

## POSITIVE SOLUTIONS FOR SUPERDIFFUSIVE MIXED PROBLEMS

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ABSTRACT. We study a semilinear parametric elliptic equation with superdiffusive reaction and mixed boundary conditions. Using variational methods, together with suitable truncation techniques, we prove a bifurcation-type theorem describing the nonexistence, existence and multiplicity of positive solutions.

### 1. INTRODUCTION

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$ -boundary  $\partial\Omega$  and let  $\Sigma_1, \Sigma_2 \subseteq \partial\Omega$  be two  $(N - 1)$ -dimensional  $C^2$ -submanifolds of  $\partial\Omega$  such that  $\partial\Omega = \Sigma_1 \cup \Sigma_2$ ,  $\Sigma_1 \cap \Sigma_2 = \emptyset$ ,  $|\Sigma_1|_{N-1} \in (0, |\partial\Omega|_{N-1})$ , and  $\overline{\Sigma_1} \cap \overline{\Sigma_2} = \Gamma$ . Here,  $|\cdot|_{N-1}$  denotes the  $(N - 1)$ -dimensional Hausdorff (surface) measure and  $\Gamma \subset \partial\Omega$  is a  $(N - 2)$ -dimensional  $C^2$ -submanifold of  $\partial\Omega$ .

In this paper, we study the following logistic-type elliptic problem:

$$(P_\lambda) \quad \left\{ \begin{array}{l} -\Delta u(z) = \lambda u(z)^{q-1} - f(z, u(z)) \quad \text{in } \Omega, \\ u|_{\Sigma_1} = 0, \quad \frac{\partial u}{\partial n} \Big|_{\Sigma_2} = 0, \quad u > 0, \quad \lambda > 0. \end{array} \right\}$$

When  $f(z, x) = x^{r-1}$  with  $r \in (2, 2^*)$ , we get the classical logistic equation, which is important in biological models (see Gurtin & Mac Camy [8]). Depending on the value of  $q > 1$ , we distinguish three cases: (i)  $1 < q < 2$  (subdiffusive logistic equation); (ii)  $2 = q < r$  (equidiffusive logistic equation); (iii)  $2 < q < r$  (superdiffusive logistic equation). In this paper, we deal with the third situation (superdiffusive case), which exhibits bifurcation-type phenomena for large values of the parameter  $\lambda > 0$ .

Let  $E_{\Sigma_1} = \{u \in H^1(\Omega) : u|_{\Sigma_1} = 0\}$ . This space is defined as the closure of  $C_c^1(\Omega \cup \Sigma_1)$  with respect to the  $H^1(\Omega)$ -norm. Since  $|\Sigma_1|_{N-1} > 0$ , we know that for the space  $E_{\Sigma_1}$ , the Poincaré inequality holds (see Gasinski & Papageorgiou [7, Problem 1.139, p. 58]). So,  $E_{\Sigma_1}$  is a Hilbert space equipped with the norm  $\|u\| = \|Du\|_2$ . Let  $\mathcal{A} \in \mathcal{L}(E_{\Sigma_1}, E_{\Sigma_1}^*)$  be defined by  $\langle \mathcal{A}(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz$  for all  $u, h \in E_{\Sigma_1}$ . We denote by  $N_f$  the Nemitsky map associated with  $f$ , that is,  $N_f(u)(\cdot) = f(\cdot, u(\cdot))$  for all  $u \in E_{\Sigma_1}$ .

The hypotheses on the perturbation term  $f(z, x)$  are the following:

$H(f) : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that for almost all  $z \in \Omega$ ,  $f(z, 0) = 0$ ,  $f(z, x) \geq 0$  for all  $x > 0$ , and

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- (i)  $f(z, x) \leq a(z)(1+x^{r-1})$  for almost all  $z \in \Omega$  and all  $x \geq 0$ , with  $a \in L^\infty(\Omega)$ ,  $2 < q < r < 2^*$ ;
- (ii)  $\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{q-1}} = +\infty$  uniformly for almost all  $z \in \Omega$ , and the mapping  $x \mapsto \frac{f(z, x)}{x}$  is nondecreasing on  $(0, +\infty)$  for almost all  $z \in \Omega$ ;
- (iii)  $0 \leq \liminf_{x \rightarrow 0^+} \frac{f(z, x)}{x} \leq \limsup_{x \rightarrow 0^+} \frac{f(z, x)}{x} \leq \hat{\eta}$  uniformly for almost all  $z \in \Omega$ ;
- (iv) for every  $\rho > 0$ , there exists  $\hat{\xi}_\rho > 0$  such that for almost all  $z \in \Omega$  the function  $x \mapsto \hat{\xi}_\rho x - f(z, x)$  is nondecreasing on  $[0, \rho]$ .

The following functions satisfy hypotheses  $H(f)$ : (i)  $f(x) = x^{r-1}$  for all  $x \geq 0$  with  $2 < q < r < 2^*$ ; (ii)  $f(x) = x^{q-1} \left[ \ln(1+x) + \frac{1}{q} \frac{x}{1+x} \right]$  for all  $x \geq 0$ , with  $2 < q < 2^*$ .

Let  $\mathcal{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution}\}$  and let  $S(\lambda)$  denote the set of positive solutions of problem  $(P_\lambda)$ . Let  $\lambda_* = \inf \mathcal{L}$  (if  $\mathcal{L} = \emptyset$ , then  $\inf \emptyset = +\infty$ ).

By a solution of problem  $(P_\lambda)$ , we understand a function  $u \in E_{\Sigma_1}$  such that  $u \geq 0$ ,  $u \neq 0$  and  $\langle A(u), h \rangle = \int_\Omega [\lambda u^{q-1} - f(z, u)] h dz$  for all  $h \in E_{\Sigma_1}$ .

We refer to Bonanno, D'Agui & Papageorgiou [3], Filippucci, Pucci & Rădulescu [6], and Li, Ruf, Guo & Niu [9] for related results. We also refer to the monograph by Pucci & Serrin [11] for more results concerning the abstract setting of this paper.

## 2. A BIFURCATION-TYPE THEOREM

**Proposition 1.** *If hypotheses  $H(f)$  hold, then  $S(\lambda) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$  with  $\alpha \in (0, 1/2)$ . For all  $u \in S(\lambda)$  we have  $u(z) > 0$  for all  $z \in \Omega$  and  $\lambda_* > 0$ .*

*Proof.* From DiBenedetto [5] and Colorado & Peral [4], we know that if  $u \in S(\lambda)$  then  $u \in C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$  with  $\alpha \in (0, 1/2)$ . Moreover, using Harnack's inequality, we deduce that if  $u \in S(\lambda)$  then  $u(z) > 0$  for all  $z \in \Omega$ . Let  $\hat{\lambda}_1$  be the smallest eigenvalue of  $-\Delta$  with mixed boundary conditions. From Colorado & Peral [4, p. 482], we know that  $\hat{\lambda}_1 = \inf \left\{ \frac{\|Du\|_2^2}{\|u\|_2^2} : u \in E_{\Sigma_1} \setminus \{0\} \right\} > 0$ . By  $H(f)(i), (iii)$ , there exists  $\lambda_0 > 0$  such that

$$(1) \quad \lambda_0 x^{q-1} - f(z, x) \leq \hat{\lambda}_1 x \text{ for almost all } z \in \Omega, \text{ and all } x \geq 0$$

(recall that  $2 < q < r$ ). Let  $\lambda \in (0, \lambda_0)$  and suppose that  $\lambda \in \mathcal{L}$ . Then there exists  $u_\lambda \in S(\lambda)$  and by using Green's identity, we get

$$(2) \quad A(u_\lambda) = \lambda u_\lambda^{q-1} - N_f(u_\lambda) \text{ in } E_{\Sigma_1}^*.$$

We act on (2) with  $u_\lambda \in E_{\Sigma_1}$  and obtain  $\|Du_\lambda\|_2^2 = \lambda \|u_\lambda\|_q^q - \int_\Omega f(z, u_\lambda) u_\lambda dz < \hat{\lambda}_1 \|u_\lambda\|_2^2$  (see (1) and recall that  $\lambda < \lambda_0, u_\lambda(z) > 0$  for all  $z \in \Omega$ ), which contradicts the definition of  $\hat{\lambda}_1$ . Therefore  $\lambda \notin \mathcal{L}$  and we have  $0 < \lambda_0 \leq \lambda_* = \inf \mathcal{L}$ .  $\square$

**Proposition 2.** *If hypotheses  $H(f)$  hold, then  $\mathcal{L} \neq \emptyset$  and " $\lambda \in \mathcal{L}, \eta > \lambda \Rightarrow \eta \in \mathcal{L}$ ".*

*Proof.* Fix  $\lambda > 0$  and let  $\varphi_\lambda : E_{\Sigma_1} \rightarrow \mathbb{R}$ ,  $\varphi_\lambda(u) = \frac{1}{2} \|Du\|_2^2 - \frac{\lambda}{q} \|u^+\|_q^q + \int_\Omega F(z, u) dz$ , where  $F(z, x) = \int_0^x f(z, s) ds$ . Then  $\varphi_\lambda \in C^1(E_{\Sigma_1})$  and  $\varphi_\lambda$  is sequentially weakly lower semicontinuous. Hypotheses  $H(f)(i), (ii)$  imply that given  $\xi > 0$ , we can find  $c_1 = c_1(\xi) > 0$  such that  $F(z, x) \geq \frac{\xi}{q} x^q - c_1$  for almost all  $z \in \Omega$  and for all  $x \geq 0$ . Thus, for all  $u \in E_{\Sigma_1}$  we have  $\varphi_\lambda(u) \geq \frac{1}{2} \|Du\|_2^2 + \frac{\xi - \lambda}{2} \|u^+\|_q^q - c_1 |\Omega|_N$ . Choosing

$\xi > \lambda$ , we deduce that  $\varphi_\lambda$  is coercive. So, by the Weierstrass-Tonelli theorem, there exists  $u_\lambda \in E_{\Sigma_1}$  such that

$$(3) \quad \varphi_\lambda(u_\lambda) = \inf\{\varphi_\lambda(u) : u \in E_{\Sigma_1}\} = m_\lambda.$$

Fix  $\bar{u} \in E_{\Sigma_1} \cap C(\bar{\Omega})$  with  $u(z) > 0$  for all  $z \in \Omega$ . For large enough  $\lambda > 0$  we have  $\varphi_\lambda(\bar{u}) < 0$ , hence  $\varphi_\lambda(u_\lambda) = m_\lambda < 0 = \varphi_\lambda(0)$  (see (3)). Thus,  $u_\lambda \neq 0$ . By (3),  $\varphi'_\lambda(u_\lambda) = 0$ , hence

$$(4) \quad A(u_\lambda) = \lambda(u_\lambda^+)^{q-1} - N_f(u_\lambda) \text{ in } E_{\Sigma_1}^*.$$

We act on (4) with  $-u_\lambda^- \in E_{\Sigma_1}$  and obtain  $\|Du_\lambda^-\|_2^2 = 0$ , hence  $u_\lambda \geq 0$ . So, relation (4) becomes  $A(u_\lambda) = \lambda u_\lambda^{q-1} - N_f(u_\lambda)$ . By Green's identity,  $u_\lambda \in S(\lambda)$ , hence  $\lambda \in \mathcal{L} \neq \emptyset$ .

Next, let  $\lambda \in \mathcal{L}$  and  $\eta > \lambda$ . Choose  $\vartheta \in (0, 1)$  such that  $\lambda = \vartheta^{q-2}\eta$  (recall that  $2 < q$ ). Also, let  $u_\lambda \in S(\lambda) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\bar{\Omega})$  with  $\alpha \in (0, 1/2)$ . Let  $\underline{u} = \vartheta u_\lambda$ . Then

$$(5) \quad A(\underline{u}) = \vartheta A(u_\lambda) = \vartheta \left[ \lambda u_\lambda^{q-1} - N_f(u_\lambda) \right] \text{ in } E_{\Sigma_1}^*.$$

From hypothesis  $H(f)(ii)$  and since  $u_\lambda(z), \underline{u}(z) > 0$  for all  $z \in \Omega$ , we have for a.a.  $z \in \Omega$

$$(6) \quad \frac{f(z, \underline{u}(z))}{\underline{u}(z)} \leq \frac{f(z, u_\lambda(z))}{u_\lambda(z)} \Rightarrow f(z, \underline{u}(z)) \leq \vartheta f(z, u_\lambda(z)) \text{ (recall that } \underline{u} = \vartheta u_\lambda).$$

Using (5) in (6) and since  $\vartheta \in (0, 1)$ , we obtain

$$(7) \quad A(\underline{u}) \leq \vartheta^{q-1} \eta u_\lambda^{q-1} - N_f(\underline{u}) \leq \eta \underline{u}^{q-1} - N_f(\underline{u}) \text{ in } E_{\Sigma_1}^*.$$

We introduce the following Carathéodory truncation of the reaction term in problem  $(P_\eta)$

$$(8) \quad g_\eta(z, x) = \begin{cases} \eta \underline{u}(z)^{q-1} - f(z, \underline{u}(z)) & \text{if } x \leq \underline{u}(z) \\ \eta x^{q-1} - f(z, x) & \text{if } \underline{u}(z) < x. \end{cases}$$

Let  $G_\eta(z, x) = \int_0^x g_\eta(z, s) ds$  and define  $\hat{\varphi}_\eta : E_{\Sigma_1} \rightarrow \mathbb{R}$  by  $\hat{\varphi}_\eta(u) = \frac{1}{2} \|Du\|_2^2 - \int_\Omega G_\eta(z, u) dz$ .

Hypotheses  $H(f)(i), (ii)$  imply that given  $\xi > 0$ , we can find  $c_2 = c_2(\xi) > 0$  such that

$$(9) \quad \eta x^{q-1} - f(z, x) \leq (\eta - \xi)x^{q-1} + c_2 \text{ for almost all } z \in \Omega \text{ and all } x \geq 0.$$

Then for all  $u \in E_{\Sigma_1}$ , we have

$$(10) \quad \hat{\varphi}_\eta(u) \geq \frac{1}{2} \|Du\|_2^2 + \frac{\xi - \eta}{q} \|u^+\|_q^q - c_3 \text{ for some } c_3 > 0 \text{ (see (8), (9)).}$$

Choosing  $\xi > \eta$ , we see from (10) that  $\hat{\varphi}_\eta$  is coercive. This function is also sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, there exists  $u_\eta \in E_{\Sigma_1}$  such that  $\hat{\varphi}_\eta(u_\eta) = \inf\{\hat{\varphi}_\eta(u) : u \in E_{\Sigma_1}\}$ , hence  $\hat{\varphi}'_\eta(u_\eta) = 0$ . We deduce that

$$(11) \quad A(u_\eta) = N_{g_\eta}(u_\eta) \text{ in } E_{\Sigma_1}^*.$$

We act on (11) with  $(\underline{u} - u_\eta)^+ \in E_{\Sigma_1}$ . By (8) and (7) we have

$$\langle A(u_\eta), (\underline{u} - u_\eta)^+ \rangle = \int_\Omega [\eta \underline{u}^{q-1} - f(z, \underline{u})](\underline{u} - u_\eta)^+ dz \geq \langle A(\underline{u}), (\underline{u} - u_\eta)^+ \rangle$$

$$(12) \quad \Rightarrow \langle A(\underline{u} - u_\eta), (\underline{u} - u_\eta)^+ \rangle \leq 0 \Rightarrow \|D(\underline{u} - u_\eta)^+\|_2^2 \leq 0 \Rightarrow \underline{u} \leq u_\eta.$$

Using (8) and (12) we see that relation (11) becomes  $A(u_\lambda) = \eta u_\eta^{q-1} - N_f(u_\eta)$  in  $E_{\Sigma_1}^*$ . Thus, by Proposition 1, we have  $u_\eta \in S(\eta) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ . Therefore  $\eta \in \mathcal{L}$ . We also observe that Proposition 2 implies  $(\lambda_*, +\infty) \subseteq \mathcal{L}$ .  $\square$

**Proposition 3.** *If hypotheses  $H(f)$  hold and  $\lambda > \lambda_*$ , then problem  $(P_\lambda)$  has at least two positive solutions  $u_0, \hat{u} \in E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$  for  $\alpha \in (0, 1/2)$  with  $0 < u_0(z), \hat{u}(z)$  for all  $z \in \Omega$ .*

*Proof.* Let  $\mu \in (\lambda_*, \lambda)$ . By Proposition 2 we know that  $\mu \in \mathcal{L}$ . Hence we can find  $u_\mu \in S(\mu) \subseteq E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$  with  $\alpha \in (0, 1/2)$ ,  $u_\mu(z) > 0$  for all  $z \in \Omega$ . We have  $A(u_\mu) = \mu u_\mu^{q-1} - N_f(u_\mu)$  in  $E_{\Sigma_1}^*$ . Next, we define the following Carathéodory function

$$(13) \quad \hat{h}_\lambda(z, x) = \begin{cases} \lambda u_\mu(z)^{q-1} - f(z, u_\mu(z)) & \text{if } x \leq u_\mu(z) \\ \lambda x^{q-1} - f(z, x) & \text{if } u_\mu(z) < x. \end{cases}$$

Let  $\hat{H}_\lambda(z, x) = \int_0^x \hat{h}_\lambda(z, s) ds$  and let  $\hat{\psi}_\lambda : E_{\Sigma_1} \rightarrow \mathbb{R}$ ,  $\hat{\psi}_\lambda(u) = \frac{1}{2} \|Du\|_2^2 - \int_\Omega \hat{H}_\lambda(z, u) dz$ . Then  $\hat{\psi}_\lambda$  is coercive and sequentially weakly lower semicontinuous. Thus, we can find  $u_0 \in E_{\Sigma_1}$  such that  $\hat{\psi}_\lambda(u_0) = \inf\{\hat{\psi}_\lambda(u) : u \in E_{\Sigma_1}\}$ , hence  $\hat{\psi}'_\lambda(u_0) = 0$ . Thus,  $A(u_0) = N_{\hat{h}_\lambda}(u_0)$ . Using (13) and reasoning as in the proof of Proposition 2 we deduce that  $u_\mu \leq u_0$ . By Colorado & Peral [4, Theorem 6.6], we have  $u_0 \in E_{\Sigma_1} \cap C^{0,\alpha}(\overline{\Omega})$  with  $\alpha \in (0, 1/2)$  and  $u_0 > 0$  in  $z\Omega$  (by Harnack's inequality).

Let  $\rho_0 = \|u_0\|_\infty$  and let  $\hat{\xi}_{\rho_0} > 0$  be as postulated in hypothesis  $H(f)(iv)$ . We have

$$(14) \quad \left\{ \begin{array}{l} -\Delta u_0(z) + \hat{\xi}_{\rho_0} u_0(z) = \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \hat{\xi}_{\rho_0} u_0(z) \text{ in } \Omega, \\ u_0|_{\Sigma_1} = 0, \frac{\partial u_0}{\partial n} \Big|_{\Sigma_2} = 0 \end{array} \right\}$$

and

$$(15) \quad \left\{ \begin{array}{l} -\Delta u_\mu(z) + \hat{\xi}_{\rho_0} u_\mu(z) = \mu u_\mu(z)^{q-1} - f(z, u_\mu(z)) + \hat{\xi}_{\rho_0} u_\mu(z) \text{ in } \Omega, \\ \hat{u}_\mu|_{\Sigma_1} = 0, \frac{\partial u_\mu}{\partial n} \Big|_{\Sigma_2} = 0. \end{array} \right\}$$

Let  $\hat{y} = u_0 - u_\mu \geq 0$ . Since  $\lambda > \mu$ ,  $u_0 \geq u_\eta$ , from (14), (15), and  $H(f)(iv)$  we have

$$\begin{aligned} -\Delta \hat{y}(z) + \hat{\xi}_{\rho_0} \hat{y}(z) &= \lambda u_0(z)^{q-1} - \mu u_\mu(z)^{q-1} + [\hat{\xi}_{\rho_0} u_0(z) - f(z, u_0(z))] - \\ &\quad - [\hat{\xi}_{\rho_0} u_\mu(z) - f(z, u_\mu(z))] \geq 0 \text{ in } \Omega. \end{aligned}$$

Let  $v_1 \in E_{\Sigma_1}$  be the unique function satisfying  $-\Delta v(z) + \hat{\xi}_{\rho_0} v(z) = 1$   $\Omega$ ,  $v|_{\Sigma_1} = 0$ , and  $\frac{\partial v}{\partial n} \Big|_{\Sigma_2} = 0$ . Then  $v_1 \in C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$  with  $\alpha \in (0, 1/2)$  (see [4, 5]) and  $v_1 > 0$  in  $\Omega$ . By Lemma 2.1 of Barletta, Livrea & Papageorgiou [2] (see also Lemma 5.3 of Colorado & Peral [4]), we can find  $\vartheta > 0$  such that

$$(16) \quad \vartheta v_1(z) \leq u_\mu(z) \text{ and } \vartheta v_1(z) \leq \hat{y}(z) \Rightarrow \vartheta v_1(z) \leq u_\mu(z) \leq u_0(z) - \vartheta v_1(z) \text{ for all } z \in \overline{\Omega}.$$

Let  $\hat{C}_1 = \left\{ y \in E_{\Sigma_1} \cap C(\overline{\Omega}) : \left\| \frac{y}{v_1} \right\|_\infty < \infty \right\}$  and  $[u_\mu] = \{u \in E_{\Sigma_1} : u_\mu \leq u(z), \text{ a.a. } z \in \Omega\}$ . We claim that if  $\bar{B}_1(0) := \{y \in \hat{C}_1 : \left\| \frac{y}{v_1} \right\|_\infty \leq 1\}$ , then  $u_0 - \vartheta \bar{B}_1(0) \subseteq [u_\mu] \cap \hat{C}_1$ . To see this, let  $y \in \bar{B}_1(0)$ . Then

$$(17) \quad -v_1(z) \leq y(z) \leq v_1(z) \text{ for all } z \in \overline{\Omega}.$$

Fix  $z \in \bar{\Omega}$ . If  $y(z) > 0$ , then  $0 \leq u_\mu(z) \leq u_\mu(z) + \vartheta y(z) \leq u_\mu(z) + \vartheta v_1(z) \leq u_0(z)$  (see (16), (17)), hence  $u_\mu(z) \leq u_0(z) - \vartheta y(z)$ . If  $y(z) < 0$ , then  $0 \leq u_\mu(z) - \vartheta v_1(z) \leq u_\mu(z) + \vartheta y(z) \leq u_\mu(z) + \vartheta v_1(z) \leq u_0(z)$  (see (16), (17)), hence  $u_\mu(z) \leq u_0(z) - \vartheta y(z)$ . We conclude that  $u_\mu \in u_0 - \vartheta \bar{B}_1(0)$ , which argues our above claim. It follows that

$$(18) \quad u_0 \in \text{int}_{\hat{C}_1} [u_\mu] \cap C(\bar{\Omega}).$$

By (13) it is clear that

$$(19) \quad \hat{\psi}_\lambda(u) = \varphi_\lambda(u) + c_4 \text{ for some } c_4 \in \mathbb{R} \text{ and for all } u \in [u_\mu].$$

It follows from (18) and (19) that  $u_0$  is a local  $\hat{C}_1$ -minimizer of  $\varphi_\lambda$ .

**Claim.**  $u_0$  is a local  $E_{\Sigma_1}$ -minimizer of  $\varphi_\lambda$ . Suppose that this assertion is not true. Then for every  $\rho > 0$ , we have  $\inf\{\varphi_\lambda(u_0 + y) : y \in E_{\Sigma_1}, \|y\| \leq \rho\} < \varphi_\lambda(u_0)$ . By the Weierstrass-Tonelli theorem, there exists  $y_\rho \in E_{\Sigma_1} \setminus \{0\}$ ,  $\|y_\rho\| \leq \rho$  such that  $\varphi_\lambda(u_0 + y_\rho) = \inf\{\varphi_\lambda(u_0 + y) : y \in E_{\Sigma_1}, \|y\| \leq \rho\} < \varphi_\lambda(u_0)$ . By the Lagrange multiplier rule, there exists  $\vartheta \leq 0$  such that  $(1 - \vartheta) \langle A(u_\rho), h \rangle = \lambda \int_\Omega (u_\rho^+)^{q-1} h dz - \int_\Omega f(z, u_\rho) h dz$  for all  $h \in E_{\Sigma_1}$ , with  $u_\rho = u_0 + y_\rho \in E_{\Sigma_1}$ . It follows that  $\Delta u_\rho(z) = \frac{1}{1-\vartheta} [\lambda u_\rho^+(z)^{q-1} - f(z, u_\rho(z))]$  in  $\Omega$ , hence

$$(20) \quad -\Delta u_\rho(z) + \hat{\xi}_{\rho_0} u_\rho(z) = \frac{1}{1-\vartheta} [\lambda u_\rho^+(z)^{q-1} + f(z, u_\rho(z))] + \hat{\xi}_{\rho_0} u_\rho(z) \text{ in } \Omega,$$

with  $\hat{\xi}_{\rho_0} > 0$  as before resulting from hypothesis  $H(f)(iv)$  (recall that  $\rho_0 = \|u_0\|_\infty$ ). Also,

$$(21) \quad -\Delta u_0(z) + \hat{\xi}_{\rho_0} u_0(z) = \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \hat{\xi}_{\rho_0} u_0(z) \text{ in } \Omega.$$

From (20) and (21) we obtain

$$(22) \quad -\Delta y_\rho(z) + \hat{\xi}_{\rho_0} y_\rho(z) = g_\lambda^\rho(z) \text{ in } \Omega$$

with  $g_\lambda^\rho(z) = \frac{1}{1-\vartheta} [\lambda u_\rho^+(z)^{q-1} - f(z, u_\rho(z))] - \lambda u_0(z)^{q-1} + f(z, u_0(z)) + \hat{\xi}_{\rho_0} y_\rho(z)$ . By (22) and Colorado & Peral [4], there exist  $c_5 > 0$  and  $\alpha \in (0, 1/2)$  such that

$$(23) \quad y_\rho \in C^{0,\alpha}(\bar{\Omega}) \text{ and } \|y_\rho\|_{C^{0,\alpha}(\bar{\Omega})} \leq c_5 \text{ for all } \rho \in (0, 1].$$

Exploiting the compact embedding of  $C^{0,\alpha}(\bar{\Omega})$  into  $C(\bar{\Omega})$ , we have  $y_\rho \rightarrow 0$  in  $C(\bar{\Omega})$  as  $\rho \rightarrow 0^+$ . Thus, by the definition of  $g_\lambda^\rho$ , there exists  $\tau_\rho^* > 0$  such that

$$(24) \quad \|g_\lambda^\rho\|_\infty \leq \tau_\rho^* \text{ for all } \rho \in (0, 1] \text{ and } \tau_\rho^* \rightarrow 0^+ \text{ as } \rho \rightarrow 0^+.$$

Let  $\hat{y}_\rho = \frac{1}{\tau_\rho^*} y_\rho$ . Then by (24)  $-\Delta(\hat{y}_\rho - v_1)(z) + \hat{\xi}_{\rho_0}(\hat{y}_\rho - v_1)(z) = \frac{1}{\tau_\rho^*} g_\lambda^\rho(z) - 1 \leq 0$ . We deduce that  $\|D(\hat{y}_\rho - v_1)^+\|_2^2 + \hat{\xi}_{\rho_0} \|(\hat{y}_\rho - v_1)^+\|_2^2 \leq 0$ , hence  $y_\rho \leq \tau_\rho^* v_1$ .

Also, we have  $-\Delta(-\hat{y}_\rho - v_1)(z) + \hat{\xi}_{\rho_0}(-\hat{y}_\rho - v_1)(z) = -\frac{1}{\tau_\rho^*} g_\lambda^\rho(z) - 1 \leq 0$  in  $\Omega$  and so as above we obtain that  $-\tau_\rho^* v_1 \leq y_\rho$ . Therefore we have proved that  $-\tau_\rho^* v_1 \leq y_\rho \leq \tau_\rho^* v_1$ . These relations show that  $y_\rho \in \hat{C}_1$  and  $\left\| \frac{y_\rho}{v_1} \right\|_\infty \leq \tau_\rho^*$  for all  $\rho \in (0, 1]$ , hence  $y_\rho \rightarrow 0$  in  $\hat{C}_1$  as  $\rho \rightarrow 0^+$ . Therefore for small  $\rho \in (0, 1]$  we have  $\varphi_\lambda(u_0 + y_\rho) < \varphi_\lambda(u_0)$ , which contradicts the fact that  $u_0$  is a local  $\hat{C}_1$ -minimizer of  $\varphi_\lambda$ . This proves the claim.

Since  $f \geq 0$ , for all  $u \in E_{\Sigma_1}$  we have  $\varphi_\lambda(u) \geq \frac{1}{2} \|Du\|_2^2 - \frac{\lambda}{q} \|u^+\|_q^q \geq \frac{1}{2} \|Du\|_2^2 - c_6 \|Du\|_2^q$  for some  $c_6 > 0$ . Since  $q > 2$ , we deduce that  $u = 0$  is a local minimizer of  $\varphi_\lambda$ . We assume that the set of critical points of  $\varphi_\lambda$  is finite (otherwise we already have an infinity of positive solutions for  $(P_\lambda)$  for  $\lambda > \lambda_*$  and so we are

done) and that  $\varphi_\lambda(0) \leq \varphi_\lambda(u_0)$  (the reasoning is similar if the opposite inequality holds). The claim implies that we can find small enough  $\rho \in (0, \|u_0\|)$  such that  $0 = \varphi_\lambda(0) \leq \varphi_\lambda(u) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda^\rho$ . Thus, we can apply the mountain pass theorem. So, there exists  $\hat{u} \in E_{\Sigma_1}$  such that  $\varphi'_\lambda(\hat{u}) = 0$  and  $m_\lambda^\rho \leq \varphi_\lambda(\hat{u})$ , hence  $\hat{u} \notin \{0, u_0\}$ ,  $\hat{u} \in S_\lambda \subseteq E_{\Sigma_1} \cap C^{0,\alpha}(\bar{\Omega})$ , and  $\hat{u} > 0$  in  $\Omega$ .  $\square$

**Proposition 4.** *If hypotheses  $H(f)$  hold, then  $\lambda_* \in \mathcal{L}$ , that is,  $\mathcal{L} = [\lambda^*, +\infty)$ .*

*Proof.* Let  $\{\lambda_n\}_{n \geq 1} \subseteq (\lambda_*, +\infty)$  be such that  $\lambda_n \downarrow \lambda_*$ . We find  $u_n \in S(\lambda_n)$  such that

$$(25) \quad A(u_n) = \lambda u_n^{q-1} - N_f(u_n) \text{ in } E_{\Sigma_1}^* \text{ for all } n \in \mathbb{N}.$$

Hypotheses  $H(f)(i), (ii)$  imply that given any  $\xi > 0$ , we find  $c_7 = c_7(\xi) > 0$  such that

$$(26) \quad f(z, x) \geq \xi x^{q-1} - c_7 \text{ for almost all } z \in \Omega \text{ and all } x \geq 0.$$

We act on (25) with  $u_n \in E_{\Sigma_1}$  and then use (26). We obtain  $\|Du_n\|_2^2 \leq (\lambda_n - \xi)\|u_n\|_q^q + c_7|\Omega|_N$ . Choosing  $\xi > \lambda_1 \geq \lambda_n$  for all  $n \in \mathbb{N}$ , we have  $\|Du_n\|_2^2 \leq c_7|\Omega|_N$  for all  $n \in \mathbb{N}$ , hence  $\{u_n\}_{n \geq 1} \subseteq E_{\Sigma_1}$  is bounded. By passing to a subsequence if necessary, we may assume that

$$(27) \quad u_n \xrightarrow{w} u_* \text{ in } E_{\Sigma_1} \text{ and } u_n \rightarrow u \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.$$

In (25) we pass to the limit as  $n \rightarrow \infty$  and use (27). Then  $A(u_*) = \lambda_* u_*^{q-1} - N_f(u_*)$ . Thus,  $u_* \in E_{\Sigma_1}$  and  $u_* \geq 0$  is a solution of  $(P_{\lambda_*})$ . We also notice that  $\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u_* \rangle = 0$ , hence  $\|Du_n\|_2 \rightarrow \|Du_*\|_2$ . Using the Kadec-Klee property we deduce that  $u_n \rightarrow u_*$  in  $E_{\Sigma_1}$ .

**Claim.**  $u_* \neq 0$ . Arguing by contradiction, suppose that  $u_* = 0$ . Then  $\|u_n\| \rightarrow 0$ . Let  $y_n = \frac{u_n}{\|u_n\|}$ ,  $n \in \mathbb{N}$ . Then  $\|y_n\| = 1$ ,  $y_n \geq 0$  for all  $n \in \mathbb{N}$ . From (25) we have

$$(28) \quad A(y_n) = \lambda_n u_n^{q-2} y_n - \frac{N_f(u_n)}{\|u_n\|} \text{ for all } n \in \mathbb{N}.$$

From hypotheses  $H(f)(i), (iii)$ , we see that we can find  $\eta > \hat{\eta}$  and  $c_8 > 0$  such that

$$(29) \quad f(z, x) \leq \eta x + c_8 x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0 \Rightarrow \{N_f(u_n)\}_{n \geq 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

By [4], there exist  $\alpha \in (0, 1/2)$  and  $c_9 > 0$  such that  $u_n \in C^{0,\alpha}(\bar{\Omega})$ ,  $\|u_n\|_{C^{0,\alpha}(\bar{\Omega})} \leq c_9$  for all  $n \in \mathbb{N}$ . Since  $C^{0,\alpha}(\bar{\Omega})$  is compactly embedded compactly in  $C(\bar{\Omega})$ , we deduce that

$$(30) \quad u_n \rightarrow 0 \text{ in } C(\bar{\Omega}).$$

Recall that  $\|y_n\| = 1$ ,  $y_n \geq 0$  for all  $n \in \mathbb{N}$ . So, we may assume that

$$(31) \quad y_n \xrightarrow{w} y \text{ in } E_{\Sigma_1} \text{ and } y_n \rightarrow y \text{ in } L^2(\Omega), y \geq 0.$$

It follows from (29), (30) and (31) that  $\left\{ \frac{N_f(u_n)}{\|u_n\|} \right\}_{n \geq 1} \subseteq L^2(\Omega)$  is bounded. Thus, by hypothesis  $H(f)(iii)$ , we have at least for a subsequence,

$$(32) \quad \frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} \eta_0 y \text{ in } L^2(\Omega) \text{ with } 0 \leq \eta_0(z) \leq \hat{\eta} \text{ for almost all } z \in \Omega.$$

We act on (28) with  $y_n - y \in E_{\Sigma_1}$  and pass to the limit as  $n \rightarrow \infty$ . Using (30), (31) and (32) we obtain  $\lim_{n \rightarrow \infty} \langle A(y_n), y_n - y \rangle = 0$ . By the Kadec-Klee property we

have  $y_n \rightarrow y$ , hence  $\|y\| = 1$ ,  $y \geq 0$ . In (28) we pass to the limit as  $n \rightarrow \infty$  and use (30), (32). Then  $A(y) = -\eta_0 y$ . Thus, by (32) we have  $\|Dy\|_2^2 = -\int_{\Omega} \eta_0 y^2 dz \leq 0$ , hence  $y = 0$ , a contradiction. This shows that the claim is true. Hence  $u_* \in S(\lambda_*) \subseteq E_{\Sigma_1} \cap C(\bar{\Omega})$  and so  $\lambda_* \in \mathcal{L}$ .  $\square$

Summarizing, we can state the following bifurcation-type theorem.

**Theorem 5.** *If hypotheses  $H(f)$  hold, then there exists  $\lambda_* > 0$  such that*

- (a) *for all  $\lambda > \lambda_*$ , problem  $(P_\lambda)$  has at least two positive solutions  $u_0, \hat{u} \in E_{\Sigma_1} \cap C(\bar{\Omega})$ ;*
- (b) *for  $\lambda = \lambda_*$ , problem  $(P_\lambda)$  has at least one positive solution  $u_* \in E_{\Sigma_1} \cap C(\bar{\Omega})$ ;*
- (c) *for  $\lambda \in (0, \lambda_*)$ , problem  $(P_\lambda)$  has no positive solutions.*

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## REFERENCES

- [1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, *Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints*, Memoirs Amer. Math. Soc., Vol. 196, No. 915 (2008), pp. 70.
- [2] G. Barletta, R. Livrea, N.S. Papageorgiou, Bifurcation phenomena for the positive solutions of semilinear elliptic problems with mixed boundary conditions, *J. Nonlin. Convex Anal.* **17** (2016), 1497-1516.
- [3] G. Bonanno, G. D'Agui, N.S. Papageorgiou, Infinitely many solutions for mixed elliptic problems involving the  $p$ -Laplacian, *Adv. Nonlinear Stud.* **15** (2015), 939-950.
- [4] E. Colorado, I. Peral, Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions, *J. Functional Anal.* **199** (2003), 468-507.
- [5] E. DiBenedetto,  $C^{1,\alpha}$  local regularity of weak solutions of degenerate elliptic equations, *Nonlin. Anal.* **7** (1983), 827-850.
- [6] R. Filippucci, P. Pucci, V. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, *Comm. Partial Differential Equations* **33** (2008), 706-717.
- [7] L. Gasinski, N.S. Papageorgiou, *Exercises in Analysis. Part 2: Nonlinear Analysis*, Springer, 2016.
- [8] M.E. Gurtin, R.C. Mac Camy, On the diffusion of biological populations, *Math. Biosci.* **33** (1977), 35-49.
- [9] Y. Li, B. Ruf, Q. Guo, P. Niu, Positive solutions for singular elliptic equations with mixed Dirichlet-Neumann boundary conditions, *Math. Nachr.* **287** (2014), 374-397.
- [10] N.S. Papageorgiou, P. Winkert, On parametric nonlinear Dirichlet problems with subdiffusive and equidiffusive reaction, *Adv. Nonlin. Studies* **14** (2014), 569-591.
- [11] P. Pucci, J. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007.

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