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NONLINEAR SECOND ORDER EVOLUTION INCLUSIONS WITH NONCOERCIVE VISCOSITY TERM

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Abstract. In this paper we deal with a second order nonlinear evolution inclusion, with a nonmonotone, noncoercive viscosity term. Using a parabolic regularization (approximation) of the problem and *a priori* bounds that permit passing to the limit, we prove that the problem has a solution.

1. Introduction

Let $T = [0, b]$ and let (X, H, X^*) be an evolution triple of spaces, with the embedding of X into H being compact (see Section 2 for definitions).

In this paper, we study the following nonlinear evolution inclusion:

$$(1) \qquad \left\{ \begin{array}{l} u''(t) + A(t, u'(t)) + Bu(t) \in F(t, u(t), u'(t)) \text{ for almost all } t \in T, \\ u(0) = u_0,\ u'(0) = u_1. \end{array} \right\}$$

In the past, such multi-valued problems were studied by Gasinski [3], Gasinski and Smolka [6, 7], Migórski *et al.* [11, 12, 13, 14], Ochal [15], Papageorgiou, Rădulescu and Repovš [16, 17], Papageorgiou and Yannakakis [18, 19]. The works of Gasinski [3], Gasinski and Smolka [6, 7] and Ochal [15], all deal with hemivariational inequalities, that is, $F(t, x, y) = \partial J(x)$ with $J(\cdot)$ being a locally Lipschitz functional and $\partial J(\cdot)$ denoting the Clarke subdifferential of $J(\cdot)$. In Papageorgiou and Yannakakis [18, 19], the multivalued term $F(t, x, y)$ is general (not necessarily of the subdifferential type) and depends also on the time derivative of the unknown function $u(\cdot)$. With the exception of Gasinski and Smolka [7], in all the other works the viscosity term $A(t, \cdot)$ is assumed to be coercive or zero. In the work of Gasinski and Smolka [7], the viscosity term is autonomous (that is, time independent) and $A : X \to X^*$ is linear and bounded.

In this work, the viscosity term $A : T \times X \to X^*$ is time dependent, noncoercive, nonlinear and nonmonotone in $x \in X$. In this way, we extend and improve the result of Gasinski and Smolka [7]. Our approach uses a kind of parabolic regularization of the inclusion, analogous to the one used by Lions [10, p. 346] in the context of semilinear hyperbolic equations.

2. Mathematical Background and Hypotheses

Let V, Y be Banach spaces and assume that V is embedded continuously and densely into Y (denoted by $V \hookrightarrow Y$). Then we have the following properties:
(i) Y^* is embedded continuously into V^*;
(ii) if V is reflexive, then $Y^* \hookrightarrow V^*$.

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The following notion is a useful tool in the theory of evolution equations.

Definition 1. *By an "evolution triple" (or "Gelfand triple") we understand a triple of spaces* (X, H, X^*) *such that*

(a) X *is a separable reflexive Banach space and* X^* *is its topological dual;*
(b) H *is a separable Hilbert space identified with its dual* H^**, that is,* $H = H^*$ *(pivot space);*
(c) $X \hookrightarrow H$.

Then from the initial remarks we have

$$X \hookrightarrow H = H^* \hookrightarrow X^*.$$

In what follows, we denote by $||\cdot||$ the norm of X, by $|\cdot|$ the norm of H and by $||\cdot||_*$ the norm of X^*. Evidently we can find $\hat{c}_1, \hat{c}_2 > 0$ such that

$$|\cdot| \leqslant \hat{c}_1||\cdot|| \text{ and } ||\cdot||_* \leqslant \hat{c}_2|\cdot|.$$

By $(\cdot,\cdot)$ we denote the inner product of H and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X). We have

$$\langle\cdot,\cdot\rangle|_{H\times X} = (\cdot,\cdot). \tag{2}$$

Let $1 < p < \infty$. The following space is important in the study of problem (1):

$$W_p(0,b) = \left\{u \in L^p(T,X) : u' \in L^{p'}(T,X^*)\right\} \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

Here u' is understood in the distributional sense (weak derivative). We know that $L^p(T,X)^* = L^{p'}(T,X^*)$ (see, for example, Gasinski and Papageorgiou [4, p. 129]). Suppose that $u \in W_p(0,b)$. If we view $u(\cdot)$ as an X^*-valued function, then $u(\cdot)$ is absolutely continuous, hence differentiable almost everywhere and this derivative coincides with the distributional one. So, $u' \in L^{p'}(T,X^*)$ and we can say

$$W_p(0,b) \subseteq AC^{1,p'}(T,X^*) = W^{1,p'}((0,b),X^*).$$

The space $W_p(0,b)$ is equipped with the norm

$$||u||_{W_p} = \left[||u||^p_{L^p(T,X)} + ||u'||^p_{L^{p'}(T,X^*)}\right]^{\frac{1}{p}} \text{ for all } u \in W_p(0,b).$$

Evidently, another equivalent norm on $W_p(0,b)$ is

$$|u|_{W_p} = ||u||_{L^p(T,X)} + ||u'||_{L^p(T,X^*)} \text{ for all } u \in W_p(0,b).$$

With any of the above norms, $W_p(0,b)$ becomes a separable reflexive Banach space. We have that

$$W_p(0,b) \hookrightarrow C(T,H); \tag{3}$$

$$W_p(0,b) \hookrightarrow L^p(T,H) \text{ and the embedding is compact.} \tag{4}$$

The elements of $W_p(0,b)$ satisfy an integration by parts formula which will be useful in our analysis.

Proposition 2. *If* $u, v \in W_p(0,b)$ *and* $\xi(t) = (u(t), v(t))$ *for all* $t \in T$*, then* $\xi(\cdot)$ *is absolutely continuous and* $\frac{d\xi}{dt}(t) = \langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle$ *for almost all* $t \in T$.

Now suppose that (Ω, Σ, μ) is a finite measure space, Σ is $\mu-complete$ and Y is a separable Banach space. A multifunction (set-valued function) $F : \Omega \to 2^Y \backslash \{\emptyset\}$ is said to be "graph measurable", if

$$\operatorname{Gr} F = \{(\omega, y) \in \Omega \times Y : y \in F(\omega)\} \in \Sigma \times B(Y),$$

with $B(Y)$ being the Borel σ-field of Y.

If $F(\cdot)$ has closed values, then graph measurability is equivalent to saying that for every $y \in Y$ the $\mathbb{R}_+$-valued function

$$\omega \mapsto d(y, F(\omega)) = \inf\{||y-v||_Y : v \in F(\omega)\}$$

is Σ-measurable.

Given a graph measurable multifunction $F : \Omega \to 2^Y \backslash \{\emptyset\}$, the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [8, p. 158]) implies that $F(\cdot)$ admits a measurable selection, i.e. that there exists $f : \Omega \to Y$ a Σ-measurable function such that $f(\omega) \in F(\omega)$ μ-almost everywhere. In fact, we can find an entire sequence $\{f_n\}_{n \geqslant 1}$ of measurable selections such that $F(\omega) \subseteq \overline{\{f_n(\omega)\}}_{n \geqslant 1}$ μ-almost everywhere.

For $1 \leqslant p \leqslant \infty$, we define

$$S_F^p = \{f \in L^p(\Omega, Y) : f(\omega) \in F(\omega) \ \mu\text{-almost everywhere}\}.$$

It is easy to see that $S_F^p \neq \emptyset$ if and only if $\omega \mapsto \inf\{||v||_Y : v \in F(\omega)\}$ belongs to $L^p(\Omega)$. This set is "decomposable" in the sense that if $(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$, then

$$\chi_A f_1 + \chi_{A^c} f_2 \in S_F^p.$$

Finally, for a sequence $\{C_n\}_{n \geqslant 1}$ of nonempty subsets of Y, we define

$$w-\limsup_{n\to\infty} C_n = \{y \in Y : y = w - \lim_{k\to\infty} y_{n_k}, y_{n_k} \in C_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}.$$

For more details on the notions discussed in this section, we refer to Gasinski and Papageorgiou [4], Roubiček [20], Zeidler [21] (for evolution triples and related notations) and Hu and Papageorgiou [8] (for measurable multifunctions).

Let V be a reflexive Banach space and $A : V \to V^*$ a map. We say that A is "pseudomonotone", if A is continuous from every finite dimensional subspace of V into V_w^* (= the dual V^* equipped with the weak topology) and if

$$v_n \xrightarrow{w} v \text{ in } V, \ \limsup_{n\to\infty} \langle A(v_n), v_n - v\rangle \leqslant 0$$

then

$$\langle A(v), v - y\rangle \leqslant \liminf_{n\to\infty} \langle A(v_n), v_n - y\rangle \text{ for all } y \in V.$$

An everywhere defined maximal monotone operator is pseudomonotone. If V is finite dimensional, then every continuous map $A : V \to V^*$ is pseudomonotone.

In what follows, for any Banach space Z, we will use the following notations:

$$P_{f(c)}(Z) = \{C \subseteq Z : C \text{ is nonempty, closed (and convex)}\},$$

$$P_{(w)k(c)}(Z) = \{C \subseteq Z : C \text{ is nonempty, (weakly-) compact (and convex)}\}.$$

The hypotheses on the data of problem (1) are the following:

$H(A) : A : T \times T \to X^*$ is a map such that

(i) for all $y \in X, t \mapsto A(t, y)$ is measurable;
(ii) for almost all $t \in T$, the map $y \mapsto A(t, y)$ is pseudomonotone;

(iii) $||A(t,y)||_* \leqslant a_1(t) + c_1||y||^{p-1}$ for almost all $t \in T$ and all $y \in X$, with $a_1 \in L^{p'}(T)$, $c_1 > 0$, $2 \leqslant p < \infty$;
(iv) $\langle A(t,y), y\rangle \geqslant 0$ for almost all $t \in T$ and all $y \in X$.

$H(B)$: $B \in \mathscr{L}(X, X^*)$, $\langle Bx, y\rangle = \langle x, By\rangle$ for all $x, y \in X$ and $\langle Bx, x\rangle \geqslant c_0||x||^2$ for all $x \in X$ and some $c_0 > 0$.

$H(F)$: $F : T \times H \times H \to P_{f_c}(H)$ is a multifunction such that

(i) for all $x, y \in H$, $t \mapsto F(t,x,y)$ is graph measurable;
(ii) for almost all $t \in T$, the graph $\mathrm{Gr}\, F(t,\cdot,\cdot)$ is sequentially closed in $H \times H_w \times H_w$ (here H_w denotes the Hilbert space H furnished with the weak topology);
(iii) $|F(t,x,y)| = \sup\{|h| : h \in F(t,x,y)\} \leqslant a_2(t)(1 + |x| + |y|)$ for almost all $t \in T$ and all $x, y \in H$ with $a_2 \in L^2(T)_+$.

Definition 3. *We say that $u \in C(T,X)$ is a "solution" of problem (1) with $u_0 \in X$, $u_1 \in H$, if*

- *$u' \in W_p(0,b)$ and*
- *there exists $f \in S^2_{F(\cdot,u(\cdot),u'(\cdot))}$ such that*

$$\left\{ \begin{array}{l} u''(t) + A(t,u'(t)) + Bu(t) = f(t) \text{ for almost all } t \in T, \\ u(0) = u_0, u'(0) = u_1. \end{array} \right\}$$

In what follows, we denote by $S(u_0,u_1)$ the set of solutions of problem (1). Recalling that $W_p(0,b) \hookrightarrow C(T,H)$ (see (3)), we have that

$$S(u_0,u_1) \subseteq C^1(T,H).$$

By Troyanski's renorming theorem (see Gasinski and Papageorgiou [4, p. 911]) we may assume without loss of generality that both X and X^* are locally uniformly convex. Let $\mathcal{F} : X \to X^*$ be the duality map of X defined by

$$\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x\rangle = ||x||^2 = ||x^*||_*^2\}.$$

We know that $\mathcal{F}(\cdot)$ is single-valued and a homeomorphism (see Gasinski and Papageorgiou [4, p. 316] and Zeidler [21, p. 861]).

For every $r \geqslant p$, let $K_r : X \to X^*$ be the map defined by

$$K_r(y) = ||y||^{r-2}\mathcal{F}(y) \text{ for all } y \in X.$$

3. Existence Theorem

Given $\epsilon > 0$, we consider the following perturbation (parabolic regularization) of problem (1):

$$(5)\quad \left\{ \begin{array}{l} u''(t) + A(t,u'(t)) + \epsilon K_r(u'(t)) + Bu(t) \in F(t,u(t),u'(t)) \text{ for a.a. } t \in T, \\ u(0) = u_0,\ u'(0) = u_1. \end{array} \right\}$$

Consider the map $A_\epsilon : T \times X \to X^*$ defined by

$$A_\epsilon(t,y) = A(t,y) + \epsilon K_r(y) \text{ for all } t \in T, \text{ and all } y \in X.$$

This map has the following properties:

(i) for all $y \in X$, the map $t \mapsto A_\epsilon(t,y)$ is measurable;
(ii) for almost all $t \in T$, the map $y \mapsto A_\epsilon(t,y)$ is pseudomonotone;
(iii) $||A_\epsilon(t,y)||_* \leqslant \hat{a}_1(t) + \hat{c}_1||y||^{r-1}$ for almost all $t \in T$, all $y \in X$ and with $\hat{a}_1 \in L^{p'}(T)$, $\hat{c}_1 > 0$ (recall that $r \geqslant p$ and $\dfrac{1}{r} + \dfrac{1}{r'} = 1$);

(iv) $\langle A_\epsilon(t,y),y\rangle \geqslant \epsilon||y||^r$ for all $t\in T$, all $y\in X$.

So, in problem (1) the viscosity term $A_\epsilon(t,\cdot)$ is coercive. Therefore we can apply Theorem 1 of Papageorgiou and Yannakakis [18] and we obtain the following existence result for the approximate (regularized) problem (5).

Proposition 4. *If hypotheses $H(A)$, $H(B)$, $H(F)$ hold and $u_0\in X, u_1\in H$, then problem (5) admits a solution $u_\epsilon\in W^{1,r}((0,b),X)\cap C^1(T,H)$ with*

$$u'_\epsilon\in W_r(0,b).$$

To produce a solution for the original problem (1), we have to pass to the limit as $\epsilon\to 0^+$. To do this, we need to have *a priori* bounds for the solutions $u_\epsilon(\cdot)$ which are independent of $\epsilon\in(0,1]$ and $r\geqslant p$.

Proposition 5. *If hypotheses $H(A),H(B),H(F)$ hold, $u_0\in X,u_1\in H$ and $u(\cdot)$ is a solution of (5), then there exists $M_0>0$ which is independent of $\epsilon\in(0,1]$ and $r\geqslant p$ for which we have*

$$||u||_{C(T,X)},\ ||u'||_{C(T,H)},\ \epsilon^{\frac{1}{r}}||u'||_{L^r(T,X)},\ ||u''||_{L^2(T,X^*)}\leqslant M_0.$$

Proof. It follows from Proposition 4 that $u'\in W_r(0,b)$ and that there exists $f\in S^2_{F(\cdot,u(\cdot),u'(\cdot))}$ such that

$$u''(t)+A(t,u'(t))+\epsilon K_r(u'(t))+Bu(t)=f(t)\text{ for almost all }t\in T.$$

We act with $u'(t)\in X$. Then

$$\begin{aligned}\langle u''(t),u'(t)\rangle+\langle A(t,u'(t)),u'(t)\rangle+\epsilon\langle K_r(u'(t)),u'(t)\rangle=(f(t),u'(t))\\ \text{for almost all }t\in T\text{ (see (2)).}\end{aligned}\tag{6}$$

We examine separately each summand on the left-hand side of (6). Recall that $u'_r\in W_r(0,b)$. So from Proposition 2 (the integration by parts formula), we have

$$\langle u''(t),u'(t)\rangle=\frac{1}{2}\frac{d}{dt}|u'(t)|^2\text{ for almost all }t\in T.\tag{7}$$

Hypothesis $H(A)(iv)$ and the definition of the duality map, imply that

$$\langle A(t,u'(t)),u'(t)\rangle+\epsilon\langle K_r(u'(t)),u'(t)\rangle\geqslant\epsilon||u'(t)||^r\text{ for almost all }t\in T.\tag{8}$$

By hypothesis $H(B)$, we have

$$\langle Bu(t),u'(t)\rangle=\frac{1}{2}\frac{d}{dt}\langle Bu(t),u(t)\rangle\text{ for almost all }t\in T.\tag{9}$$

We return to (6) and use (7), (8), (9). We obtain

$$\frac{1}{2}\frac{d}{dt}|u'(t)|_2+\epsilon||u'(t)||^r+\frac{1}{2}\frac{d}{dt}\langle Bu(t),u(t)\rangle\leqslant(f(t),u'(t))\text{ for a.a. }t\in T,$$

$$\Rightarrow\ \frac{1}{2}|u'(t)|^2+\epsilon\int_0^t||u'(s)||^r ds+c_0||u(t)||^2$$

$$\leqslant\int_0^t(f(s),u'(s))ds+\frac{1}{2}|u_1|^2+\frac{1}{2}||B||_{\mathscr{L}}||u_0||^2\text{ (see hypothesis }H(B)).\tag{10}$$

Using hypothesis $H(F)(iii)$, we get

$$\begin{aligned} & \int_0^t (f(s), u'(s))ds \\ \leqslant & \int_0^t [a_2(s) + a_2(s)(|u(s)| + |u'(s)|)]\,|u'(s)|ds \\ \leqslant & \int_0^t |u'(s)|^2 ds + \int_0^t a_2(s)^2 ds + \int_0^t a_2(s)^2 \left[|u(s)|^2 + |u'(s)|^2\right] ds. \end{aligned} \tag{11}$$

Recall that $u \in W^{1,r}((0,b),X)$ (see Proposition 4). So, $u \in AC^{1,r}(T,H)$ and we can write

$$u(t) = u_0 + \int_0^t u'(s)ds \text{ for all } t \in T$$

$$\Rightarrow \quad |u(t)|^2 \leqslant 2|u_0|^2 + 2b\int_0^t |u'(s)|^2 ds \text{ for all } t \in T \text{ (using Jensen's inequality).} \tag{12}$$

We use (12) in (11) and obtain

$$\begin{aligned} & \int_0^t (f(s), u'(s))ds \\ \leqslant & \ ||a_2||_2^2 + \int_0^t \left[1 + a_2(s)^2\right]|u'(s)|^2 ds + \int_0^t 2a_2(s)^2 \left[|u_0|^2 + b\int_0^s |u'(\tau)|^2 d\tau\right] ds \\ \leqslant & \ c_2 + \int_0^t \eta(s)|u'(s)|^2 ds + 2b\int_0^t a_2(s)^2 \int_0^s |u'(\tau)|^2 d\tau ds \end{aligned} \tag{13}$$

for some $c_2 > 0$ and $\eta \in L^1(T)$.

We use (13) in (10) and have

$$\begin{aligned} & \frac{1}{2}|u'(t)|^2 + \epsilon\int_0^t ||u'(s)||^p ds + c_0||u(t)||^2 \\ \leqslant & \ c_3 + \int_0^t \eta(s)|u'(s)|^2 ds + 2b\int_0^t a_2(s)^2 \int_0^s |u'(\tau)|^2 d\tau ds \text{ for some } c_3 > 0. \end{aligned} \tag{14}$$

Invoking Proposition 1.7.87 of Denkowski, Migórski and Papageorgiou [2, p. 128] we can find $M > 0$ (independent of $\epsilon \in (0,1]$ and $r \geqslant p$) such that

$$\begin{aligned} & |u'(t)|^2 \leqslant M \text{ for all } t \in T, \\ \Rightarrow \quad & ||u'||_{C(T,H)} \leqslant M_1 = M^{\frac{1}{2}}. \end{aligned}$$

Using this bound in (14), we can find $M_2 > 0$ (independent of $\epsilon \in (0,1]$ and $r \geqslant p$) such that

$$||u||_{C(T,X)} \leqslant M_2 \text{ and } \epsilon^{\frac{1}{r}}||u'||_{L^r(T,X)} \leqslant M_2.$$

Finally, directly from (5), we see that there exists $M_3 > 0$ (independent of $\epsilon \in (0,1]$ and $r \geqslant p$) such that

$$||u''||_{L^{r'}}(T, X^*) \leqslant M_3.$$

We set $M_0 = \max\{M_1, M_2, M_3\} > 0$ and get the desired bound. □

The bounds produced in Proposition 5 permit passing to the limit as $\epsilon \to 0^+$ to produce a solution for problem (1).

Theorem 6. *If hypotheses $H(A), H(B), H(F)$ hold and $u_0 \in X, u_1 \in H$, then $S(u_0, u_1) \neq \emptyset$.*

Proof. Let $\epsilon_n \to 0^+$ and let $u_n = u_{\epsilon_n}$ be solutions of the "regularized" problem (5) (see Proposition 4). Because of the bounds established in Proposition 5 and by passing to a suitable subsequence if necessary, we can say that

$$\left\{ \begin{array}{l} u_n \xrightarrow{w^*} u \text{ in } L^\infty(T,X),\ u_n \xrightarrow{w} u \text{ in } C(T,H),\ u_n \to u \text{ in } L^r(T,H) \\ u_n' \xrightarrow{w^*} y \text{ in } L^\infty(T,H),\ u_n'' \xrightarrow{w} v \text{ in } L^{r'}(T,X^*) \text{ (see (3) and (4)).} \end{array} \right\} \tag{15}$$

Recall that $u_n \in AC^{1,r}(T,H)$ for all $n \in \mathbb{N}$ and so

$$\begin{aligned} & u_n(t) = u_0 + \int_0^t u_n'(s)ds \text{ for all } t \in T, \\ \Rightarrow\quad & u(t) = u_0 + \int_0^t y(s)ds \text{ for all } t \in T \text{ (see (15))}, \\ \Rightarrow\quad & u \in AC^{1,r}(T,H) \text{ and } u' = y. \end{aligned}$$

Since $u_n \in W_r(0,b)$ for all $n \in \mathbb{N}$, we have

$$v = y' = u'' \in L^{r'}(T,X^*) \text{ (see Hu and Papageorgiou [9, p. 6]).}$$

Let $a : L^r(T,X) \to L^{r'}(T,X^*)$ be the nonlinear map defined by

$$a(u)(\cdot) = A(\cdot, u(\cdot)) \text{ for all } u \in L^r(T,X).$$

Also, let $\hat{K}_r : L^r(T,X) \to L^{r'}(T,X^*)$ be defined by

$$\hat{K}_r(u)(\cdot) = ||u(\cdot)||^{r-2}\mathscr{F}(u(\cdot)) \text{ for all } u \in L^r(T,X).$$

Both maps are continuous and monotone, hence maximal monotone (see Gasinski and Papageorgiou [4, Corollary 3.2.32, p. 320]).

Finally, let $\hat{B} \in \mathscr{L}(L^r(T,X), L^{r'}(T,X^*))$ be defined by

$$\hat{B}(u)(\cdot) = B(u(\cdot)) \text{ for all } u \in L^r(T,X).$$

We have

$$\begin{aligned} u_n'' + a(u_n') + \epsilon_n \hat{K}_r(u_n') + \hat{B}u_n = f_n \text{ in } L^r(T,X^*) \\ \text{with } f_n \in S^2_{F(\cdot,u_n(\cdot),u_n'(\cdot))} \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{16}$$

From (15) we have

$$\begin{aligned} & u_n \xrightarrow{w} u \text{ in } L^r(T,X), \\ \Rightarrow\quad & \hat{B}u_n \xrightarrow{w} \hat{B}u \text{ in } L^{r'}(T,X^*) \text{ as } n \to \infty. \end{aligned} \tag{17}$$

Also, we have

$$\begin{aligned} & ||\hat{K}_r(u_n')||_{L^{r'}(T,X^*)} = ||u_n'||^{r-1}_{L^r(T,X)}, \\ \Rightarrow\quad & \epsilon_n ||\hat{K}_r(u_n')||_{L^{r'}(T,X^*)} = \epsilon_n^{\frac{1}{r}} \left(\epsilon_n^{\frac{1}{r}} ||u_n'||_{L^r(T,X)} \right)^{r-1} \text{ (recall that } \frac{1}{r} + \frac{1}{r'} = 1) \\ & \leqslant \epsilon_n^{\frac{1}{r}} M_0^{r-1} \text{ for all } n \in \mathbb{N} \text{ (see Proposition 5)} \end{aligned}$$

$$(18) \Rightarrow\quad \epsilon_n ||\hat{K}_r(u_r')||_{L^{r'}(T,X^*)} \to 0 \text{ as } n \to \infty$$

From (15) and since $v = u''$, we have

$$u_n'' \xrightarrow{w} u'' \text{ in } L^{r'}(T,X^*). \tag{19}$$

Finally, hypothesis $H(F)(iii)$ and Proposition 5 imply that

$$\{f_n\}_{n\geqslant 1} \subseteq L^2(T,H) \text{ is bounded.}$$

By passing to a subsequence if necessary, we may assume that

$$f_n \xrightarrow{w} f \text{ in } L^2(T,H).$$

Invoking Proposition 3.9 of Hu and Papageorgiou [8, p. 694], we have

$$\begin{aligned} f(t) &\in \overline{\text{conv}}\, w - \limsup_{n\to\infty}\{f_n(t)\} \\ &\leqslant \overline{\text{conv}}\, w - \limsup_{n\to\infty} F(t,u_n(t),u_n'(t)) \text{ for almost all } t\in T \text{ (see (16))}. \end{aligned} \tag{20}$$

From (15) we see that

$$u_n' \xrightarrow{w} u' \text{ in } W^{1,r'}((0,b),X^*).$$

Recall that $W^{1,r'}((0,b),X^*) \hookrightarrow C(T,X^*)$. So, it follows that

$$\begin{aligned} & u_n' \xrightarrow{w} u' \text{ in } C(T,X^*) \\ \Rightarrow\ & u_n'(t) \xrightarrow{w} u'(t) \text{ in } X^* \text{ for all } t\in T. \end{aligned} \tag{21}$$

On the other hand, by Proposition 5 we have

$$|u_n'(t)| \leqslant M_0 \text{ for all } t\in T, \text{ all } n\in\mathbb{N}.$$

So, by passing to a subsequence (*a priori* the subsequence depends on $t\in T$), we have

$$\begin{aligned} & u_n'(t) \xrightarrow{w} \hat{y}(t) \text{ in } H \\ \Rightarrow\ & \hat{y}(t) = u'(t) \text{ for all } t\in T \text{ (see (21))}. \end{aligned}$$

Hence for the original sequence we have

$$u_n'(t) \xrightarrow{w} u'(t) \text{ in } H \text{ for all } t\in T. \tag{22}$$

We know that $\{u_n\}_{n\geqslant 1}\subseteq W_r(0,b)$ is bounded (see Proposition 5) and recall that $W_r(0,b)\hookrightarrow L^r(T,H)$ compactly (see (4)). From this compact embedding and from (22), we obtain

$$u_n(t)\to u(t) \text{ in } H \text{ for all } t\in T \text{ as } n\to\infty. \tag{23}$$

From (20), (22), (23) and hypothesis $H(F)(iii)$ we infer that

$$\begin{aligned} & f(t)\in F(t,u(t),u'(t)) \text{ for almost all } t\in T, \\ \Rightarrow\ & f\in S^2_{F(\cdot,u(\cdot),u'(\cdot))}. \end{aligned}$$

In what follows, we denote by $((\cdot,\cdot))$ the duality brackets for the pair

$$(L^r(T,X^*),L^r(T,X)).$$

Acting with $u_n'-u'\in L^r(T,X)$ on (16), we have

$$((u_n'',u_n'-u'))+((a(u_n'),u_n'-u'))+((\epsilon_n\hat{K}_r(u_n'),u_r'-u'))+((\hat{B}u_n,u_n'-u')) = \int_0^b (f_n,u_n'-u')dt \text{ for all } n\in\mathbb{N}. \tag{24}$$

Note that

$$\begin{aligned}
((u_n'', u_n' - u')) &= \int_0^b \langle u_n'', u_n' - u' \rangle dt \\
&= \int_0^b \langle u_n'' - u'', u_n' - u' \rangle dt + ((u'', u_n' - u')) \\
&= \int_0^b \frac{1}{2}\frac{d}{dt}|u_n' - u'|^2 dt + ((u'', u_n' - u')) \text{ (see Proposition 2)} \\
&= \frac{1}{2}|u_n'(b) - u'(b)|^2 + ((u'', u_n' - u')) \\
&\quad \text{(since } u_n'(0) = u'(0) = u_1 \text{ for all } n \in \mathbb{N}\text{, see (22))}
\end{aligned}$$

$$\Rightarrow \liminf_{n\to\infty}((u_n'', u_n' - u')) = \frac{1}{2}\liminf_{n\to\infty}|u_n'(b) - u'(b)|^2 \geqslant 0. \tag{25}$$

Also we have

$$((\hat{B}(u_n - u), u_n' - u')) = \int_0^b \frac{1}{2}\frac{d}{dt}\langle B(u_n - u), u_n - u\rangle dt$$

$$\frac{1}{2}\langle B(u_n - u)(b), (u_n - u)(b)\rangle \geqslant 0 \text{ (see hypothesis } H(B))$$

$$\Rightarrow ((\hat{B}u, u_n' - u')) \leqslant ((\hat{B}u_n, u_n' - u')) \text{ for all } n \in \mathbb{N}. \tag{26}$$

Recall that

$$\epsilon_n^{\frac{1}{2}}||u_n||_{L^r(T,X)} \leqslant M_0 \text{ for all } n \in \mathbb{N} \text{ all } r \geqslant p \text{ (see Proposition 5)}.$$

Suppose that $r_m \to +\infty$, $r_m \geqslant p$ for all $m \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, $\epsilon_n^{\frac{1}{r_m}} \to 1$ as $m \to \infty$. Invoking Problem 1.175 of Gasinski and Papageorgiou [5], we can find $\{m_n\}_{n\geqslant 1}$ with $m_n \to +\infty$ such that

$$\epsilon_n^{\frac{1}{r_{m_n}}} \to 1 \text{ as } n \to \infty.$$

Therefore there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2} \leqslant \epsilon_n^{\frac{1}{r_{m_n}}} \text{ for all } n \geqslant n_0,$$

$$\frac{1}{2}||u_n'||_{L^{r_{m_n}}(T,X)} \leqslant M_0 \text{ for all } n \geqslant n_0,$$

$$\Rightarrow ||u_n'||_{L^p(T,X)} \leqslant 2M_0 \text{ for all } n \geqslant n_0 \text{ (recall that } r_{m_n} \geqslant p).$$

On account of (15) and since $y = u'$, we have

$$u_n' \xrightarrow{w} u' \text{ in } L^p(T,X). \tag{27}$$

Then from (26) and (27) it follows that

$$0 \leqslant \liminf_{n\to\infty}((\hat{B}u_n, u_n' - u')). \tag{28}$$

In addition, we have

$$\epsilon_n \hat{K}_p(u_n') \to 0 \text{ in } L^{p'}(T, X^*) \text{ as } n \to \infty \text{ (see (18))}. \tag{29}$$

By Proposition 5 and (27) it follows that

$$\{u_n'\}_{n\geqslant 1} \subseteq W_p(0,b) \text{ is bounded,}$$

$$\Rightarrow \{u_n'\}_{n\geqslant 1} \subseteq L^p(T,H) \text{ is relatively compact (see (4)).}$$

Therefore we have

$$\begin{aligned} & u_n' \to u' \text{ in } L^p(T,H) \text{ (see (27))}, \\ \Rightarrow \quad & \int_0^b (f_n, u_n' - u')dt \to 0 \text{ as } n \to \infty \text{ (recall that } p \geqslant 2). \end{aligned} \tag{30}$$

If in (24) we pass to the limit as $n \to \infty$ and use (25), (28), (29), (30), then

$$\limsup_{n\to\infty}((a(u_n'), u_n' - u')) \leqslant 0.$$

Invoking Theorem 2.35 of Hu and Papageorgiou [9, p. 41], we have

$$a(u_n) \xrightarrow{w} a(u') \text{ in } L^{p'}(T, X^*) \text{ as } n \to \infty. \tag{31}$$

In (24) we pass to the limit as $n \to \infty$ and use (15) (with $v = u''$) (27), (29), (31). We obtain

$$\begin{aligned} & u'' + a(u') + \hat{B}u = f,\ u(0) = u_0, u'(0) = u_1, f \in S^2_{F(\cdot,u(\cdot),u'(\cdot))}, \\ \Rightarrow \quad & u \in S(u_0, u_1) \neq \emptyset. \end{aligned}$$

The proof is now complete. □

3.1. **An example.** We illustrate the main abstract result of this paper with a hyperbolic boundary value problem. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. We consider the following boundary value problem

$$\left\{ \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(a(t,z)|Du_t|^{p-2}Du_t) + \beta(z)u_t - \Delta u = f(t,z,u) + \gamma u_t \text{ in } T\times\Omega, \\ u|_{T\times\partial\Omega} = 0,\ u(0,z) = u_0(z),\ u_t(0,z) = u_1(z), \end{array} \right\} \tag{32}$$

with $u_t = \dfrac{\partial u}{\partial t}$, $2 \leqslant p \leqslant \infty$, $\gamma > 0$.

The forcing term $f(t,z,\cdot)$ need not to be continuous. So, following Chang [1], to deal with (32), we replace it by a multivalued problem (partial differential inclusion), by filling in the gaps at the discontinuity points of $f(t,z,\cdot)$. So we define

$$f_l(t,z,x) = \liminf_{x'\to x} f(t,z,x') \text{ and } f_u(t,z,x) = \limsup_{x'\to x} f(t,z,x').$$

Then we replace (32) by the following partial differential inclusion

$$\left\{ \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(a(t,z)|Du_t|^{p-2}Du_t) + \beta(z)u_t - \Delta u \in [f_l(t,z,u), f_u(t,z,u)] \text{ in } T\times\Omega, \\ u|_{T\times\partial\Omega} = 0,\ u(0,z) = u_0(z),\ u_t(0,z) = u_1(z). \end{array} \right\} \tag{33}$$

Our hypotheses on the data of (33) are the following:

$H(a)$: $a \in L^\infty(T\times\Omega), a(t,z) \geqslant 0$ for almost all $(t,z) \in T\times\Omega$.

$H(\beta)$: $\beta \in L^\infty(\Omega)$, $\beta(z) \geqslant 0$ for almost all $z \in \Omega$.

$H(f)$: $f : T\times\Omega\times\mathbb{R} \to \mathbb{R}$ is a function such that

(i) f_l, f_u are superpositionally measurable (that is, for all $u : T\times\Omega \to \mathbb{R}$ measurable, the functions $(t,z) \mapsto f_l(t,z,u(t,z))$, $f_u(t,z,u(t,z))$ are both measurable);

(ii) there exists $a \in L^2(T\times\Omega)$ such that

$$|f(t,z,x)| \leqslant a_2(t,z)(1+|x|) \text{ for almost all } (t,z) \in T\times\Omega, \text{ all } x \in \mathbb{R}.$$

Let $X = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $X^* = W^{-1,p'}(\Omega)$. Then (X, H, X^*) is an evolution triple with $X \hookrightarrow H$ compactly (by the Sobolev embedding theorem).

Let $A : T \times X \to X^*$ be defined by

$$\langle A(t,u), h\rangle = \int_\Omega a(t,z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz + \int_\Omega \beta(z)uh dz \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

Then $A(t,u)$ is measurable in $t \in T$, continuous and monotone in $u \in W_0^{1,p}(\Omega)$ (hence, maximal monotone) and $\langle A(t,u), u\rangle \geqslant 0$ for almost all $t \in T$, all $u \in W_0^{1,p}(\Omega)$.

Let $B \in \mathscr{L}(X, X^*)$ be defined by

$$\langle Bu, h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\,dz \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

Clearly, B satisfies hypothesis $H(B)$.

Finally, let $G(t,z,x) = [f_l(t,z,x), f_u(t,z,x)]$ and set

$$F(t,u,v) = S^2_{G(t,\cdot,u(\cdot))} + \gamma v \text{ for all } u, v \in L^2(\Omega).$$

Hypothesis $H(f)$ implies that F satisfies $H(F)$.

Using $A(t,u), Bu$ and $F(t,u,v)$ as defined above, we can rewrite problem (33) as the equivalent second order nonlinear evolution inclusion (1). Assuming that $u_0 \in W_0^{1,p}(\Omega)$ and that $u_1 \in L^2(\Omega)$, we can use Theorem 6 and infer that problem (30) has a solution $u \in C^1(T, L^2(\Omega)) \cap C(T, W^{1,p}(\Omega))$ with $\dfrac{\partial u}{\partial t} \in L^p(\Omega, W_0^{1,p}(\Omega))$ and $\dfrac{\partial^2 u}{\partial t} \in L^{p'}(\Omega, W^{-1,p'}(\Omega))$.

Note that if $a = 0, f(t,z,x) = x$ and $\gamma = 0$, then we have the Klein-Gordon equation. If $f(t,z,x) = f(x) = \eta \sin x$ with $\eta > 0$, then we have the sine Gordon equation.

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