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DOUBLE-PHASE PROBLEMS AND A DISCONTINUITY PROPERTY OF THE SPECTRUM

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Abstract. We consider a nonlinear eigenvalue problem driven by the sum of p and q-Laplacian. We show that the problem has a continuous spectrum. Our result reveals a discontinuity property for the spectrum of a parametric (p, q)-differential operator as the parameter $\beta \to 1^-$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear nonhomogeneous eigenvalue problem:

$$(P_\lambda) \qquad \left\{ \begin{array}{l} -\alpha\Delta_p u(z) - \beta\Delta_q u(z) = \lambda|u(z)|^{q-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \alpha > 0, \beta > 0, \lambda > 0, 1 < p, q < \infty, p \neq q. \end{array} \right\}$$

For every $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \text{ for all } u \in W_0^{1,r}(\Omega).$$

Equations driven by the sum of a p-Laplacian and of q-Laplacian, known as (p, q)-equations, arise in many problems of mathematical physics such as particle physics, see Benci, D'Avenia, Fortunato & Pisani [1] and nonlinear elasticity, see Zhikov [19]. Zhikov [19] introduced models for strongly anisotropic materials in the context of homogenization. For this purpose defined and studied the double phase functional

$$J_{p,q}(u) = \int_\Omega (|Du|^p + a(z)|Du|^q)\, dz \; 0 \leqslant a(z) \leqslant M, 1 < p < q, u \in W_0^{1,q}(\Omega),$$

where the modulating coefficient $a(z)$ dictates the geometry of the composite made of two different materials with hardening exponents p and q respectively.

Such problems were studied by Chorfi & Rădulescu [2], Gasinski & Papageorgiou [3, 4], Marano, Mosconi & Papageorgiou [9, 10], Mihailescu & Rădulescu [11], Papageorgiou & Rădulescu [12, 13], Papageorgiou, Rădulescu & Repovš [14, 15, 16], Rădulescu & Repovš [17], and Yin & Yang [18], under different conditions on the data of the problem.

In the present paper, we show that problem (P_λ) has a continuous spectrum which is the half line $(\beta\hat{\lambda}_1(q), +\infty)$, with $\hat{\lambda}_1(q) > 0$ being the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. So, for every $\lambda \in (\beta\hat{\lambda}_1(q), +\infty)$, problem (P_λ) admits a nontrivial solution. Our result reveals an interesting fact better illustration in the particular case where

$$1 < p < \infty, q = 2, p \neq q, \alpha = 1 - \beta, \beta \in (0, 1).$$

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Let $L_\beta = -(1-\beta)\Delta_p u - \beta\Delta u$ and let $\hat{\sigma}(\beta)$ denote the spectrum of L_β. We have that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty) \text{ for } \beta \in (0,1).$$

The set function $\beta \to \hat{\sigma}(\beta)$ is h-continuous (Hausdorff continuous) on $(0,1]$, but at $\beta = 0$ exhibits a discontinuity since $L_0 = \Delta$ which has a discrete spectrum.

Our approach is based on the use of the Nehari manifold. So, we perform minimization under constraint.

2. Preliminaries

Let $r \in (1, +\infty)$. We recall some basic facts about the spectrum of $(-\Delta_r, W_0^{1,r}(\Omega))$. So, we consider nonlinear eigenvalue problem

$$-\Delta_r u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \text{ in } \Omega, u|_{\partial\Omega} = 0 \tag{1}$$

We say that $\hat{\lambda}$ is an eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$ if problem (1) admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$, known as an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. From the nonlinear regularity theory (see, for example, Gasinski & Papageorgiou [5, pp. 737-738]), we have that $\hat{u} \in C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}$. There is a smallest eigenvalue $\hat{\lambda}_1(r)$ which has the following properties:

- $\hat{\lambda}_1(r)$ is isolated (that is, there exists $\epsilon > 0$ such that the interval $(\hat{\lambda}_1(r), \hat{\lambda}_1(r) + \epsilon)$ contains no eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$).
- $\hat{\lambda}_1(r)$ is simple (that is, if $\hat{u}, \hat{v}$ are eigenfunction corresponding to $\hat{\lambda}_1(r)$, then $\hat{u} = \mu\hat{v}$ with $\mu \in \mathbb{R}\backslash\{0\}$).
- $\hat{\lambda}_1(r) > 0$ and admits the following variational characterization

$$\hat{\lambda}_1(r) = inf\left\{\frac{||Du||_r^r}{||u||_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right\} \tag{2}$$

The infimum in (2) is realized on the corresponding one dimensional eigenspace. The above properties imply that the elements of this eigenspace are in $C_0^1(\overline{\Omega})$ and do not change sign. By $\hat{u}_1(r)$ we denote the positive, L^r-normalized (that is, $||\hat{u}_1(r)||_r = 1$) eigenfunction corresponding to $\hat{\lambda}_1(r) > 0$. We have

$$\hat{u}_1(r) \in C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact the nonlinear maximum principle (see, for example, Gasinski & Papageorgiou [4, p. 738]), implies that

$$\hat{u}_1(r) \in \operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0\},$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ being the outward normal derivative of u. Note that if $\hat{u}$ an eigenfunction corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(r)$, then $\hat{u}$ is nodal (that is, sign changing). The Ljusternik-Schnirelmann minimax scheme gives in addition to $\hat{\lambda}_1(r)$ a whole strictly increasing sequence $\{\hat{\lambda}_k(r)\}_{k\in\mathbb{N}}$ of distinct eigenvalue such that $\hat{\lambda}_k(r) \to +\infty$. These are called “variational eigenvalues” and we do not know if they exhaust the spectrum of $(-\Delta_r, W_0^{1,r}(\Omega))$. However, if $r = 2$ (linear eigenvalue problem), then the spectrum is the sequence $\{\hat{\lambda}_k(2)\}_{k\in\mathbb{N}}$ of variational eigenvalues.

Let $r = max\{p,q\}$ and $\lambda > 0$. The energy (Euler) functional for problem (P_λ) is defined by

$$\varphi_\lambda(u) = \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{\lambda}{q}||u||_q^q \text{ for all } u \in W_0^{1,r}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W_0^{1,r}(\Omega), \mathbb{R})$.

The Nehari manifold for the functional φ_λ is the set

$$N_\lambda = \{u \in W_0^{1,r}(\Omega) : \langle \varphi_\lambda'(u), u\rangle = 0, u \neq 0\}.$$

In what follows, we denote by $\hat{\sigma}(\alpha,\beta)$ the spectrum of

$$u \to -\alpha\Delta_p u - \beta\Delta_q u \text{ for all } u \in W_0^{1,r}(\Omega).$$

So, $\lambda \in \hat{\sigma}(\alpha,\beta)$ if and only if problem (P_λ) admits a nontrivial solution $\hat{u} \in C_0^1(\overline{\Omega})$. This solution is an eigenvector for the eigenvalue λ.

In what follows for every $\tau \in (1,+\infty)$ by $||\cdot||_{1,\tau}$ we denote the norm of $W_0^{1,\tau}(\Omega)$. On account of the Poincare inequality, we have

$$||u||_{1,\tau} = ||Du||_\tau \text{ for all } u \in W_0^{1,\tau}(\Omega).$$

Also, by $A_\tau : W_0^{1,p}(\Omega) \to W_0^{-1,\tau'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{\tau} + \frac{1}{\tau'} = 1\right)$ we denote the nonlinear operator defined by

$$\langle A_\tau(u), h\rangle = \int_\Omega |Du|^{\tau-2}(Du, Dh)_{\mathbb{R}^N} dt \text{ for all } u, h \in W_0^{1,\tau}(\Omega).$$

This operator is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too).

3. The spectrum of (P_λ)

First we deal with the easy case where $1 < q < p$. As we will see in the sequel, for this case $\varphi_\lambda(\cdot)$ is coercive and so we can use the direct method of the calculus of variations.

Proposition 1. *If $1 < q < p$, then $\hat{\sigma}(\alpha,\beta) = (\beta\hat{\lambda}_1(q), +\infty)$ and the eigenvectors belong in $C_0^1(\overline{\Omega})$.*

Proof. Now $r = max\{p,q\} = p$. Evidently, if $\lambda \leqslant \beta\hat{\lambda}_1(q)$, then $\lambda \notin \hat{\sigma}(\alpha,\beta)$ ot other wise we violate (2).

Let $\lambda > \beta\hat{\lambda}_1(q)$ and $u \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
\varphi_\lambda(u) &\geqslant \frac{\alpha}{p}||Du||_p^p - \frac{\lambda}{\hat{\lambda}_1(q)q}||Du||_q^q \text{ (see (2))}\\
&\geqslant \frac{\alpha}{p}||Du||_p^p - c_1||Du||_q^q \text{ for some } c_1 > 0 \text{ (since } q < p),\\
\Rightarrow \quad \varphi_\lambda(u) &\geqslant c_2||u||^p - c_3||u||^q \text{ for some } c_2, c_3 > 0,\\
\Rightarrow \quad \varphi_\lambda(\cdot) &\quad \text{is coercive (since } q < p).
\end{aligned}$$

Also, by the Sobolev embedding theorem $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by Weierstrass-Tonelli theorem, we can find $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(\hat{u}_\lambda) = inf\{\hat{\varphi}_\lambda(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3}$$

For $t > 0$ we have

$$\begin{aligned}\varphi_\lambda(t\hat{u}_1(q)) &= \frac{t^p\alpha}{p}||D\hat{u}_1(q)||_p^p + \frac{t^q}{q}[\beta\hat{\lambda}_1(q) - \lambda] \text{ (recall that } ||\hat{u}_1(q)||_q = 1) \\ &= c_4 t^p - c_5 t^q \text{ for some } c_4, c_5 > 0 \text{ (recall that } \lambda > \beta\hat{\lambda}_1(q))\end{aligned}$$

Since $q < p$, choosing $t \in (0,1)$ small we have

$$\begin{aligned}& \varphi_\lambda(t\hat{u}_1(q)) < 0, \\ \Rightarrow\ & \varphi_\lambda(\hat{u}_\lambda) < 0 = \varphi_\lambda(0) \text{ (see (3))}, \\ \Rightarrow\ & \hat{u}_\lambda \neq 0.\end{aligned}$$

From (3) we have

$$\begin{aligned}& \varphi'_\lambda(\hat{u}_\lambda) = 0, \\ \Rightarrow\ & \langle \alpha A_p(\hat{u}_\lambda), h\rangle + \langle \beta A_q(\hat{u}_\lambda), h\rangle = \lambda\int_\Omega |\hat{u}_\lambda|^{q-2}\hat{u}_\lambda h dz \text{ for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ & -\alpha\Delta\hat{u}_\lambda(z) - \beta\Delta_q\hat{u}_\lambda(z) = \lambda|\hat{u}_\lambda(z)|^{q-2}\hat{u}_\lambda(z) \text{ for almost all } z \in \Omega, \hat{u}_\lambda|_{\partial\Omega} = 0, \\ \Rightarrow\ & \hat{u}_\lambda \in C_0^1(\overline{\Omega}) \text{ (by the nonlinear regularity theory, see Lieberman [8]).}\end{aligned}$$

□

When $1 < p < q$, the energy functional is no longer coercive. So, the direct method of the calculus of the variations fails and we have to use a different approach. Instead we will minimize φ_λ on the Nehair manifold N_λ.

First we show that $N_\lambda \neq \varnothing$.

Proposition 2. *$\lambda > \beta\hat{\lambda}_1(q)$ if and only if $N_\lambda \neq \varnothing$.*

Proof. As before (see the proof of Proposition 1), using (2) we see that

$$N_\lambda \neq \varnothing \Rightarrow \lambda > \beta\hat{\lambda}_1(q).$$

Now suppose that $\lambda > \beta\hat{\lambda}_1(q)$. Then on account of (2) we can find $u \in W_0^{1,q}(\Omega)$ $u \neq 0$ such that

$$||Du||_q^q < \frac{\lambda}{\beta}||u||_q^q \tag{4}$$

Consider the function $\xi_\lambda : (0, +\infty) \to \mathbb{R}$ defined by

$$\begin{aligned}\xi_\lambda(t) &= \langle \varphi'_\lambda(tu), tu\rangle \\ &= \alpha t^p||Du||_p^p + \beta t^q||Du||_q^q - \lambda t^q||u||_q^q \\ &= t^p\alpha||Du||_p^p + t^q[\beta||Du||_q^q - \alpha||u||_q^q] \\ &= c_6 t^p - c_7 t^q \text{ for some } c_6, c_7 > 0 \text{ (see (4))}.\end{aligned} \tag{5}$$

Since $q > p$, from (5) we see that

$$\xi_\lambda(t) \to -\infty \text{ as } t \to +\infty$$

On the other hand for $t \in (0,1)$ small, we have

$$\xi_\lambda(t) > 0 \text{ see (5)}.$$

Therefore, by Balzano's theorem, we can find $t_0 > 0$ such that

$$\begin{aligned} & \xi_\lambda(t_0) = 0, \\ \Rightarrow\ & \langle \varphi'_\lambda(t_0 u), t_0 u\rangle = 0 \text{ with } t_0 u \neq 0, \\ \Rightarrow\ & t_0 u \in N_\lambda \text{ and so } N_\lambda \neq \varnothing. \end{aligned}$$

□

We define

$$m_\lambda = inf\{\varphi_\lambda(u) : u \in N_\lambda\}. \tag{6}$$

For $u \in N_\lambda$, we have

$$\alpha ||Du||_p^p + \beta ||Du||_q^q = \lambda ||u||_q^q. \tag{7}$$

Therefore

$$\begin{aligned} \varphi_\lambda(u) &= \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{1}{q}[\alpha||Du||_p^p + \beta||Du||_q^q] \text{ (see (7))} \\ &= \alpha\left[\frac{1}{p} - \frac{1}{q}\right]||Du||_p^p, \end{aligned} \tag{8}$$

$\Rightarrow \quad m_\lambda \geqslant 0 \quad$ (see (6))

From (8) we infer that $\varphi_\lambda|_{N_\lambda}$ is coercive on $W_0^{1,p}(\Omega)$.

Proposition 3. *If $\lambda > \beta\hat{\lambda}_1(q)$, then every minimizing sequence of (6) is bounded in $W_0^{1,q}(\Omega)$.*

Proof. We argue by contradiction. So, suppose that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega)$ is a minimizing sequence of (6) such that

$$||u_n||_{1,q} \to +\infty.$$

We have

$$\alpha||Du_n||_p^p + \beta||Du_n||_q^q = \lambda||u_n||_q^q \text{ for all } n \in \mathbb{N}, \tag{9}$$

$$\Rightarrow \quad \beta||Du_n||_q^q = \beta||u_n||_{1,q}^q \leqslant \lambda||u_n||_q^q \text{ for all } n \in \mathbb{N}, \tag{10}$$

$$\Rightarrow \quad ||u_n||_q \to +\infty \text{ as } n \to \infty.$$

We set $y_n = \dfrac{u_n}{||u_n||_q}$ $n \in \mathbb{N}$. Then $||y_n||_q = 1$ $n \in \mathbb{N}$. Also, from (10) we have

$$\begin{aligned} & ||Dy_n||_q^q \leqslant \frac{\lambda}{\beta}||y_n||_q^q = \frac{\lambda}{\beta} \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \quad & \{y_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega) \text{ is bounded.} \end{aligned}$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,q}(\Omega) \text{ and } y_n \to y \text{ in } L^q(\Omega). \tag{11}$$

We multiply (9) with $\dfrac{1}{||u_n||_q^q}$. We obtain

$$\alpha||Dy_n||_p^p = \frac{\lambda||u_n||_q^q - \beta||Du_n||_q^q}{||u_n||_q^p}, \ n \in \mathbb{N}. \tag{12}$$

Recall that $\{u_n\}_{n\geqslant 1} \subseteq N_\lambda$ is a minimizing sequence for (6). So, we have

$$\left(\frac{1}{p} - \frac{1}{q}\right)[\lambda||u_n||_q^q - \beta||Du_n||_q^q] \to m_\lambda \text{ as } n \to \infty \text{ (see (7), (8))} \tag{13}$$

Using (13) in (12), we infer that

$$\begin{aligned} & y_n \to 0 \text{ in } W_0^{1,p}(\Omega), \\ \Rightarrow \quad & y_n \to 0 \text{ in } L^q(\Omega) \text{ (see (11))}, \end{aligned}$$

a contradiction since $||y_n||_q = 1$ for all $n \in \mathbb{N}$.

Therefore we conclude that every minimizing sequence of (6) is bounded in $W_0^{1,q}(\Omega)$. □

We have already seen that $m_\lambda \geqslant 0$. We can say more.

Proposition 4. *If $\lambda > \beta\hat{\lambda}_1(q)$, then $m_\lambda > 0$.*

Proof. Arguing by contradiction, suppose that $m_\lambda = 0$. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq N_\lambda$ such that $\varphi_{\lambda_-}(u_n) \to 0^+$. From (8) we have

$$\begin{aligned} & \alpha \left[\frac{1}{p} - \frac{1}{q}\right] ||Du_n||_p^p \to 0, \\ \Rightarrow \quad & u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \end{aligned} \tag{14}$$

Then from (14) and Proposition 3, we infer that

$$u_n \xrightarrow{w} 0 \text{ in } W_0^{1,q}(\Omega) \text{ and } u_n \to 0 \text{ in } L^q(\Omega) \tag{15}$$

From (14), (15) and (7), it follows that

$$\begin{aligned} & ||Du_n||_q \to 0, \\ \Rightarrow \quad & u_n \to 0 \text{ in } W_0^{1,q}(\Omega). \end{aligned} \tag{16}$$

Let $v_n = \dfrac{u_n}{||u_n||_q}$ $n \in \mathbb{N}$. Then $||v_n||_q = 1$ $n \in \mathbb{N}$. We have

$$\lambda||v_n||_q^q - \beta||Dv||_q^q = \frac{\alpha}{||u_n||_q^{q-p}}||Dv_n||_p^p > 0 \text{ for all } n \in \mathbb{N}, \tag{17}$$

$$\Rightarrow \quad ||Dv_n||_q^q \leqslant \frac{\lambda}{\beta} \text{ for all } n \in \mathbb{N},$$

$$\Rightarrow \quad \{v_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega) \text{ is bounded} \tag{18}$$

Then from (17) and (18) it follows that

$$\begin{aligned} & \frac{\alpha}{||u_n||_q^{q-p}}||Dv_n||_p^p \leqslant c_8 \text{ for some } c_8 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \quad & ||Dv_n||_p \to 0 \text{ (see (16) and recall that } p < q), \\ \Rightarrow \quad & v_n \to 0 \text{ in } W_0^{1,p}(\Omega) \end{aligned} \tag{19}$$

From (18) and (19), we infer that

$$v_n \to 0 \text{ in } L^q(\Omega),$$

a contradiction since $||v_n||_q = 1$ $n \in \mathbb{N}$. From this we conclude that $m_\lambda > 0$. □

Proposition 5. *If $\lambda > \beta\hat{\lambda}_1(q)$, then there exists $\hat{u}_\lambda \in N_\lambda$ such that $m_\lambda = \varphi_\lambda(\hat{u}_\lambda)$.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq N_\lambda$ such that $\varphi_\lambda(u_n) \to m_\lambda$. According to Proposition 3, $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,q}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \hat{u}_\lambda \text{ in } W_0^{1,q}(\Omega) \text{ and } u_n \to \hat{u}_\lambda \text{ in } L^q(\Omega) \tag{20}$$

Since $u_n \in N_\lambda$ $n \in \mathbb{N}$, we have

$$\alpha||Du_n||_p^p + \beta||Du_n||_q^q = \lambda||u_n||_q^q \text{ for all } n \in \mathbb{N}. \tag{21}$$

Passing to the limit as $n \to \infty$ and using (20) and the weak lower semicontinuity of the functional in a Banach space, we obtain

$$\alpha||D\hat{u}_\lambda||_p^p \leqslant \lambda||\hat{u}_\lambda||_q^q - \beta||D\hat{u}_\lambda||_q^q. \tag{22}$$

Note that $\lambda||\hat{u}_\lambda||_q^q - \beta||D\hat{u}_\lambda||_q^q \neq 0$ or otherwise from (18), we have

$$\begin{aligned} & ||Du_n||_p \to 0, \\ \Rightarrow \quad & u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \end{aligned}$$

Recall that

$$\varphi_\lambda(u_n) = \alpha\left[\frac{1}{p} - \frac{1}{q}\right]||Du_n||_p^p \text{ for all } n \in \mathbb{N} \text{ (see (8))}.$$

So, it follows that

$$\begin{aligned} & \varphi_\lambda(u_n) \to 0 \text{ as } n \to \infty, \\ \Rightarrow \quad & m_\lambda = 0, \end{aligned}$$

which contradicts Proposition 4. Therefore

$$\begin{aligned} & \lambda||\hat{u}_\lambda||_q^q - \beta||D\hat{u}_\lambda||_q^q \neq 0, \\ \Rightarrow \quad & \hat{u}_\lambda \neq 0. \end{aligned}$$

Also, exploiting the sequential weak lower semicontinuity of $\varphi_\lambda(\cdot)$, we have

$$\varphi_\lambda(\hat{u}_\lambda) \leqslant \lim_{n\to\infty} \varphi_\lambda(u_n) = m_\lambda \text{ (see (20))}.$$

If we show that $\hat{u}_\lambda \in N_\lambda$, then $\varphi_\lambda(\hat{u}_\lambda) = m_\lambda$ and this will conclude the proof. To this end, let

$$\hat{\xi}_\lambda(t) = \langle\varphi_\lambda'(t\hat{u}_\lambda, t\hat{u}_\lambda)\rangle \text{ for all } t \in [0,1].$$

Evidently $\hat{\xi}_\lambda(\cdot)$ is a continuous function. Arguing indirectly, suppose that $\hat{u}_\lambda \notin N_\lambda$. Then since $u_n \in N_\lambda$ for all $n \in \mathbb{N}$, from (20) we infer that

$$\hat{\xi}_\lambda(1) < 0. \tag{23}$$

On the other hand, note that since $\lambda > \beta\hat{\lambda}_1(q)$, we have

$$\begin{aligned} & \hat{\xi}_\lambda(t) \geqslant c_9 t^p - c_{10} t^q \text{ for some } c_9, c_{10} > 0, \\ \Rightarrow \quad & \hat{\xi}_\lambda(t) > 0 \text{ for all } t \in (0,\epsilon) \text{ with } \epsilon \in (0,1) \text{ small (recall } p < q). \end{aligned} \tag{24}$$

From (23), (24) and Bolzano's theorem, we see that there exists $t^* \in (0,1)$ such that

$$\begin{aligned} & \hat{\xi}_\lambda(t^*\hat{u}_\lambda) = 0, \\ \Rightarrow \quad & t^*\hat{u}_\lambda \in N_\lambda. \end{aligned}$$

Then using (8) we have

$$\begin{aligned} m_\lambda \leqslant \varphi_\lambda(t^*\hat{u}_\lambda) &= \alpha\left[\frac{1}{p}-\frac{1}{q}\right](t^*)^p||D\hat{u}_\lambda||_p^p \\ &< \alpha\left[\frac{1}{p}-\frac{1}{q}\right]||D\hat{u}_\lambda||_p^p \text{ (since } t^*\in(0,1)) \\ &\leqslant \alpha\left[\frac{1}{p}-\frac{1}{q}\right]\liminf_{n\to\infty}||Du_n||_p^p \text{ (see (20))} \\ &= m_\lambda, \end{aligned}$$

a contradiction. Therefore $\hat{u}_\lambda \in N_\lambda$ and this finishes the proof. ☐

So, we can state the following theorem concerning problem (P_λ).

Theorem 6. *If $\lambda > \beta\hat{\lambda}_1(q)$ then λ is an eigenvalue of problem (P_λ) with eigenfunction $\hat{\lambda} \in C_0^1(\overline{\Omega})$.*

Proof. For $1 < q < p$, this follows from Proposition 1.

For $1 < q < p$, let $h \in W_0^{1,q}(\Omega)$. Choose $\epsilon > 0$ such that $\hat{u}_\lambda + sh \not\equiv 0$ for $s \in (-\epsilon, \epsilon)$. We set

$$t(s) = \left[\frac{\lambda||\hat{u}_\lambda + sh||_q^q - \beta||D(\hat{u}_\lambda + sh)||_q^q}{\alpha||D(\hat{u}_\lambda + sh)||_p^p}\right]^{\frac{1}{p-q}}, \ s \in (-\epsilon, \epsilon).$$

Then we have that $s \to t(s)$ is a curve in N_λ and it is differentiable. Let $\hat{\xi}_\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ be defined by

$$\hat{\xi}_\lambda(s) = \varphi_\lambda(t(s)(\hat{u}_\lambda + sh)), \ s \in (-\epsilon, \epsilon).$$

Evidently $s = 0$ is a minimizer of $\hat{\xi}_\lambda(\cdot)$ and so

$$\begin{aligned} 0 &= \hat{\xi}_\lambda(0) \\ &= \langle \varphi_\lambda'(\hat{u}_\lambda), t'(0)\hat{u}_\lambda + h\rangle \text{ (by the chain rule)} \\ &= t'(0)\langle \varphi_\lambda'(\hat{u}_\lambda), \hat{u}_\lambda\rangle + \langle \varphi_\lambda'(\hat{u}_\lambda), h\rangle \\ &= \langle \varphi_\lambda'(\hat{u}_\lambda), h\rangle \text{ (since } \hat{u}_\lambda \in N_\lambda), \\ \Rightarrow \quad & \alpha\langle A_p(\hat{u}_\lambda), h\rangle + \beta\langle A_q(\hat{u}_\lambda), h\rangle = \lambda\int_\Omega |\hat{u}_\lambda|^{q-2}\hat{u}_\lambda hzd, \\ \Rightarrow \quad & -\alpha\Delta_p\hat{u}_\lambda(z) - \beta\Delta_q\hat{u}_\lambda(z) = \lambda|\hat{u}_\lambda(z)|^{q-2}\hat{u}_\lambda(z) \text{ for almost all } z \in \Omega, \ \hat{u}_\lambda|_{\partial\Omega} = 0, \\ \Rightarrow \quad & \hat{u}_\lambda \neq 0 \text{ is an eigenfunction with eigenvalue } \lambda > \beta\hat{\lambda}_1(q). \end{aligned}$$

Then nonlinear regularity theory of Lieberman [8, p. 320], implies that $\hat{u}_\lambda \in C_0^1(\overline{\Omega})$. ☐

Remark 1. *In the terminology of critical point theory, the above proof shows that the Nehari manifold, is a natural constant for the functional φ_λ (see Gasinski & Papageorgiou* [5, p. 812]*).*

Now suppose that $\alpha = (1-\beta), \beta \in (0,1)$. Let $L_\beta = -(1-\beta)\Delta_p - \beta\Delta_q$ and let $\hat{\sigma}(\beta)$ be the spectrum of L_β. From Theorem 6, we know that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty).$$

Evidently $\hat{\sigma}(\cdot)$ is Hausdorff and Vietoris continuous on $(0,1)$ (see Hu & Papageorgiou [6]), but at $\beta = 1$, it exhibits a discontinuity since

$$\hat{\sigma}(1) = \text{the spectrum of } (-\Delta_q, W_0^{1,q}(\Omega))$$

and from Section 2, we know that $\hat{\lambda}_1(q) > 0$ is isolated and so $\hat{\sigma}(1) \neq (\hat{\lambda}_1(q), +\infty)$. This is more emphatically illustrated when $q = 2$. Then

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(2), +\infty) \text{ for all } \beta \in (0,1)$$

but at $\beta = 1$, we have

$$\hat{\sigma}(1) = \{\hat{\lambda}_k(2)\}_{k\geqslant 1} \text{ (discrete spectrum).}$$

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