we infer that u^* is a positive solution of problem (11). From Papageorgiou & Rădulescu [20], we have

$$u^* \in L^\infty(\Omega).$$

Now the nonlinear regularity theory of Lieberman [10] implies that

$$u^* \in C_+ \backslash \{0\}.$$

From (16) and (17), we have

$$\left\{ \begin{array}{l} -\mathrm{div}\, a(Du^*(z)) + |\xi(z)| u^*(z)^{p-1} = k(z, u^*(z)) \text{ for almost all } z \in \Omega, \\ \dfrac{\partial u^*}{\partial n_a} + \beta(z) u^* = 0 \text{ on } \partial\Omega \end{array} \right\}$$

(see Papageorgiou & Rădulescu [18])

$$\begin{aligned} \Rightarrow\quad & -\mathrm{div}\, a(Du^*(z)) + |\xi(z)| u^*(z)^{p-1} \geqslant -c_6 u^*(z)^{r-1} \text{ for almost all } z \in \Omega \text{ (see (10))}, \\ \Rightarrow\quad & \mathrm{div}\, a(Du^*(z)) \leqslant \left[c_6 ||u^*||_\infty^{r-p} + ||\xi||_\infty\right] u^*(z)^{p-1} \text{ for almost all } z \in \Omega \\ & \text{(see hypothesis } H(\xi)), \\ \Rightarrow\quad & u^* \in D_+ \text{ (see Pucci \& Serrin [24, p. 120])}. \end{aligned}$$

Next, we show the uniqueness of this solution. To this end, let $\hat{i} : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$\hat{i}(u) = \left\{ \begin{array}{ll} \displaystyle\int_\Omega G(Du^{\frac{1}{q}}) dz + \frac{1}{p} \int_\Omega |\xi(z)| u^{\frac{p}{q}} dz + \frac{1}{p} \int_{\partial\Omega} \beta(z) u^{\frac{p}{q}} d\sigma & \text{if } u \geqslant 0, u^{\frac{1}{q}} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{array} \right.$$

From Papageorgiou & Winkert [23] (see the proof of Proposition 3.3), we know that $\hat{i}(\cdot)$ is convex and if $u^*, v^* \in D_+$ are two positive solutions of (11), then

$$\begin{aligned} \hat{i}'((u^*)^q)(h) = & \ \frac{1}{q} \int_\Omega \frac{-\mathrm{div}\, a(Du^*) + |\xi(z)|(u^*)^{p-1}}{(u^*)^{q-1}} h dz \\ \hat{i}'((v^*)^q)(h) = & \ \frac{1}{q} \int_\Omega \frac{-\mathrm{div}\, a(Dv^*) + |\xi(z)|(v^*)^{p-1}}{(v^*)^{q-1}} h dz \text{ for all } h \in C^1(\overline{\Omega}). \end{aligned}$$

The convexity of $\hat{i}(\cdot)$ implies the monotonicity of $\hat{i}'(\cdot)$. Hence

$$\begin{aligned} 0 \quad & \leqslant \int_\Omega \left[\frac{-\mathrm{div}\, a(Du^*) + |\xi(z)|(u^*)^{p-1}}{(u^*)^{q-1}} - \frac{\mathrm{div}\, a(Dv^*) + |\xi(z)|(v^*)^{p-1}}{(v^*)^{q-1}} \right] ((u^*)^q - (v^*)^q) dz \\ & = \int_\Omega c_6 \left[(v^*)^{r-q} - (v^*)^{r-q} \right] ((u^*)^q - (v^*)^q) dz \text{ (see (10))}, \\ \Rightarrow \quad & u^* = v^* \text{ (since } q \leqslant p < r). \end{aligned}$$

This proves the uniqueness of the positive solution $u^* \in D_+$ of (11). Since problem (11) is odd, it follows that $v^* = -u^* \in -D_+$ is the unique negative solution of problem (11). □

Consider the following Robin problem:

$$\left\{ \begin{array}{l} -\mathrm{div}\, a(Du(z)) + \xi(z)|u(z)|^{p-2} u(z) = \hat{f}(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2} u = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{18}$$

We denote by S^+ (respectively S^-) the set of positive (respectively negative) solutions of problem (18) which are in the order interval $[0, \eta] = \{u \in W^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant \eta$ for almost all $z \in \Omega\}$ (respectively in $[-\eta, 0] = \{ v \in W^{1,p}(\Omega) : -\eta \leqslant v(z) \leqslant 0$ for almost all $z \in \Omega$ $\}$). From Papageorgiou, Rădulescu & Repovš [22], we know that

- S^+ is downward directed (that is, if $u_1,\ u_2 \in S^+$, then we can find $u \in S^+$ such that $u \leqslant u_1, u \leqslant u_2$).
- S^- is upward directed (that is, if $v_1,\ v_2 \in S^-$, then we can find $v \in S^-$ such that $v_1 \leqslant v, v_2 \leqslant v$).

Moreover, reasoning as in the proof of Proposition 5 (with $k(z,x)$ replaced by $\hat{f}(z,x)$), we show that

$$\emptyset \neq S^+ \subseteq D_+ \text{ and } \emptyset \neq S^- \subseteq -D_+.$$

Proposition 6. *If hypotheses $H(a), H(\xi), H(\beta), H(f)$ hold, then $u^* \leqslant u$ for all $u \in S^+$ and $v \leqslant v^*$ for all $v \in S^-$.*

Proof. Let $u \in S_+$ and let $\hat{k}(z,x)$ be given by (12). We introduce the following truncation of $\hat{k}(z,\cdot)$:

$$e_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ \hat{k}(z,x) & \text{if } 0 \leqslant x \leqslant u(z) \\ \hat{k}(z,u(z)) & \text{if } u(z) < x. \end{cases} \tag{19}$$

This is a Carathéodory function. We set $E_+(z,x) = \int_0^x e_+(z,s)ds$ and consider the C^1-functional $\Psi_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi_+(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega [|\xi(z)|+1]|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega E_+(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently, $\Psi_+(\cdot)$ is coercive (see (19)) and sequentially weakly lower semicontinuous. So, we can find $\hat{u}^* \in W^{1,p}(\Omega)$ such that

$$\Psi_+(\hat{u}^*) = \inf\left\{\Psi_+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{20}$$

As in the proof of Proposition 5, using hypotheses $H(a)(iv)$ and $H(f)(iii)$, we show that

$$\begin{aligned} & \Psi_+(\hat{u}^*) < 0 = \Psi_+(0), \\ \Rightarrow\ & \hat{u}^* \neq 0. \end{aligned}$$

From (20) we have

$$\begin{aligned} & \Psi'_+(\hat{u}^*) = 0, \\ \Rightarrow\ & \langle A(\hat{u}^*), h\rangle + \int_\Omega [|\xi(z)|+1]|\hat{u}^*|^{p-2}\hat{u}^* h dz + \int_{\partial\Omega} \beta(z)|\hat{u}^*|^{p-2}\hat{u}^* h d\sigma = \\ & \int_\Omega e_+(z,\hat{u}^*)hdz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{21}$$

In (21), we choose $h = -(u^*)^- \in W^{1,p}(\Omega)$. Then using Lemma 2(c), we have

$$\begin{aligned} & \frac{c_1}{p-1}||D(\hat{u}^*)^-||_p^p + \int_\Omega [|\xi(z)+1|]((\hat{u}^*)^-)^p dz \leqslant 0 \text{ (see hypothesis } H(\beta) \text{ and (19))} \\ \Rightarrow\ & \hat{u}^* \geqslant 0,\ \hat{u}^* \neq 0. \end{aligned}$$

Next, in (21) we choose $h = (\hat{u}^* - u)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
& \langle A(\hat{u}^*), (\hat{u}^* - u)^+\rangle + \int_\Omega [|\xi(z)+1|](\hat{u}^*)^{p-1}(\hat{u}^* - u)^+ dz + \int_{\partial\Omega} \beta(z)(u^*)^{p-1}(\hat{u}^* - u)^+ d\sigma \\
= \; & \int_\Omega [\mu u^{q-1} - c_6 u^{r-1} + u^{p-1}](\hat{u}^* - u)^+ dz \text{ (see (19), (12), (10) and recall that } u \in S^+) \\
\leqslant \; & \int_\Omega [\hat{f}(z,u) + u^{p-1}](\hat{u}^* - u)^+ dz \text{ (see (9))} \\
= \; & \langle A(u), (\hat{u}^* - u)^+\rangle + \int_\Omega [|\xi(z)+1|]u^{p-1}(\hat{u}^* - u)^+ dz + \int_{\partial\Omega} \beta(z) u^{p-1}(\hat{u}^* - u)^+ d\sigma \\
& \text{(since } u \in S^+) \\
\Rightarrow \; & \hat{u}^* \leqslant u.
\end{aligned}$$

So, we have proved that

$$\hat{u}^* \in [0,u] = \{y \in W^{1,p}(\Omega) : 0 \leqslant y(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}.$$

This fact, together with (10), (12), (19), (21), imply that

$$\begin{aligned}
& -\mathrm{div}\, a(D\hat{u}^* z) + |\xi(z)|\hat{u}^*(z)^{p-1} = k(z, \hat{u}^*(z)) \text{ for almost all } z \in \Omega, \\
& \frac{\partial \hat{u}^*}{\partial n_a} + \beta(z)(\hat{u}^*)^{p-1} = 0 \text{ on } \partial\Omega \text{ (see Papageorgiou \& Rădulescu [18])}, \\
\Rightarrow \; & \hat{u}^* = u^* \text{ (see Proposition 5)}, \\
\Rightarrow \; & u^* \leqslant u \text{ for all } u \in S^+.
\end{aligned}$$

Similarly, we show that

$$v \leqslant v^* \text{ for all } v \in S^-.$$

This completes the proof. □

Now we can establish the existence of extremal constant sign solutions for problem (18), that is, we show that problem (18) has a smallest positive solution and a biggest negative solution.

Proposition 7. *If hypotheses $H(a), H(\beta), H(\xi), H(f)$ hold, then there exists a smallest positive solution $u_+ \in S^+ \subseteq D_+$ and a biggest negative solution $v_+ \in S^- \subseteq -D_+$.*

Proof. Invoking Lemma 3.10 of Hu & Papageorgiou [6, p. 178], we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq S^+$ such that

$$\inf S^+ = \inf_{n\geqslant 1} u_n.$$

Evidently, $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_+ \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_+ \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{22}$$

We have

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega \hat{f}(z, u_n) h dx \quad \text{for all } h \in W^{1,p}(\Omega), n \in \mathbb{N}. \tag{23}$$

In (23) we choose $h = u_n - u_+ \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (22). Then

$$\begin{aligned}
& \lim_{n\to\infty} \langle A(u_n), u_n - u_+\rangle = 0, \\
\Rightarrow \; & u_n \to u_+ \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 4)}.
\end{aligned} \tag{24}$$

In (23) we pass to the limit as $n \to \infty$ and use (24). Then

(25)
$$\langle A(u_+), h\rangle + \int_\Omega \xi(z) u_+^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_+^{p-1} h d\sigma = \int_\Omega \hat{f}(z, u_+) h dz \text{ for all } h \in W^{1,p}(\Omega).$$

From Proposition 6, we have

$$(26) \qquad \begin{array}{rl} & u^* \leqslant u_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow & u^* \leqslant u_+ \text{ (see (24)), hence } u_+ \neq 0. \end{array}$$

It follows from (25) and (26) that

$$u_+ \in S^+ \subseteq D_+, \ u_+ = \inf S^+.$$

Similarly, we produce

$$v_- \in S^- \subseteq -D_+, \ v_- = \sup S^-.$$

□

Let $\tau > ||\xi||_\infty$ and consider the following truncation-perturbation of $\hat{f}(z, \cdot)$:

$$(27) \qquad f_0(z, x) = \begin{cases} \hat{f}(z, v_-(z)) + \tau |v_-(z)|^{p-2} v_-(z) & \text{if } x < v_-(z) \\ \hat{f}(z, x) + \tau |x|^{p-2} x & \text{if } v_-(z) \leqslant x \leqslant u_+(z) \\ \hat{f}(z, u_+(z)) + \tau u_+(z)^{p-1} & \text{if } u_+(z) < x. \end{cases}$$

We set $F_0(z, x) = \displaystyle\int_0^x f_0(z, s) ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du) dz + \frac{1}{p} \int_\Omega [\xi(z) + \tau] |u|^p dz + \frac{1}{p} \int_\Omega \beta(z) |u|^p d\sigma - \int_\Omega F_0(z, u) dz$$
$$\text{for all } u \in W^{1,p}(\Omega).$$

Evidently, $\varphi_0(\cdot)$ is coercive (see (27) and recall that $\tau > ||\xi||_\infty$). So, $\varphi_0(\cdot)$ is bounded below and satisfies the PS-condition (see Marano & Papageorgiou [14, 15]).

Proposition 8. *If hypotheses $H(a), H(\xi), H(\beta), H(f)$ hold and $V \subseteq W^{1,p}(\Omega)$ is a finite dimensional linear subspace, then there exists $\rho_V > 0$ such that*

$$\sup \{\varphi_0(u) : u \in V, ||u|| = \rho_V\} < 0.$$

Proof. Recall that $u_+ \in D_+$ and $v_- \in -D_+$. So, $m_0 = \min\{\min_{\overline{\Omega}} u_+, -\max_{\overline{\Omega}} v_-\} > 0$. We set $\epsilon_0 = \min\{\epsilon, m_0\}$ (where $\epsilon > 0$ is from (4)). On account of hypothesis $H(f)(iii)$, given any $\mu > 0$, we can find $\delta = \delta(\mu) > 0 \in (0, \epsilon_0)$ such that

(28)
$$\begin{aligned} F_0(z, x) = \hat{F}(z, x) + \frac{\tau}{p}|x|^p \ & = F(z, x) + \frac{\tau}{p}|x|^p \\ & \geqslant \frac{\mu}{q}|x|^q + \frac{\tau}{p}|x|^p \\ & \text{(for almost all } z \in \Omega, \text{ and all } |x| \leqslant \delta, \text{ see (4) and (27)).} \end{aligned}$$

Moreover, on account of hypothesis $H(a)(iv)$ and Corollary 3, we have

$$(29) \qquad G(y) \leqslant c_{10}[|y|^q + |y|^p] \text{ for some } c_{10} > 0, \text{ and all } y \in \mathbb{R}^N.$$

Since the subspace $V \subseteq W^{1,p}(\Omega)$ is finite dimensional, all norms are equivalent. So, we can find $\rho_V \in (0, 1]$ such that

$$(30) \qquad u \in V, \ ||u|| \leqslant \rho_V \Rightarrow |u(z)| \leqslant \delta \text{ for all } z \in \overline{\Omega}.$$

Then for every $u \in V$ with $||u|| \leqslant \rho_V$, we have

$$\varphi_0(u) \leqslant c_{11}||u||^q - \mu c_{12}||u||^q \text{ for some } c_{11}, c_{12} > 0$$
$$\text{(see (27), (28), (29), (30) and recall that } \rho_V \leqslant 1,\ q \leqslant p)$$

Since $\mu > 0$ is arbitrary, we choose $\mu > \dfrac{c_{11}}{c_{12}}$ and conclude that

$$\varphi_0(u) < 0 \text{ for all } u \in V \text{ with } ||u|| = \rho_V.$$

The proof is now complete. □

We now obtain the following multiplicity theorem for the nodal solutions of problem (1).

Theorem 9. *Assume that hypotheses $H(a), H(\xi), H(\beta), H(f)$ hold. Then there exists a sequence $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ of nodal solutions of problem (1) such that*

$$u_n \to 0 \textit{ in } C^1(\overline{\Omega}).$$

Proof. We know that $\varphi_0(\cdot)$ is even, bounded below, satisfies the PS-condition, and $\varphi_0(0) = 0$. Moreover, using (27) as before, we can check that

$$K_{\varphi_0} \subseteq [v_-, u_+] \cap C^1(\overline{\Omega}). \tag{31}$$

The aforementioned properties of $\varphi_0(\cdot)$ and Proposition 8 permit us to apply Theorem 1. So, we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ such that

$$u_n \in K_{\varphi_0} \subseteq [v_-, u_+] \cap C^1(\overline{\Omega}) \text{ (see (31)) and } u_n \to 0 \text{ in } W^{1,p}(\Omega). \tag{32}$$

The nonlinear regularity theory of Lieberman [10] implies that we can find $\gamma \in (0,1)$ and $c_{13} > 0$ such that

$$u_n \in C^{1,\gamma}(\overline{\Omega}),\ ||u_n||_{C^{1,\gamma}(\overline{\Omega})} \leqslant c_{13} \text{ for all } n \in \mathbb{N}. \tag{33}$$

We know that $C^{1,\gamma}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$. So, it follows from (32) and (33) that

$$\begin{aligned} & u_n \to 0 \text{ in } C^1(\overline{\Omega}), \\ \Rightarrow\ & -\epsilon_0 \leqslant u_n(z) \leqslant \epsilon_0 \text{ for all } z \in \overline{\Omega}, \text{ and all } n \geqslant n_0 \\ & \text{(recall that } \epsilon_0 = \min\{\epsilon, m_0\} > 0, \text{ see the proof of Proposition 8).} \end{aligned}$$

From (4), (32) and the extremality of u_+, v_-, we get that $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ are nodal solutions of (1) and we have $u_n \to 0$ in $C^1(\overline{\Omega})$. □

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