This postprint is published in:
Nonlinear Analysis, Theory, Methods and Applications, Volume 190, Art. 111598, January 2020
DOI: 10.1016/j.na.2019.111598

Existence and multiplicity results for a new $p(x)$-Kirchhoff problem

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Abstract

In this work, we study the existence and multiplicity results for the following nonlocal $p(x)$-Kirchhoff problem:

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) div(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u + g(x,u) \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{0.1}$$

where $a \geq b > 0$ are constants, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $N > p(x) > 1$, λ is a real parameter and g is a continuous function. The analysis developed in this paper proposes an approach based on the idea of considering a new nonlocal term which presents interesting difficulties.

Keywords: Variable exponent; New nonlocal Kirchhoff equation; $p(x)$-Laplacian operator; Palais-Smale condition; Mountain Pass theorem; Fountain theorem.
Math. Subj. Class. (2010): Primary: 35J55, 35J65; Secondary: 35B65.

1. Introduction and statement of the main results

In this work, we study the existence and multiplicity results for the following nonlocal $p(x)$-Kirchhoff problem:

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) div(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{p(x)-2}u + g(x,u) \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $N > p(x) > 1$, $a, b > 0$ are constants, g is a continuous function satisfying conditions which will be stated later, $\lambda > 0$ is a real parameter and $div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, that is,

$$\Delta_{p(x)} = div(|\nabla u|^{p(x)-2}\nabla u) = \sum_{i=1}^{N}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right),$$

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which is not homogeneous and is related to the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. These facts imply some difficulties. For example, some classical theories and methods, including the Lagrange multiplier theorem and the theory of Sobolev spaces, cannot be applied. We call (1.1) a problem of Kirchhoff type because of the appearance of the term

$$b\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx$$

which makes the study of (1.1) interesting.

In the previous decades, the Kirchhoff type problem (1.1) with $p(x)\equiv 2$ has been the object of intensive research due to its strong relevance in applications (see [27, 26, 37]). Indeed, the study of Kirchhoff type problems, which arise in various models of physical and biological systems, has received more and more attention in recent years. More precisely, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ, p_0, h, E, L are constants which represent some physical meanings respectively. Eq. (1.2) extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations.

Since the variable exponent spaces have been thoroughly studied by Kováčik and Rákosník [25], they have been used in the previous decades to model various phenomena. In the studies of a class of non-standard variational problems and PDEs, variable exponent spaces play an important role for example, in electrorheological fluids [36, 35, 34], thermorheological fluids [6], image processing [1, 9, 28], etc. In recent years, there has been a great deal of work done on problem (1.1), especially concerning the existence, multiplicity, uniqueness and regularity of solutions. Some important and interesting results can be found, for example, in [4, 5, 3, 2, 7, 10, 8, 11, 12, 13, 15, 18, 21, 19, 22, 23, 24, 25, 29, 30, 31, 32, 33, 40] and references therein.

At first, the eigenvalues of $p(x)$-Laplacian Dirichlet problem were studied in [17], i.e., if $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, the Rayleigh quotient

$$\lambda_{p(.)}=\inf_{u\in W_0^{1,p(x)}(\Omega)\backslash\{0\}}\frac{\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx}{\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}dx} \tag{1.3}$$

is zero in general, and only under some special conditions $\lambda_{p(.)}>0$ holds. For example, when $\Omega\subset\mathbb{R}$ ($N=1$) is an interval, results show that $\lambda_{p(.)}>0$ if and only if $p(.)$ is monotone. It is well known that $\lambda_p>0$ plays a very important role in the study of p-Laplacian problems.

Motivated by the papers mentioned above, our main purpose is to consider the perturbed problem (1.1) with a new nonlocal term

$$a-b\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx$$

which presents interesting difficulties. The key argument in our main result is the proof that the energy functional J (which appeared in (2.2)) of problem (1.1) possesses a Mountain Pass energy c.

To deal with the difficulty caused by the noncompactness due to the Kirchhoff function term, we must estimate precisely the value of c and give a threshold value (see Lemma 3.1) under which the Palais–Smale condition at the level c for J is satisfied. So the variational technique for problem (1.1) becomes more delicate. We obtain a nontrivial weak solution by using the Mountain Pass theorem. To the best of our knowledge, the present papers results have not been covered yet in the literature.

Suppose that the nonlinearity $g(x,t)\in C(\overline{\Omega}\times\mathbb{R})$ satisfies the following assumptions:

g_1: the subcritical growth condition:

$$|g(x,s)|\leq C(1+|s|^{q(x)-1}),\ \text{for all } (x,s)\in\Omega\times\mathbb{R},$$

where $C>0$ and $p(x)<q(x)<p^*(x)$;

g_2: $\lim\limits_{s\to 0} \dfrac{g(x,s)}{|s|^{p(x)-2}s} = 0$;

g_3: there exist $s_A > 0$ and $\theta \in (p^+, \dfrac{2p^{-2}}{p^+})$ such that

$$0 < \theta G(x,s) \leq sg(x,s), \text{ for all } |s| \geq s_A, \ x \in \Omega,$$

where $G(x,s) = \displaystyle\int_0^s g(x,t)dt$;

g_4: $g(x,-s) = -g(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}$.

Now we can state our main results:

Theorem 1.1. *Suppose that the function $q \in C(\overline{\Omega})$ satisfies*

$$1 < p^- < p(x) < p^+ < 2p^- < q^- < q(x) < p^*(x). \tag{1.4}$$

Then for any $\lambda \in \mathbb{R}$, with (g_1)–(g_3) satisfied, problem (1.1) *has a nontrivial weak solution.*

Theorem 1.2. *Suppose that the function $q \in C(\overline{\Omega})$ satisfies*

$$1 < p^- < p(x) < p^+ < 2p^- < q^- < q(x) < p^*(x).$$

Then for any $\lambda \in \mathbb{R}$, with (g_1)–(g_4) satisfied, problem (1.1) *has infinitely many solutions $\{u_n\}$ such that $I(u_n) \to \infty$ as $n \to \infty$.*

Remark 1.1. *Hypothesis (g_3) is known as the Ambrosetti–Rabinowitz's superlinear condition (see [11]). Moreover, condition (g_3) ensures that the Euler–Lagrange functional associated with problem* (1.1) *possesses the geometry of Mountain Pass theorem and it also guarantees the boundedness of the Palais–Smale sequence corresponding to the Euler–Lagrange's functional.*

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we prove the Palais-Smale compactness condition. In Section 4, we prove Theorem 1.1 via the Mountain Pass theorem. In Section 5, we prove Theorem 1.2 via the Fountain theorem. In this paper, $|\cdot|$ denotes the Lebesgue measure on Ω, and C (respectively, C_ϵ) always denotes a generic positive constant independent of n and ϵ (respectively, n).

2. Preliminaries on variable exponent spaces

In order to discuss problem (1.1), we need some theory on spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ which we shall call generalized Lebesgue Sobolev spaces. Let Ω be a bounded domain of $\mathbb{R}^N$, denote $C_+(\overline{\Omega}) = \{p(x); p(x) \in C(\overline{\Omega}), p(x) > 1, \text{ for all } x \in \overline{\Omega}\}$ and $p^- = \inf\limits_{\Omega} p(x) \leq p(x) \leq p^+ = \sup\limits_{\Omega} p(x) < N$.

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$||u||_{L^{p(x)}(\Omega)} = |u|_{p(.)} = \inf \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [20, 25, 39].

Proposition 2.1 ([39]). *The space* $(L^{p(x)}(\Omega), |.|_{p(x)})$ *is separable, uniformly convex, reflexive, and its conjugate space is* $(L^{q(x)}(\Omega), |.|_{q(x)})$, *where* $q(x)$ *is the conjugate function of* $p(x)$ *i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \textit{for all } x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$, *the Hölder type inequality*

$$\Big| \int_{\Omega} uv dx \Big| \leq \left(\frac{1}{p^{-}} + \frac{1}{q^{-}} \right) |u|_{p(x)} |v|_{q(x)}$$

holds.

The inclusion between Lebesgue spaces also generalizes the classical framework, namely, if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1 \leq p_2$ in Ω, then there exists a continuous embedding $L^{p_2(x)}(\Omega) \to L^{p_1(x)}(\Omega)$.

An important role in working with the generalized Lebesgue–Sobolev spaces is played by the $m(\cdot)$-modular of the $L^{p(\cdot)}(\Omega)$ space, which is the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u|^{p(x)}\, dx.$$

Lemma 2.1 ([14]). *Suppose that* $u_n, u \in L^{p(\cdot)}$ *and* $p_+ < +\infty$. *Then the following properties hold:*

1. $|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p-}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^+}_{p(\cdot)}$;
2. $|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^-}_{p(\cdot)}$;
3. $|u|_{p(\cdot)} < 1$ *(respectively,* $= 1; >1$*)* $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ *(respectively,* $=1; >1$*);*
4. $|u_n|_{p(\cdot)} \to 0$ *(respectively,* $\to +\infty$*)* $\Longleftrightarrow \rho_{p(\cdot)}(u_n) \to 0$ *(respectively,* $\to +\infty$*);*
5. $\lim\limits_{n\to\infty} |u_n - u|_{p(x)} = 0 \Longleftrightarrow \lim\limits_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0.$

The Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \Big\{ u : \Omega \subset \mathbb{R}^N \to \mathbb{R} : u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \Big\},$$

equipped with the norm

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)}.$$

Then $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $||u||_{1,p(x)}$. In this way, $L^{p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ become separable and reflexive Banach spaces. For more details, we refer to [14, 16, 18]. Moreover, we define

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

The following results were proved in [18].

Proposition 2.2 (Sobolev Embedding [18]). *For* $p, q \in C_+(\overline{\Omega})$ *such that* $1 \leq q(x) \leq p^*(x)$ *for all* $x \in \overline{\Omega}$, *there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we replace $\leq$ *with* $<$, *the embedding is compact.*

Proposition 2.3 (Poincaré Inequality [18]). *There is a constant $C > 0$, such that*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)} \tag{2.1}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Remark 2.1. *By Proposition 2.3, we know that $\|\nabla u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.*

Lemma 2.2 ([21]). *Denote*

$$A(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

Then $A(u) \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and the derivative operator A' of A is

$$\langle A'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx \textit{ for all } u, v \in W_0^{1,p(x)}(\Omega),$$

and the following holds:

1. *A is a convex functional;*
2. *$A' : W_0^{1,p(x)}(\Omega) \to (W^{-1,p'(x)}(\Omega)) = \left(W_0^{1,p(x)}(\Omega)\right)^*$ is a bounded homeomorphism and strictly monotone operator, and the conjugate exponent satisfies $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$;*
3. *A' is a mapping of type S_+, namely, $u_n \rightharpoonup u$ and $\limsup\langle A'(u_n), u_n - u\rangle \leq 0$, imply $u_n \to u$ (strongly) in $W_0^{1,p(x)}(\Omega)$.*

Definition 2.1. *We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if*

$$\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx - \lambda\int_\Omega |u|^{p(x)-2}u\varphi dx = \int_\Omega g(x,u)\varphi dx,$$

where $\varphi \in W_0^{1,p(x)}(\Omega)$.

The energy functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ associated with problem (1.1)

$$J(u) = a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx, \tag{2.2}$$

for all $u \in W_0^{1,p(x)}(\Omega)$ is well defined and of C^1 class on $W_0^{1,p(x)}(\Omega)$. Moreover, we have

$$\begin{aligned}\langle J'(u), \varphi\rangle &= \left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx - \lambda\int_\Omega |u|^{p(x)-2}u\varphi dx \\ &\quad - \int_\Omega g(x,u)\varphi dx, \end{aligned} \tag{2.3}$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$. Hence, we can observe that the critical points of the functional J are the weak solutions for problem (1.1). In order to simplify the presentation we will denote the norm of $W_0^{1,p(x)}(\Omega)$ by $\|.\|$, instead of $\|\cdot\|_{W_0^{1,p(x)}(\Omega)}$.

3. The Palais-Smale Compactness Condition

Recall now the definition of the Palais-Smale compactness condition.

Definition 3.1. *Let $(W_0^{1,p(x)}(\Omega),\ \|.\|)$ be a Banach space and $J \in C^1(W_0^{1,p(x)}(\Omega))$. Given $c \in \mathbb{R}$, we say that J satisfies the Palais–Smale condition at the level $c \in \mathbb{R}$ ("$(PS)_c$ condition", for short) if every sequence $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ satisfying*

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \text{ in } W^{-1,p'(x)}(\Omega) \text{ as } n \to \infty, \tag{3.1}$$

has a convergent subsequence.

First, we investigate the compactness conditions for the functional J.

Lemma 3.1. *Assume that (g_1)–(g_3) hold. Then the functional J satisfies the $(PS)_c$ condition, where precisely $c < \dfrac{a^2}{2b}$.*

Proof. We proceed in two steps.

Step 1. We prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence such that $c < \dfrac{a^2}{2b}$.

• For $\lambda \leq 0$. From (3.1) and (g_3), for n large enough, we have

$$\begin{aligned}
&C + \|u_n\| \\
&\geq \theta J(u_n) - \langle J'(u_n), u_n\rangle \\
&\geq \theta\left(a\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}dx - \int_\Omega G(x,u_n)dx\right) \\
&-\left(\left[a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right]\int_\Omega |\nabla u_n|^{p(x)}dx - \lambda\int_\Omega |u_n|^{p(x)}dx - \int_\Omega g(x,u_n)u_n dx\right) \\
&\geq a\left(\frac{\theta}{p^+} - 1\right)\int_\Omega |\nabla u_n|^{p(x)}dx + b\left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 - \lambda\left(\frac{\theta}{p^-} - 1\right)\int_\Omega |u_n|^{p(x)}dx - C|\Omega|,
\end{aligned}$$

where $|\Omega| = \displaystyle\int_\Omega dx$. Since $\lambda \leq 0$, we can deduce that

$$C + \|u_n\| \geq a\left(\frac{\theta}{p^+} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+}\right)\|u_n\|^{2p^-} - C|\Omega|.$$

It follows from (1.4) that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

• For $\lambda > 0$. Arguing by contradiction, we assume that, passing eventually to a subsequence, still denoted by $\{u_n\}$, we have $\|u_n\| \to +\infty$ as $n \to +\infty$. By (3.1) and (g_3), for n large enough, we have

$$\begin{aligned}
C \quad + \quad & \|u_n\| \geq \theta J(u_n) - \langle J'(u_n), u_n\rangle \\
\geq \quad & a\left(\frac{\theta}{p^+} - 1\right)\int_\Omega |\nabla u_n|^{p(x)}dx + b\left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^2 - \lambda\left(\frac{\theta}{p^-} - 1\right)\int_\Omega |u_n|^{p(x)}dx - C|\Omega|,
\end{aligned}$$

Therefore the last inequality, together with (2.1), implies that

$$C + \|u_n\| + \lambda C\left(\frac{\theta}{p^-} - 1\right)\|u_n\|^{p^+} \geq a\left(\frac{\theta}{p^+} - 1\right)\|u_n\|^{p^-} + b\left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+}\right)\|u_n\|^{2p^-} - C|\Omega|.$$

Dividing the above inequality by $\|u_n\|^{p^+}$, taking into account (1.4) holds and passing to the limit as $n \to \infty$, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Step 2. Here, we will prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. It follows from Proposition 2.2 that the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact, where $1 \le s(x) < p(x)^*$. Passing, if necessary, to a subsequence, there exists $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega),\ u_n \to u \text{ in } L^{s(x)}(\Omega),\quad u_n(x) \to u(x), \text{ a.e. in } \Omega. \tag{3.2}$$

By Hölder's inequality and (3.2), we have

$$\begin{aligned} \left|\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)dx\right| &\le \int_\Omega |u_n|^{p(x)-1}|u_n-u|dx \\ &\le \left||u_n|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}} |u_n-u|_{p(x)} \\ &\to 0 \text{ as } n\to\infty \end{aligned}$$

and thus,

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)dx = 0. \tag{3.3}$$

By virtue of conditions (g_1) and (g_2), one has that for every $\epsilon \in (0,1)$ there exists $C_\epsilon > 0$ such that

$$|g(x,u_n)| \le \epsilon |u_n|^{p(x)-1} + C_\epsilon |u_n|^{q(x)-1}. \tag{3.4}$$

By (3.4) and Proposition 2.2, it follows that

$$\begin{aligned} \left|\int_\Omega g(x,u_n)(u_n-u)dx\right| &\le \int_\Omega \epsilon|u_n|^{p(x)-1}|u_n-u| + C_\epsilon|u_n|^{q(x)-1}|u_n-u|dx \\ &\le \epsilon\left||u_n|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}|u_n-u|_{p(x)} + C_\epsilon\epsilon\left||u_n|^{q(x)-1}\right|_{\frac{q(x)}{q(x)-1}}|u_n-u|_{q(x)} \\ &\to 0 \text{ as } n\to\infty \end{aligned}$$

which shows that

$$\lim_{n\to\infty}\int_\Omega g(x,u_n)(u_n-u)dx = 0. \tag{3.5}$$

By (3.1), we have

$$\langle J'(u_n), u_n-u\rangle \to 0.$$

Therefore

$$\begin{aligned} \langle J'(u_n), u_n-u\rangle &= \left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx \\ &\quad -\lambda\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)dx - \int_\Omega g(x,u_n)(u_n-u)dx \to 0. \end{aligned}$$

So, we can deduce from (3.3) and (3.5) that

$$\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx \to 0. \tag{3.6}$$

Since $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to t_0 \ge 0 \text{ as } n\to\infty.$$

Case 1. If $t_0 = 0$ then $\{u_n\}$ strongly converges to $u = 0$ in $W_0^{1,p(x)}(\Omega)$ and the proof is finished.
Case 2. If $t_0 > 0$ we need to consider two subcases:
Subcase 1. If $t_0 \ne \dfrac{a}{b}$ then $a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0$ is not true and no subsequence of $\{a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to$

$0\}$ converges to zero. Therefore, there exists $\delta > 0$ such that $\left|a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right| > \delta > 0$ when n is large enough. So, it is clear that

$$\{a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0\} \text{ is bounded.} \tag{3.7}$$

Subcase 2.[1] If $t_0 = \frac{a}{b}$ then $a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0$.

We define

$$\varphi(u) = \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega G(x,u)dx, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Then

$$\langle\varphi'(u), v\rangle = \lambda\int_\Omega |u|^{p(x)-2}uvdx + \int_\Omega g(x,u)vdx, \text{ for all } u, v \in W_0^{1,p(x)}(\Omega).$$

It follows that

$$\langle\varphi'(u_n) - \varphi'(u), v\rangle = \lambda\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)vdx + \int_\Omega (g(x,u_n) - g(x,u))vdx.$$

To complete the argument we need the following lemma.

Lemma 3.2. *Let $u_n, u \in W_0^{1,p(x)}(\Omega)$ such that (3.2) holds. Then, passing to a subsequence, if necessary, the following properties hold:*

(i) $\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)vdx = 0$;

(ii) $\lim_{n\to\infty}\int_\Omega |g(x,u_n) - g(x,u)||v|dx = 0$;

(iii) $\langle\varphi'(u_n) - \varphi'(u), v\rangle \to 0,\ v \in W_0^{1,p(x)}(\Omega)$.

Proof. By (3.2), we have $u_n \to u$ in $L^{p(x)}(\Omega)$ which implies that

$$|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u \text{ in } L^{\frac{p(x)}{p(x)-1}}(\Omega). \tag{3.8}$$

Due to Hölder's inequality, we have

$$\begin{aligned}\left|\int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)vdx\right| &\le \int_\Omega ||u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u||v|dx \\ &\le \left||u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right|_{\frac{p(x)}{p(x)-1}}|v|_{p(x)} \\ &\le C\left||u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right|_{\frac{p(x)}{p(x)-1}}||v|| \\ &\to 0.\end{aligned} \tag{3.9}$$

By a slight modification of the proof above, we can also prove part (*ii*) so we omit the details.

$$\int_\Omega |g(x,u_n) - g(x,u)||v|dx \le \int_\Omega [\epsilon(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u) + C_\epsilon(|u_n|^{q(x)-1} - |u|^{q(x)-1})]|v|dx \to 0.$$

Finally, part (*iii*) follows by combining parts (*i*) and (*ii*). Consequently, $||\varphi'(u_n) - \varphi'(u)||_{W^{-1,p'(x)}} \to 0$ and $\varphi'(u_n) \to \varphi'(u)$.

We can now complete the proof of Subcase 2:

[1] This case does not exist if the Kirchhoff function is given by $a + b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx$.

By Lemma 3.2 and since $\langle J'(u), u\rangle = \left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx - \langle\varphi'(u), v\rangle$, $\langle J'(u), u\rangle \to 0$ and $a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \to 0$, it follows that $\varphi'(u_n) \to 0$ $(n \to \infty)$, i.e.,

$$\langle\varphi'(u), v\rangle = \lambda\int_\Omega |u|^{p(x)-2}uvdx + \int_\Omega g(x, u)vdx, \text{ for all } v \in W_0^{1,p(x)}(\Omega),$$

and therefore

$$\lambda|u(x)|^{p(x)-2}u(x) + g(x, u(x)) = 0 \text{ for a.e. } x \in \Omega$$

by the fundamental lemma of the variational method (see [38]). It follows that $u = 0$. So

$$\varphi(u_n) = \lambda\int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}dx + \int_\Omega G(x, u_n)dx \to \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega G(x, u)dx = 0.$$

Hence, we see that for $t_0 = \frac{a}{b}$ we have

$$J(u_n) = a\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}dx - \int_\Omega G(x, u_n)dx \to \frac{a^2}{2b}.$$

This is a contradiction since $J(u_n) \to c < \frac{a^2}{2b}$, then $a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0$ is not true and similarly to Subcase 1, we have that

$$\{a - b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0\} \text{ is bounded.} \tag{3.10}$$

So, it follows from the two cases above that

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \to 0.$$

Invoking the S_+ condition (see Lemma 2.2), we can now deduce that $\|u_n\| \to \|u\|$ as $n \to \infty$, which means that J satisfies the $(PS)_c$ condition. □

Remark 3.1. *The $(PS)_c$ condition is not satisfied for $c > \frac{a^2}{2b}$.*

Indeed,

$$J(u) \le a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \le \frac{a^2}{2b}$$

and so if $\{u_n\}$ is a $(PS)_c$ sequence of J, then we have $c \le \frac{a^2}{2b}$, which is a contradiction.

4. Proof of Theorem 1.1

To verify the conditions of the Mountain Pass theorem (see e.g., [38]), we first need to prove two lemmas.

Lemma 4.1. *Assume that g satisfies (g_1) and (g_2). Then there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \ge \alpha > 0$, for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.*

Proof.

• For $\lambda \le 0$. By assumptions (g_1) and (g_2), we have

$$|G(x, u)| \le \frac{\epsilon}{p(x)}|u|^{p(x)} + \frac{C_\epsilon}{q(x)}|u|^{q(x)}. \tag{4.1}$$

Let $\epsilon = \frac{1}{8}a\lambda_{p(x)}$ and $u \in W_0^{1,p(x)}(\Omega)$ be such that $\|u\| = \rho \in (0,1)$. By considering Lemma 2.1, Proposition 2.2 and (1.4), we can deduce that

$$
\begin{aligned}
J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx \\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \epsilon\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx - C_\epsilon\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx. \\
&\geq (a - \frac{\epsilon}{\lambda_{p(x)}})\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \frac{CC_\epsilon}{q^-}\int_\Omega |\nabla u|^{q(x)}dx \\
&\geq \frac{1}{p^+}(a - \frac{\epsilon}{\lambda_{p(x)}})\rho_{p(x)}(\nabla u) - \frac{b}{2p^{-2}}(\rho_{p(x)}(\nabla u))^2 - \frac{CC_\epsilon}{q^-}\rho_{q(x)}(\nabla u) \\
&\geq \frac{1}{p^+}(a - \frac{\epsilon}{\lambda_{p(x)}})\|u\|^{p^+} - \frac{b}{2p^{-2}}\|u\|^{2p^-} - \frac{CC_\epsilon}{q^-}\|u\|^{q^-} \\
&\geq \left(\frac{7a}{8p^+} - \frac{b}{2p^{-2}}\|u\|^{2p^- - p^+} - \frac{CC_\epsilon}{q^-}\|u\|^{q^- - p^+}\right)\|u\|^{p^+}.
\end{aligned}
$$

We can choose ρ sufficiently small (i.e. ρ is such that $\frac{7a}{8p^+} - \frac{b}{2p^{-2}}\rho^{2p^- - p^+} - \frac{CC_\epsilon}{q^-}\rho^{q^- - p^+} > 0$), so that

$$
I(u) \geq \rho^{p^+}(\frac{7a}{8p^+} - \frac{b}{2p^{-2}}\rho^{2p^- - p^+} - \frac{CC_\epsilon}{q^-}\rho^{q^- - p^+}) -: \alpha > 0.
$$

• For $\lambda > 0$. Let $\epsilon > 0$ be small enough so that $\frac{1}{2p^+}(a - \frac{\lambda}{\lambda_{p(x)}}) = \frac{\epsilon}{\lambda_{p(x)}p^-}$.

Let $\rho \in (0,1)$ and $u \in W_0^{1,p(x)}(\Omega)$ be such that $\|u\| = \rho$. By considering Lemma 2.1, Proposition 2.2, (1.4), and (4.1), we can deduce that

$$
\begin{aligned}
J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx \\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \frac{\lambda}{\lambda_{p(x)}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) \\
&\quad -\epsilon\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx - C_\epsilon\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx \\
&\geq (a - \frac{\lambda}{\lambda_{p(x)}})\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \\
&\quad -\frac{\epsilon}{\lambda_{p(x)}}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{CC_\epsilon}{q^-}\int_\Omega |\nabla u|^{q(x)}dx \\
&\geq (\frac{1}{p^+}(a - \frac{\lambda}{\lambda_{p(x)}}) - \frac{\epsilon}{\lambda_{p(x)}p^-})\rho_{p(x)}(\nabla u) - \frac{b}{2p^{-2}}(\rho_{p(x)}(\nabla u))^2 - \frac{CC_\epsilon}{q^-}\rho_{q(x)}(\nabla u) \\
&\geq (\frac{1}{p^+}(a - \frac{\lambda}{\lambda_{p(x)}}) - \frac{\epsilon}{\lambda_{p(x)}p^-})\|u\|^{p^+} - \frac{b}{2p^{-2}}\|u\|^{2p^-} - \frac{CC_\epsilon}{q^-}\|u\|^{q^-} \\
&\geq \left(\frac{1}{2p^+}(a - \frac{\lambda}{\lambda_{p(x)}}) - \frac{b}{2p^{-2}}\|u\|^{2p^- - p^+} - \frac{CC_\epsilon}{q^-}\|u\|^{q^- - p^+}\right)\|u\|^{p^+}.
\end{aligned}
$$

Set

$$
\lambda^* = \frac{qp^{-2}\lambda_{p(x)}a - bp^+q^-\rho^{2p^- - p^+} - 2CC_\epsilon p^{-2}\rho^{q^- - p^+}}{q^-p^{-2}} \text{ and } \alpha = \lambda^*\rho^{p^+}. \tag{4.2}
$$

We can conclude that for any $\lambda \in (0,\lambda^*)$, there exists $\alpha > 0$ such that for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$ we have $J(u) \geq \alpha > 0$. □

Lemma 4.2. *Assume that g satisfies* (g_3)*. Then there exists* $e \in W_0^{1,p(x)}(\Omega)$ *with* $\|e\| > \rho$ *(where* ρ *is given by Lemma 4.1) such that* $J(e) < 0$*.*

Proof. In view of (g_3) we know that for all $A > 0$, there exists $C_A > 0$ such that

$$G(x,u) \geq A|u|^{\theta} - C_A, \text{ for all } (x,u) \in \Omega \times \mathbb{R}. \tag{4.3}$$

Let $\psi \in C_0^{\infty}(\Omega)$, $\psi > 0$, and $t > 1$. By (4.3) we have

$$\begin{aligned} J(t\psi) &= a\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|t\psi|^{p(x)}dx - \int_\Omega G(x,t\psi)dx \\ &\leq a\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|t\psi|^{p(x)}dx \\ &- At^{\theta}\int_\Omega |t\psi|^{\theta}dx + C_A|\Omega| \\ &\leq \frac{at^{p^+}}{p^-}\int_\Omega |\nabla\psi|^{p(x)}dx - \frac{bt^{2p^-}}{2p^{+2}}\left(\int_\Omega |\nabla\psi|^{p(x)}dx\right)^2 - \frac{\lambda}{p^+}t^{p^-}\int_\Omega |\psi|^{p(x)}dx - At^{\theta}\int_\Omega |\psi|^{\theta}dx + C_A|\Omega|. \end{aligned}$$

Since $\theta > 2p^- > p^+ > p^-$, we obtain $J(t\psi) \to -\infty$ $(t \to +\infty)$. Then for $t > 1$ large enough, we can take $e = t\psi$ so that $\|e\| > \rho$ and $J(e) < 0$. □

Proof of Theorem 1.1

By Lemmas 3.1–4.2 and the fact that $J(0) = 0$, J satisfies the Mountain Pass theorem (see e.g., [38]). Therefore, problem (1.1) has indeed a nontrivial weak solution. □

5. The proof of Theorem 1.2

The proof mainly rests on an application of the Fountain theorem. Since $X :- W_0^{1,p(x)}(\Omega)$ is a separable and reflexive real Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \ldots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \ldots\}}$$

and

$$\langle e_j^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Theorem A (Fountain Theorem, see [38]). *Suppose that an even functional* $\Phi \in C^1(X, \mathbb{R})$ *satisfies the* $(PS)_c$ *condition for every* $c > 0$*, and that there is* $k_0 > 0$ *such that for every* $k \geq k_0$ *there exists* $\rho_k > r_k > 0$ *so that the following properties hold:*

(i) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$*;*

(ii) $b_k = \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to +\infty$ *as* $k \to \infty$*.*

Then Φ *has a sequence of critical points* $\{u_k\}$ *such that* $\Phi(u_k) \to +\infty$*.*

Lemma 5.1. *Assume that* $\alpha \in C_+(\overline{\Omega})$*,* $\alpha(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$*, and denote*

$$\beta_k = \sup_{u \in Z_k, \|u\| = 1} |u|_{\alpha(x)}.$$

Then $\lim_{k \to \infty} \beta_k = 0$*.*

Proof. Obviously, $0 < \beta_{k+1} \le \beta_k$, so $\beta_k \to \beta \ge 0$. Let $u_k \in Z_k$ satisfy

$$\|u_k\| = 1, \quad 0 \le \beta_k - |u_k|_{\alpha(x)} < \frac{1}{k}.$$

Then there exists a subsequence of $\{u_k\}$ (which we still denote by u_k) such that $u_k \rightharpoonup u$, and

$$\langle e_j^*, u\rangle = \lim_{k\to\infty}\langle e_j^*, u_k\rangle = 0, \ \text{ for all } e_j^*,$$

which implies that $u = 0$, and so $u_k \rightharpoonup 0$. Since the embedding from $W_0^{1,p(x)}(\Omega)$ to $L^{\alpha(x)}(\Omega)$ is compact, it follows that $u_k \to 0$ in $L^{\alpha(x)}(\Omega)$. Hence, we get $\beta_k \to 0$ as $k \to \infty$. Proof of Lemma 5.1 is thus complete. □

Proof of Theorem 1.2. By Lemma 3.1, the functional J satisfies the $(PS)_c$ condition where precisely $c < \dfrac{a^2}{2b}$. Now we shall verify that J satisfies the conditions of Theorem A item by item.

(i) By (g_3), there exist $C_1 > 0$, $M > 0$ such that

$$G(x,s) \ge C_1|s|^\theta, \quad \text{for all } |s| \ge M, \quad x \in \Omega. \tag{5.1}$$

Note that by (g_1),

$$\begin{aligned} |G(x,s)| &\le \int_0^1 |g(x,zs)s|dz \\ &\le \int_0^1 C(1+|zs|^{q(x)-1})|s|dz \le C|s| + C|s|^{q(x)}, \quad \text{for all } (x,t) \in \Omega\times\mathbb{R}. \end{aligned} \tag{5.2}$$

Therefore, if $|s| \le M$, there exists $C_2 > 0$ such that

$$|G(x,s)| \le |s|(C + C|s|^{q(x)-1}) \le C_2|s|.$$

Combining this with (5.1), we find

$$G(x,s) \ge C_1|s|^\theta - C_2|s|, \quad \text{for all } (x,t) \in \Omega\times\mathbb{R}.$$

For $u \in Y_k$, when $\|u\| > 1$,

$$\begin{aligned} J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx \\ &\le a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - C_1\int_\Omega |u|^\theta dx + C_2\int_\Omega |u|dx. \end{aligned}$$

Consequently, because when $\|u\| > 1$, all norms on the finite-dimensional space Y_k, are equivalent, there is $C_W > 0$ such that

$$\int_\Omega |u|^{p(x)}dx \ge C_W\|u\|^{p^-}, \quad \int_\Omega |u|^\theta dx \ge C_W\|u\|^\theta \ \text{ and } \ \int_\Omega |u|dx \ge C_W\|u\|.$$

Hence, we get

$$J(u) \le \frac{a}{p^-}\|u\|^{p^+} - \frac{b}{2p^{-2}}\|u\|^{2p^-} - \frac{\lambda C_W}{p^-}\|u\|^{p^-} - C_1C_W\|u\|^\theta + C_2C_W\|u\|.$$

Since $\theta > 2p^- > p^+ > p^-$, it follows that for some $\rho_k = \|u\| > 0$ large enough we can deduce that

$$a_k = \max_{u\in Y_k, \|u\|=\rho_k} J(u) \le 0.$$

Hence, condition (i) of Theorem A holds.

(ii) By (g_1) and (g_2), there exist $C_3, C_4 > 0$ such that

$$|G(x,u)| \leq \frac{C_3}{p(x)}|u|^{p(x)} + \frac{C_4}{q(x)}|u|^{q(x)}.$$

By computation, we obtain for any $u \in Z_k$ with $|u| \leq 1$,

$$\begin{aligned} J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \lambda\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega G(x,u)dx \\ &\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega \frac{\lambda}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{C_3}{p(x)}|u|^{p(x)}dx \\ &\quad - \int_\Omega \frac{C_4}{q(x)}|u|^{q(x)}dx \\ &\geq \frac{a}{p^+}\|u\|^{p^+} - \frac{b}{2p^{-2}}\|u\|^{p^{2-}} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}\|u\|^{p^-} - \frac{C_4}{q^-}\beta_k^{q^-}\|u\|^{q^-}. \end{aligned}$$

Let $\varphi \in Z_k$, $\|\varphi\| = 1$ and $0 < t < 1$. Then it follows that

$$\begin{aligned} J(t\varphi) &\geq \frac{a}{p^+}t^{p^+} - \frac{b}{2p^{-2}}t^{p^{2-}} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}t^{p^-} - \frac{C_4}{q^-}\beta_k^{q^-}t^{q^-} \\ &\geq (\frac{a}{p^+} - \frac{b}{2p^{-2}})t^{p^{2-}} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}t^{p^-} - \frac{C_4}{q^-}\beta_k^{q^-}t^{q^-}. \end{aligned}$$

Conditions $a \geq b$ and $p^+ < 2p^{-2}$ imply that $\frac{a}{p^+} - \frac{b}{2p^{-2}} = \frac{2p^{-2}a - bp^+}{2p^{-2}p^+} > 0$.

Hence, we get

$$\begin{aligned} J(t\varphi) &\geq (\frac{2p^{-2}a - bp^+}{2p^{-2}p^+})t^{p^{2-}} - \frac{C_4}{q^-}\beta_k^{q^-}t^{q^-} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}t^{p^-} \\ &\geq (\frac{2p^{-2}a - bp^+}{2p^{-2}p^+} - \frac{C_4}{q^-}\beta_k^{q^-})t^{q^-} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}t^{p^-}. \end{aligned}$$

Choosing $\frac{C_4}{q^-}\beta_k^{q^-} < \frac{2p^{-2}a - bp^+}{4p^{-2}p^+}$, we can deduce

$$J(t\varphi) \geq \frac{2p^{-2}a - bp^+}{4p^{-2}p^+}t^{q^-} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}t^{p^-}.$$

Obviously, there exists a large enough k such that

$$J(t\varphi) \geq t^{p^-}\Big(\frac{2p^{-2}a - bp^+}{4p^{-2}p^+}t^{q^- - p^-} - (\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}\Big).$$

Put $\rho_k :- \left(\frac{4p^{-2}p^+}{2p^{-2}a - bp^+}(\lambda + C_3)\frac{\beta_k^{p^-}}{p^-}\right)^{\frac{1}{q^- - p^-}}$. Then for sufficiently large k, $\rho_k < 1$. When $t = \rho_k$, $\rho_k \in Z_k$ with $\|\varphi\| = 1$, we have $J(t\varphi) \geq 0$. Therefore, condition (ii) of Theorem A holds. This completes the proof of Theorem 1.2. □

Acknowledgments

Hamdani would like to express his deepest gratitude to the Military School of Aeronautical Specialities, Sfax (ESA) for providing an excellent atmosphere for work. Mtiri would like to express his gratitude to the Department of Mathematics, Faculty of Sciences and Arts, King Khalid University, Muhayil Asir, for supporting his work. Repovš was supported by the Slovenian Research Agency grants P1-0292, J1-0831, N1-0064, N1-0083, and N1-0114. The authors wish to acknowledge thes referees for useful comments and valuable suggestions which have helped to improve the presentation.

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