An intermediate-complexity model for four-dimensional variational data assimilation including moist processes

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The article presents a new Moist Atmosphere Dynamics Data Assimilation Model (MADDAM), an intermediate-complexity system for the four-dimensional variational (4D-Var) data assimilation. The prognostic model equations simulate non-linear moisture advection, precipitation and the impact of condensational heating on circulation. The 4D-Var assimilation applies the incremental approach and uses a transformed relative humidity as control variable. In contrast to the model dynamical variables, which are analyzed in the multivariate fashion using the equatorial wave theory, moisture data is assimilated univariately. MADDAM is applied to study the extraction of wind information from the time series of moisture observations in the Tropics, where the lack of wind information is most critical.

Results show that the wind tracing in unsaturated atmosphere largely depends on the ability of the assimilation model to resolve spatial gradients in the moisture field, which is determined by the spatial density and accuracy of observations. In the saturated atmosphere, a combined assimilation of moisture and temperature data is shown to significantly improve wind analyses, as the intensity of the condensation process is susceptible to slightest changes in saturation humidity and thus temperature. Moreover, the perfect-model 4D-Var with moisture observations can extract wind information even in the precipitating regions and strongly non-linear flow provided sufficient observations of humidity gradients.

MADDAM is envisaged to serve as a testbed for new developments in 4D-Var assimilation with focus on interactions between moist processes and dynamics across many scales.

Key Words: moist 4D-Var, wind tracing, moisture observations, humidity control variable, tropical data assimilation

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1. Introduction

Simplified numerical prediction models are valuable testbeds for data assimilation research. An idealized framework, with respect to numerical weather prediction (NWP) models, aids understanding and simplifies research. Simplified models allow to 1) develop and implement new algorithms faster than in the NWP case, and to 2) perform numerical experiments in a controlled environment in which various issues, difficult to grasp in a real NWP, can more easily be understood.

At the same time, simplified models should still be complex enough to capture main dynamical and physical aspects of the phenomena of interest in order to explain the observed features of circulation and to be of any value for NWP. An important such model is based on the rotating non-linear shallow water equations (e.g. Vallis 2006) which include both balanced (vorticity dominated) dynamics and gravity-wave dynamics as well as their

interactions. Shallow water models (SWMs) were applied in a number of data assimilation studies to develop new concepts and to study the value of mass-field and wind-field observations (e.g. Žagar *et al.* 2004a, and references therein). This question is especially important in the Tropics where the wind field information is of greater value for the assimilation than the mass field information, even in the case of perfect-model four-dimensional variational data assimilation (Žagar *et al.* 2004b).

However, a majority of data assimilation studies with SWMs considered dry dynamics and thus vastly underestimated the intrinsic nonlinearity of the atmospheric processes. Exceptions are studies of the tropical data assimilation by Žagar (2012) and Harlim and Majda (2013) that are based on the SWMs coupled to the prognostic moisture equations. In particular, Harlim and Majda (2013) demonstrated that the tropical 3D-Var assimilation with the moist background error covariances can recover the

unobserved troposphere humidity and precipitation rate. A one-dimensional SWM for convective-scale data assimilation was recently employed in Goodliff *et al.* (2017) and Robert and Künsch (2017) based on the idea of Würsch and Craig (2014) with convection taking place when the fluid height reaches a predefined level.

This study introduces a moist non-linear model with incremental four-dimensional variational data assimilation (4D-Var) which allows a realistic representation of moisture-wind couplings in tangent-linear and adjoint model dynamics in 4D-Var. We demonstrate how the new model can be used to explain the reported significant impact of moisture observations in data assimilation systems for NWP that use 4D-Var (Bormann and Thépaut 2004; Geer *et al.* 2008).

A fundamental feature of moist dynamics is that propagating waves change properties as they move through the precipitating areas. The moisture feedback on circulation through the latent heat release is successfully represented by the model introduced by Gill (1982b) which was adapted for the four-dimensional variational (4D-Var) data assimilation in Žagar *et al.* (2008, hereafter Ž2008) and Žagar (2012) for the dry and moist assimilation, respectively. The latter presented only few preliminary results. The present article extends the previous work by presenting a full 4D-Var model for the moist data assimilation including a new moisture control variable. The new system is applied to 1) discuss internal adjustment between moisture and dynamics in 4D-Var, and to 2) quantify the extent to which the wind field in the Tropics can be determined from the time series of moisture data, alone or combined with temperature data. Specifically, we are interested in the wind tracing performance if the flow is highly nonlinear, e.g. in the regions of precipitation, where a slight inaccuracy in observed moisture or temperature field (e.g. due to saturation) can result in distinct dynamics.

These questions are addressed in the Tropics, where the lack of wind observations and a lack of efficient mass-wind constraint in the background-error covariance matrix for 4D-Var make the data assimilation especially challenging (e.g. Žagar *et al.* 2005; Baker *et al.* 2014). Without an explicit equatorial mass-wind constraint in the background term for data assimilation, mass observations affect wind analyses entirely through the internal model adjustment, i.e. during the forward and adjoint model integration in 4D-Var (Žagar *et al.* 2004b; Bonavita and Holm 2016). Details of this process and the potential role of the background error constraint based on the equatorial waves can be easier understood using a simplified model for data assimilation as described in this paper. Furthermore, the complexity and nonlinearity of physical processes and observation operators in NWP assimilation models often prevent an assessment of where the impact on the analysis wind came from.

The paper is organized as follows. In Section 2, the concept and mathematical formulation of the moist model for the Tropics is described followed by the description of the incremental 4D-Var data assimilation. Section 3 presents results of simple experiments which demonstrate the simulation capabilities of the model. In Section 4, the 4D-Var wind retrieval from multiple moisture observations is addressed and the resulting wind analyses are compared to those obtained from other data types. Discussion and outlook are given in Section 5.

2. Moist atmosphere dynamics and data assimilation model

In this section we describe the moist model and the formulation of the new 4D-Var assimilation system with a moisture control variable. The two components of the system are abbreviated MAD, the Moist Atmosphere Dynamics model, and MADDAM, Moist Atmosphere Dynamics Data Assimilation Model. The

moist model is based on the work of Gill (1982b) to represent the response of the tropical atmosphere to deep heating. The dry component of the 4D-Var system based on Gill's model was described in Ž2008, and the reader is referred to that paper for details of numerical discretization and data assimilation modelling. Here, we briefly present the model formulation before continuing with the moist component of the forecast model and introduce the moisture control variable and the moisture background error variance model.

2.1. MAD: continuous model formulation

The moist model includes four prognostic equations which describe the horizontal wind, temperature and moisture distribution in response to deep heating. In the Tropics, the main source of heating is the latent heat release by condensation. Other sources of heating can include radiative cooling or evaporative heating from the underlying ocean surface. The dynamical equations describe the potential temperature perturbations at the mid-troposphere level where the heating is maximal and associated horizontal motions in the lower troposphere. In response to the half-sinusoidal structure of heating, the potential temperature perturbations and horizontal motions have half-sinusoidal and cosinusoidal vertical structure. Therefore, motions in the upper troposphere have opposite signs, the vertical structure known as the first baroclinic mode (Gill 1980). Moisture is assumed to have an exponential vertical profile in the lower layer (as in Davey and Gill 1987), which agrees well with observed structure (e.g. Holloway and Neelin 2009). The moisture conservation equation defines the balance between the non-linear advection, evaporation and condensation.

Prognostic equations for the mid-level potential temperature θ , lower level wind components $\mathbf{v} = (u, v)$ and column-mean specific humidity q (in kg of water vapour per kg air) are

$$\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla) \theta - \frac{\theta_0 N^2 H}{g} (\nabla \cdot \mathbf{v}) = Q_{LH} + Q - \epsilon_\theta \theta, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + f \mathbf{k} \times \mathbf{v} = \frac{gH}{\theta_0} \nabla \theta - \epsilon_v \mathbf{v}, \quad (2)$$

$$\frac{\partial q}{\partial t} + \nabla \cdot (\mathbf{v} q) = E - C. \quad (3)$$

The parameters E and C represent the rate of evaporation and condensation, Q_{LH} denotes the latent heating whereas Q represents other diabatic forcings. Frictional processes for the mass and momentum are parameterized by ϵ terms. These are defined by choosing the Newtonian cooling relaxation time scale $1/\epsilon_\theta$ and the Rayleigh-wind friction relaxation time scale $1/\epsilon_v$ equal 2 months. πH is the depth of the troposphere above the boundary layer, θ_0 is the background potential temperature and N stands for the Brunt-Väisälä frequency, defined as $N^2 = (g/\theta_0) \partial \theta_0 / \partial z$ (Table 1). Other constants here have their usual meaning: ρ_0 is the background air density, g is gravity and f is the Coriolis parameter. In the Tropics, the Coriolis parameter f is replaced by the equatorial β -plane approximation: $f = \beta y$, where $\beta = \partial f / \partial y = 2.28 \cdot 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$, and R_e is the radius of the Earth.

The solutions of (1-3) consist of a spectrum of the equatorial Rossby and inertio-gravity waves including the two special tropical wave motions, the Kelvin and mixed Rossby-gravity (MRG) waves (e.g. Matsuno 1966). The gravity wave phase speed in the model is $c = NH$ (see Ž2008 for details). For typical values of tropical parameters (Table 1), $N = 0.01 \text{ s}^{-1}$, c is about 50 m s^{-1} . The presence of moisture does not change static stability, i.e. the gravity wave phase speed. It is condensational heating that reduces buoyant forces in the moist zones, decreasing

the amplitude of the waves and slowing them down, as shown analytically by Gill (1982a) using a linearized 1D SWM. The meridional trapping scale for the circulation, the equatorial Rossby deformation radius, $R_e = \sqrt{NH/(2\beta)}$ is around 1000 km. The associated time scale is $R_e/c \approx 6$ hours. Therefore, the choice of N affects dynamics so that the reduced static stability increases the equatorial trapping and changes characteristics of propagating waves.

Condensation occurs in the model when the atmosphere is saturated and there is a convergence of moisture. All excess moisture is precipitated. In the absence of evaporation, this can be expressed as

$$C = -\nabla \cdot (\mathbf{v}q) \quad \text{when} \quad q \geq q_s \quad (4)$$

and $C = 0$ otherwise. Condensation leads to the release of the latent heat Q_{LH} at the mid-tropospheric level,

$$Q_{LH} = \frac{L_c}{2c_p} C,$$

where L_c is the latent heat of condensation and c_p is the specific heat of the air at constant pressure. Factor 2 is included to relate the total column condensed moisture and total column released latent heat.

Column-mean saturated specific humidity is estimated for the bottom layer from the prescribed potential temperature and pressure profiles. For given values of other parameters, the average q_s is about 10.5 g/kg (Table 1). Even though the temperature variance in the Tropics is small, the exponential behaviour of $q_s(T)$ can still result in substantially different values of saturation specific humidity.

The fundamental moisture-dynamical coupling occurs in relation to the released heating. Until the saturation is reached, presence of mid-troposphere heating will lead to lower-level convergence towards the heated region and vertical motions. The convergence leads to saturation and precipitation, which additionally intensifies convergence in the lower level. Associated temperature and wind perturbations move along the equator as well as outside the Tropics. Moisture driven condensation processes are thus the main source of nonlinearity in the model as illustrated in the next section.

2.2. MAD: numerical formulation

The applied numerical model formulation in the Tropics follows Ž2008. A new component is associated with the spectral representation of moisture equation which can produce negative values for quantities which are physically positive definite, such as specific humidity q . The Gibbs oscillations can pose a problem for moisture at both ends, i.e. numerics can produce supersaturated humidity values and triggers “numerical rain”. Sharp gradients and discontinuities associated with precipitation and latent heat release can occur during the simulation, but were not detrimental. The treatment of moist processes in physical space involves the following steps:

1. At the end of time step n :
 - compute grid point specific humidity tendencies $\partial q/\partial t$.
 - If tendency is positive ($\partial q/\partial t > 0$) and atmosphere already locally saturated ($q \geq q_s$), then: condensation rate $C = \partial q/\partial t$, set $\partial q/\partial t = 0$, specific humidity retains saturated value $q = q_s$, else: $C = 0$.
2. At the beginning of time step $n + 1$:
 - diagnose saturated specific humidity $q_s(T(\theta))$,

- relax the extra moisture (if $q > q_s$) obtained from time integration and add it to condensation rate from the previous time step: $C = C + (q - q_s)/\Delta t$. Set $q = q_s$,
- compute latent heating rate $Q_{LH} = L_c C/(2c_p)$ and add it to potential temperature tendency $\partial \theta/\partial t$.

2.3. MADDAM: formulation of 4D-Var with moisture

The 4D-Var minimization problem is solved using the incremental approach (Courtier et al. 1994) which approximates the nonlinear cost function by a sequence of quadratic cost functions around the latest guess $\mathbf{x}_g$. This sequence is most often denoted as the outer loop. Each quadratic cost function is then minimized (so-called inner loop) to update the guess for the next outer loop iteration. The minimization problem is simplified by transforming the state increment $\delta \mathbf{x}$ to a new control variable χ ,

$$\mathbf{x} - \mathbf{x}_g = \delta \mathbf{x} = \mathbf{L}\chi, \quad (5)$$

which transforms $\mathbf{B}$ matrix into an identity matrix. Similarly, the guess increment $\delta \mathbf{x}_g$ is transformed into new variable χ_g . The minimization problem therefore becomes

$$J(\chi) = \frac{1}{2} (\chi + \chi_g)^T (\chi + \chi_g) + \frac{1}{2} \sum_{k=0}^N (\mathbf{d}_k - \mathbf{G}_k \mathbf{L}\chi)^T \mathbf{R}_k^{-1} (\mathbf{d}_k - \mathbf{G}_k \mathbf{L}\chi), \quad (6)$$

where $\mathbf{d}_k = \mathbf{y}_k - \mathbf{G}_k(\mathbf{x}_g)$ is a vector of innovations, $\mathbf{G}_k = \mathbf{M}_{t_0 \rightarrow t_k} \mathbf{H}_k$ is a Jacobian of generalised observation operator and $\mathbf{M}_{t_0 \rightarrow t_k}$ is the tangent linear model. The described incremental 4D-Var algorithm computes gradient of the cost function as

$$\nabla_{\chi} J(\chi) = \chi + \chi_g + \sum_{k=0}^N \mathbf{L}^T \mathbf{G}_k^T \mathbf{R}_k^{-1} (\mathbf{G}_k \mathbf{L}\chi - \mathbf{d}_k). \quad (7)$$

The control variable χ consists of two components:

$$\chi = [\chi_d, \chi_q]^T$$

The dynamical component, χ_d , controls the multivariate analysis of θ , u and v variables. It relies on the application of the wind-mass relationships of the equatorial linear waves to represent the coupling between the background errors in temperature and winds following Žagar et al. (2004b) and Ž2008. The transformation between χ_d and assimilation increment in physical space $\delta \mathbf{x}_d$ is defined as

$$\delta \mathbf{x}_d = \mathbf{L}_d \chi_d = F F_x^{-1} P_y^{-1} \mathbf{D}^{1/2} \chi_d, \quad (8)$$

where F_x^{-1} is the inverse 1D Fourier transform in the zonal direction, P_y^{-1} is the backward projection from the meridional eigenfunctions and $\mathbf{D}$ is a diagonal matrix with the spectral error variance densities for temperature and the wind components. The second component of the control vector χ_q performs the assimilation of moisture as presented in detail in the next subsection.

2.4. Data assimilation for the moisture variable

2.4.1. Control variable for moisture

Humidity is a physical variable which spans several orders of magnitude with its quantity decreasing approximately exponentially with elevation (and decreasing temperature). High variability of humidity poses a problem for variational data

assimilation, especially for the modelling of background error covariances. [Dee and da Silva \(2003\)](#) discussed the choice of variables for atmospheric moisture analysis and found that the best choice is relative humidity (and its variations, e.g. pseudo relative humidity):

$$\text{RH} = \frac{e}{e_s(T)} = \frac{R_d}{R_v} \frac{p}{e_{s0} \exp\left(\frac{L_c}{R_v} \left(\frac{1}{T_0} - \frac{1}{T}\right)\right)} q. \quad (9)$$

The relative humidity increment δRH can then be expressed by linearizing (9) around the latest guess relative humidity RH_g and latest guess temperature T_g :

$$\delta\text{RH} = -\frac{L_c}{R_v} \frac{\text{RH}_g}{T_g^2} \delta T + \frac{1}{q_s(T_g)} \delta q. \quad (10)$$

Here, $\delta T = \gamma \delta \theta$, where the constant coefficient $\gamma \approx 0.37$ is computed from the prescribed background potential temperature profile.

Furthermore, humidity is bounded from below, i.e. its mass can not be negative, and from above, meaning it can not exceed the maximal possible amount in the air parcel of certain temperature and at given atmospheric pressure. The humidity background error distributions close to these limits are asymmetric and non-Gaussian with long exponential tails away from the limit and clear edge near the physically plausible limit. In order to achieve relative humidity control variable with an approximately Gaussian error distributions as needed for the variational formulation of the analysis problem, we normalize the assimilation increment δRH by a standard deviation that is a function of the background relative humidity RH_b increased by half of the relative humidity assimilation increment ([Holm et al. 2002](#); [Gustafsson et al. 2011](#)):

$$\delta\text{RH}^n = \frac{\delta\text{RH}}{\sigma(\text{RH}_b + \delta\text{RH}/2)}. \quad (11)$$

The symmetrizing transform (11), which is an application of the central limit theorem, is therefore nonlinear, since it makes the moisture background error model dependent not only on background RH_b but on the increment δRH as well. We formulate the symmetrizing transform in a way that in the first incremental 4D-Var outer loop iteration, $\delta\text{RH} = 0$ in $\sigma(\text{RH}_b + \delta\text{RH}/2)$, but in the following iterations $\delta\text{RH} = \text{RH}_g - \text{RH}_b$, where RH_g denotes the latest guess relative humidity, i.e. the result of previous outer loop iteration. The inner loop cost function should remain quadratic, therefore a linearized transformation around the latest outer loop is performed, i.e. $\delta\text{RH}^n = \delta\text{RH}/\sigma(\text{RH}_g)$. After each outer loop iteration, supersaturation check is performed, which prevents the possibility of $\text{RH}_g + \delta\text{RH} > 100\%$.

The new variable δRH^n is assumed Gaussian but with a reduced variance near the physically possible limits. Figure 1 shows the empirically determined standard deviations of the specific humidity background errors as a function of background relative humidity increased by half of the the relative humidity assimilation increment, while Figure 2 presents the relative humidity background error probability distributions. Notice that this choice of control variable normalization poses a risk that observations with larger errors can be neglected near the limiting background humidity values.

2.4.2. Background error model for moisture

The background-error covariance model for the normalized relative humidity is the following. Relative humidity autocorrelations ρ are represented by the second-order auto regressive (SOAR) correlation function ([Gaspari and Cohn 1999](#)) which is better

conditioned than the Gaussian correlation model. The error correlation between two distinct points i and j is defined as

$$\rho_{ij} = \left(1 + 2\sqrt{\frac{\Delta x_{ij}^2}{L_x^2} + \frac{\Delta y_{ij}^2}{L_y^2}}\right) \exp\left(-2\sqrt{\frac{\Delta x_{ij}^2}{L_x^2} + \frac{\Delta y_{ij}^2}{L_y^2}}\right), \quad (12)$$

where $\Delta x_{ij} = x_i - x_j$ and $\Delta y_{ij} = y_i - y_j$. The large-scale correlations in the Tropics are expected to be anisotropic as the zonal flow dominates. Therefore, we empirically chose $L_x = 400$ km and $L_y = 200$ km in the zonal and meridional direction, respectively. To include two different lengthscales, we performed coordinate stretching of the otherwise isotropic SOAR correlation function.

The background error term for moisture is formulated in agreement with the rest of the assimilation model. The 2D Fourier transformation is used and the resulting covariance spectrum is shown in Figure 3. The transformation between the moisture control variable χ_{RH^n} and the moisture analysis increment $\delta\mathbf{x}_q$ is a two stage transform

$$\delta\mathbf{x}_{\text{RH}} = F\mathbf{V}^{1/2}F^{-1}\mathbf{D}_{\text{RH}^n}^{1/2}\chi_{\text{RH}^n}. \quad (13)$$

$$\delta\mathbf{x}_q = FB \begin{bmatrix} F^{-1}\delta\mathbf{x}_\theta \\ F^{-1}\delta\mathbf{x}_{\text{RH}} \end{bmatrix}. \quad (14)$$

The sequence of operators in (13) is the following. First, the matrix $\mathbf{D}_{\text{RH}^n}$ is a diagonal matrix with elements representing the normalized spectral auto-correlation model for normalized relative humidity and defines the horizontal structure of the increment. Then, the diagonal matrix $\mathbf{V}$ prescribes the grid-point background error variances for relative humidity and does the inverse transform of (11), i.e. it transforms δRH^n to δRH . When going from outer loop to inner loop, $\mathbf{V}$ is a function of RH_g and RH_b , while in the inner loop, it is only a function of RH_g . In the second stage (14), the obtained relative humidity increment $\delta\mathbf{x}_{\text{RH}}$ and potential temperature increment $\delta\mathbf{x}_\theta$ are used to compute the specific humidity increment $\delta\mathbf{x}_q$ via operator B in the grid-point space, such that:

$$\delta q = f(\text{RH}_g) \frac{L}{R_v} \frac{q_g}{T_g^2} \gamma \delta \theta + q_s(T_g) \delta\text{RH}. \quad (15)$$

Here, $f(\text{RH}_g)$ denotes the empirically determined regression function between δq and $\delta\theta$:

$$f(\text{RH}_g) = 4(\text{RH}_g - 0.8) \quad \text{if} \quad \text{RH}_g > 80\%$$

and 0 otherwise. Thus, the temperature increment would only affect the specific humidity close to saturation.

One way to illustrate the moisture background error covariance model is running a single observation experiment. Analysis increments due to two moisture observations with different background moisture but homogeneous potential temperatures (and thus also the saturation specific humidity) are presented in Figure 4. In case a), where background RH is 50 %, the background error is much larger than in case b), where the background RH is 95 %. Therefore, in the latter case, the analysis increment is small and most information comes from the background state.

2.5. Tangent linear model and adjoint model

The tangent linear model (TLM) $\mathbf{M}$ of the full nonlinear model is used to evolve assimilation increments in time over the length of the assimilation window in the inner loop of the incremental 4D-Var, i.e. $\delta\mathbf{x}_k = \mathbf{M}_{t_0 \rightarrow t_k} \delta\mathbf{x}_0$. We discuss the linearized equation of

(3) for the moisture perturbation δq as it includes on-off switches.

$$\frac{\partial \delta q}{\partial t} + \nabla \cdot (\delta \mathbf{v} q + \mathbf{v} \delta q) = 0. \quad (16)$$

The treatment of moist processes procedure, described in Subsection 2.2 is in the TL model supplemented by the following conditions:

- if q is set to certain value (either 0 or q_s), then $\delta q = 0$,
- if $\partial q / \partial t$ is set to 0, then $\partial \delta q / \partial t = 0$.

Precipitation is a diagnostic variable and is therefore not linearized and directly corrected during the assimilation process. Therefore, in case of linear dynamics, the nonlinear and the tangent linear model produce identical results. The adjoint model $\mathbf{M}^* = \mathbf{M}^T$ of TLM has been coded using line by line approach, i.e. the numerical adjoint of already discretized linearized equations has been developed. This is different from Ž2008 and earlier studies that applied the tangent-linear and adjoint of the continuous equations. The purpose of line by line approach is to adequately describe moist processes that include on-off switches (e.g. Zou 1997). Both linearized model and its adjoint have been tested to ensure their correctness.

2.6. Numerical setup for data assimilation

The setup for numerical experiments follows Ž2008. The domain is the tropical belt, spanning from 32°S to 32°N with the horizontal resolution of 1 degree and 360×65 points in the zonal and meridional directions respectively. The spectral representation includes $N_k = 119$ and $N_l = 23$ waves in the zonal and meridional direction, respectively.

The backward discrete adjoint integration is used to approximate the gradient $\nabla_{\chi} J(\chi)$ (7) of the incremental 4D-Var with the control vector χ consisting of roughly 10000 elements. The number of outer loop iterations is limited to a maximum of 10, while the number of the inner iterations in the descent minimization is limited to 20. The outer loop procedure is stopped if the cost function value at initial inner loop of outer loop $n + 1$ is larger than its initial inner loop value at outer loop n , i.e. if the cost function diverges. The inner loop procedure can be stopped earlier if the value of the Euclidean norm of the cost function gradient is decreased by a factor 1000.

The applied spectral background-error variances for dynamical variables are taken from the estimated short-range forecast errors of the ECMWF model using the ensemble approach (Žagar *et al.* 2013). This provides a realistic distribution of the background-error variances among various equatorial waves. The spectrum was derived in the global domain using the Hough harmonics and multiple vertical modes were used. As a consequence, resulting analysis increments in the Subtropics appear spurious due to the use of parabolic cylinder functions in the model. Because at this stage we focus on the development of the moist assimilation, no special effort has been devoted to tune the truncations of the applied variance spectra. Our discussion focuses on near-equatorial dynamics. The spectra for several equatorial modes are shown in Figure 5. The percentage of variance, associated with the equatorial Rossby waves is about 45%. In physical space, background error variances for the zonal and meridional wind are not the same. As discussed in Žagar *et al.* (2005), the difference comes from the anisotropic nature of the background error model.

An identical twin OSSE approach (e.g. Atlas *et al.* 1985) is used for 4D-Var experiments. The assimilation window length is mostly 12 hours and observations are updated every single hour (if not stated otherwise). Each simulated observation y is made noisy by adding zero-mean gaussian noise to the nature run value x_t , therefore $y = x_t + \mathcal{N}(0, \sigma_o^2)$. The observations error

standard deviations for each observed variable are the same as the background error standard deviations listed in Table 2, except for specific humidity whose observational error is 10 % RH.

3. Tropical moist dynamics and 4D-Var

Before going to the multi-observation experiments with the moist 4D-Var, we illustrate moist tropical dynamics in MAD and MADDAM systems by simple examples. First, the moist adjustment to a temperature perturbation in the intertropical convergence zone (ITCZ) is used to illustrate the role of moisture in the adjustment process on the equatorial β -plane. Then, a single temperature observation is assimilated at the end of a 12-hour 4D-Var window to discuss the coupling between dynamical variables and moisture in our 4D-Var system. Finally, a behaviour of the new background-error model for the moisture assimilation is shown using a single moisture experiment in a complex flow. The parameters for the experiments are listed in Table 1.

3.1. Moist adjustment to temperature perturbation in ITCZ

Figure 6 presents the model response to a Gaussian temperature perturbation centred at 8°N with amplitude of 1 K and halfwidth 5° and 1.5° in the zonal and meridional directions, respectively. The response in the unsaturated case (Figure 6a-c) substantially differs from the nearly saturated case defined by RH = 99 % in the initial state (Figure 6d-h). In the unsaturated case, the perturbation excites IG waves which effectively disperse potential energy on the advective time scale (Figure 6c). After 6 hours, there is a negligible cyclonic circulation (wind speed less than 0.1 m s⁻¹) in the region of the initial perturbation.

In the saturated case, the positive temperature perturbation induces convergence in the lower layer towards the perturbed area which advects moisture and leads to saturation and precipitation in the perturbed region (Figure 6d). The released latent heat is warming the mid-troposphere level triggering further convergence. For the selected parameters of the model, the process continues for about 6 hours. After 6 hours, cyclonic circulation (speeds around 2 m s⁻¹) begins to establish, whereas the precipitation continues (Figure 6e).

The gravity wave front is wider and has a larger amplitude in the saturated case as the latent heat increases the temperature perturbation. The amplitude of the front decays approximately inversely with the distance from the perturbation (the statement would be exact in case of f -plane approximation). The wave dispersion process is relatively slow as no significant diffusion and no friction are applied in the experiment. Note also that the gravity wave front amplitude is larger on the meridional than on the zonal side of the front in Figures 6a,d because the initial perturbation is steeper in the meridional direction.

As the radiated gravity waves reach the equator, (Figure 6f), they are channelled to move along it as seen in Figures 6g,h for +24 and +36 hour times, respectively. In the absence of saturation humidity and significant friction or diffusion, the equatorially trapped waves persist in the model for long time. Presented numerical result is in line with the 3D analytical study by Peagle (1978), who showed that tropical latent heating events with precipitation rates of a few centimeters per day can produce long lasting gravity waves with cross-isobaric flow of order of 1 m s⁻¹ outward to beyond 2000 km.

3.2. Single temperature observation in MADDAM

Now we assimilate a single temperature observation at the same location 8°N, 180°E at the end of the 12-hour assimilation window. Analysis solutions at the start, in the middle and at the end of the window are presented in Figure 7 for 2 cases.

Solutions for the case when the moisture content is low are shown in Figure 7a-c, while the case when the assimilation is carried out in nearly saturated (RH = 99.9 %) atmosphere is presented in Figure 7d-f. The observation is 2 K warmer than the background which is homogeneous and at rest.

The temperature increment at the observation time (Figure 7c,f) is elongated zonally which reflects the applied background error covariances. The associated wind increment consists of weak zonal easterlies similar to previous studies using the same multivariate background-error model (Žagar *et al.* 2005). The analysis increment at 12-hour has a bit larger amplitude in the saturated case as a result of the released latent heat during the internal adjustment process in 4D-Var. The dynamics producing this solution involves adjustment between the model and the background covariances included to the minimization process after the first evaluation of (7). During the adjustment, the IG waves are excited and appear to propagate towards the observation location at the end of the window. The associated moisture convergence in the saturated atmosphere leads to precipitation. Several precipitation maxima systems can be seen at 6 and 12 hours (Figure 7e,f). Precipitation is symmetric with respect to the observation location as it is associated with the gravity wave front.

3.3. Single humidity observation in MADDAM 4D-Var

Now we present an experiment involving a single humidity observation in a strongly meandering equatorial easterly flow (Figure 8). This experiment shows how 4D-Var implicitly evolves the moisture covariances in time and at the same time allows the moisture observation to affect other prognostic variables. In order to show this effect, a complex background flow is needed as presented in Figure 8a-c.

A single saturated humidity observation at the equator and longitude $\lambda = 192^\circ$ with the observational error $\sigma_o^{(q)} = 1.1 \text{ g kg}^{-1}$ ($\sim 10 \%$ RH) is assimilated at the end of the 12-hour assimilation window. The observation departure from the background is approximately $+2.4 \text{ g kg}^{-1}$ (22 % RH).

The assimilation produces a humidity increment at 12-hour time which is heavily stretched by the background flow (Figure 8f), influencing also the precipitation fields near the observation. The analysis increment adds just enough moisture to intensify the precipitation, which is very sensitive to small changes in initial moisture content. A single observation impacts also the temperature and the wind field. The positive temperature increment around the observation location is approximately 0.4 K at the start of the assimilation window (Figure 8g). This assimilation increment excites IG waves which can be recognised in the wind field (with speed around 0.5 m s^{-1}) in Figure 8h in the middle of the assimilation window. The temperature increment at the end of the window (Figure 8i) has an opposite sign from that at the start of the window which is a compensation effect for the latent heating triggered by the moisture added by the assimilation of the humidity observation.

This experiment also serves as a good example to examine the differences between tangent linear and nonlinear model (as shown by Janisková *et al.* 1999). The additional results, obtained by comparing $M(\mathbf{x}_g + \delta\mathbf{x}_a)$ with $M(\mathbf{x}_g) + \mathbf{M}\delta\mathbf{x}_a$, also indicate that the differences have largest amplitude in the precipitation areas, which are the main source of nonlinearities in the model. The switch on/off precipitation process in our model is extremely susceptible to even slightest differences in specific humidity and temperature fields. Due to latent heat release, these differences affect dynamics and may result in different wind analyses. In this experiment, the differences are small (less than 1 m s^{-1}).

4. Wind extraction from humidity observations in MADDAM 4D-Var

The MADDAM system is now used to assess how much humidity observations can contribute to the quality of the wind field analysis. Their impact is compared with that of other observation types. First, we discuss basic properties of the wind extraction from moisture data by using two simple cases. Then the results from strongly non-linear experiments with various observation types are shown.

4.1. Wind extraction from moisture data in a homogeneous flow

In the linear case without moisture sources and sinks, wind extraction from moisture observations might be considered an easy task, as the tangent linear and adjoint model accurately describe dynamics. The efficiency of the tracing is demonstrated using a nature run shown in Figure 9a. It describes a Gaussian moisture perturbation advected by a homogeneous easterly wind with speed of 5 m s^{-1} . The maximal amplitude of the relative humidity perturbation is 30 % (3.2 g kg^{-1}), while the surrounding relative humidity is 60 % (6.4 g kg^{-1}).

The background for the 4D-Var assimilation is at rest and the relative humidity field is 60 % everywhere. Note that in the nature run, the temperature and thus saturation specific humidity is decreasing away from the equator due to the applied mass/wind balance while these fields are constant in the background.

Perturbed observations of specific humidity are assimilated at every second or every fourth grid point in each direction in the region between 150° and 198° longitude and between 15°S and 15°N latitude. The frequency of update of observations within the 12-hour window and other assimilation settings vary with purpose to illustrate the importance of observing the spatial gradients of moisture to retrieve winds.

The resulting analyses are shown in Figure 9b-f at +6 hours (in the middle of the 12-hour assimilation window) for various experiments. First, all experiments reveal that the wind increment is constrained to the region with significant humidity innovations. In general, the larger the gradients of the humidity innovation in the zonal direction, the larger the wind analysis increment. Larger amplitude of specific humidity in the nature run also implies that absolutely equally erroneous observations provide relatively more accurate estimates of the spatial gradients of moisture field. According to (3), the moisture gradients are necessary to retrieve wind. For example, almost no wind is retrieved in the regions north/south of the tracer maxima, where the mixing ratio isolines are flow-parallel and do not change significantly in time as their curvature is low (Figure 9b). It can be also observed, that the wind information is better extracted in the area upwind of the moisture innovation maxima (between 170° and 185° longitude in Figure 9b-f) as the background error covariance matrix is implicitly stretched in the divergent wind flow, and vice versa.

The quality of the retrieved wind field depends on the accuracy of resolved moisture gradients which is determined by dynamics (i.e. moisture advection), details of the background-error model (e.g. the correlation length scale, the ratio of moisture and wind background errors), observation accuracy and their spatial and temporal density. Spatially dense and frequent observations tend to smooth out the noise from observation errors. Figure 9b shows the analysis result for the experiment where moisture data are assimilated in every second grid point every hour, a total of 13 times within the 12-hour window. We choose this one as a reference experiment. The assimilation result for the case with observations in every fourth grid point in each direction is presented in Figure 9c. In this case, the amount of observations is 25 % of that in reference experiment. Figure 9d shows the analysis obtained by the assimilation of dense (as in reference

experiment) but less frequent observations which are applied only at the beginning and at the end of the 12-hour assimilation window. The amount of observations is now only 15 % of that in reference experiment. In both cases, a significant reduction in the analysis quality is obvious, however the tracing effect is smaller in the first case with spatially sparse but frequent observations (Figure 9c) even though the total amount of moisture data is larger. This indicates that in the linear flow, the spatial resolution of observations is more important than the frequency of their updates to trace winds.

Figure 9e shows the analysis from the assimilation of humidity observations with twice smaller observation and background errors than in the experiments shown in Figure 9b. In this case, smaller moisture observation errors lead to a more accurate estimate of the moisture gradients. Furthermore, the ratio of moisture and wind background error standard deviations $\sigma_b^{(q)}/\sigma_b^{(u)}$ defines the wind increment. For small (large) ratio, the wind increment increases (decreases). The result is in line with that of Allen et al. (2013), who analytically studied wind tracing from perfect observations in a simple 1D case using 4D-Var. The analysis solution is improved by lengthening the assimilation window to 24-hour (Figure 9f) as expected for linear dynamics.

Discussion becomes more complicated when the assimilation is performed in the saturated atmosphere. An example is shown in Figure 10. The nature run in this case (Figure 10a-c) is similar to the previous case, but the moisture perturbation is meridionally more stretched and its amplitude is increased to RH=40 %. With relative humidity in the surroundings equal to 60 % as before, a part of the domain has become saturated. Furthermore, an easterly balanced zonal wind perturbation with a small amplitude 1 m s^{-1} , which coincides with that of moisture perturbation, is superposed on the mean easterly flow to make the flow slightly convergent (Figure 10a). The wind convergence initiates moisture inflow and leads to condensation after 6 hours (Figure 10b). The released latent heat triggers further convergence and leads to even increased condensation and precipitation after 12 hours (Figure 10c). The background for the assimilation experiment is a fluid at rest similar to in previous experiment and observations of specific humidity are assimilated at every second grid point in each direction and updated every hour.

The 12-hour analysis result is shown in Figure 10d-f. It suggests that the wind information can be extracted from humidity observations also in saturated regions. This result can be explained by a closer look at the underlying dynamics. Condensation acts as a moisture sink in the model and should thus pose a problem for the wind extraction. However, in saturated regions, the specific humidity is locally approximately conserved (only small variations due to changes in temperature and otherwise equal to saturation specific humidity) as the amount of condensed moisture is equal to net moisture influx from the surroundings (the actual moisture sink area). If this moisture deficit is observed but not resolved by the 4D-Var forward model dynamics, i.e. the model does not know about nearby condensation area, then the internal adjustment acts in a way to decrease moisture content by imposing divergent flow. The divergent flow in the sink areas around the condensation area results in a net convergent flow in the condensation area. Therefore, even if the model initially fails to predict condensation, in well-observed humidity regions with accurate background temperature, the perfect-model 4D-Var succeeds to correct dynamics.

The experiments in Figure 11a-c reveal the importance of accurate temperature field for wind tracing in saturated environment. Figure 11a shows an analysis at +12 hours obtained by assimilation of exactly the same humidity observations as in the experiment shown in Figure 10d-f, except the potential temperature background value is now 333 K instead of 335 K,

that is 2.3 K lower than in the nature run (in condensation area). As the saturation humidity depends on the temperature, a too low background temperature results in a lower saturation threshold and thus the excessive humidity gets released. A strongly convergent flow occurs due to local latent heat release. To account for the excessive moisture sink, the internal adjustment imposes even additional convergence in order to minimize the distance of the 4D-Var trajectory to moisture observations.

If the background temperature is too high, e.g. 337 K, then the saturation threshold is elevated and almost no condensation occurs (Figure 11b). The 4D-Var internal dynamics now imposes a slightly divergent wind field in the analysis which acts to decrease humidity instead of condensation.

Figure 11c shows an analysis at +12 hours if the humidity observations are combined with potential temperature observations, which correct too cold temperature background (333 K, same as in the experiment with analysis shown in Figure 11a). Comparing the two, the equatorial wind analysis is now closer to the truth and also the precipitation is reduced. Temperature observations additionally constrain the higher-latitude wind field through the geostrophic adjustment.

4.2. Nonlinear wind-moisture interactions in 4D-Var

To study the wind tracing in a highly nonlinear flow we use the same nature run that was used as a background in the single humidity observation experiment in the previous section (Figure 8a-c). The 12-hour evolution of the background field is shown in Figure 12 and differences between nature run and the background after 6 hours of simulation are shown in Figure 13a. Simulated observations are located in every second grid point in each direction in the region between 160° and 236° longitude and between 18°S and 18°N latitude, making a total of 741 observations assimilated at a single time instance. The observations are assimilated every hour.

The wind analysis increment at +6 hours obtained by assimilation of moisture observations (experiment denoted *q*) is shown in Figure 13b. The result reveals the basic property of the wind extraction from tracers; that is, winds are better extracted in areas with stronger humidity gradients in the flow direction. For example, the gradients in the humidity innovation (not shown) around point $\lambda = 182^\circ$, $\phi = 2^\circ$ are very weak, and thus almost no wind is extracted there. The most important result of this experiment is that the wind can be extracted from moisture observations even in saturated areas with highly nonlinear dynamics, such as in the convection region around $\lambda = 185^\circ$, $\phi = 5^\circ\text{S}$, but note that the zonal gradients in humidity innovation are substantial in this region. However, the extracted wind information is much lower than in the less nonlinear experiments presented in Subsection 4.1. The ability to extract at least some winds even in case of highly nonlinear flow is also partly the consequence of using incremental 4D-Var, which brings the full nonlinear trajectory gradually closer to observations in each subsequent outer loop iteration. Thus, for small increments, the TL hypothesis is valid almost everywhere.

The analysis root-mean-square-error (RMSE) in the middle of the 12-hour assimilation window is shown in Figure 14. For the zonal wind the error is approximately 2.8 m s^{-1} and for meridional wind 3.1 m s^{-1} . That is about 20 % smaller than for the background RMSE (3.6 m s^{-1} and 3.7 m s^{-1} for the zonal and meridional wind, respectively). Note that these verification scores apply to the domain $\lambda \in [169^\circ, 217^\circ]$ and $\phi \in [15^\circ\text{S}, 15^\circ\text{N}]$ presented in Figure 14.

The wind extraction from moisture observations can be compared to the result when temperature observations are assimilated (*T*, Figure 13c). The temperature observations provide

totally a slightly better wind analysis than specific humidity data (**q**). The RMSE is now 3.1 m s^{-1} for the zonal wind and 2.4 m s^{-1} for meridional wind. That is in total around 5 % decrease in RMSE in comparison to the experiment **q** with moisture data. According to Figure 14, adding humidity to temperature observations (experiment denoted **Tq**) further decreases the analysis RMSEs by around 25 % compared to the moisture-only case. The associated analysis increment is shown in Figure 13h for 12-hour window.

These values can not be directly compared to the NWP case as our system applies a multivariate mass-wind assimilation and the identical twin 4D-Var experiments. But, even in our case the 4D-Var assimilation of mass data does not provide wind analyses with a quality comparable to that obtained when wind observations are assimilated. This is clearly shown by RMSEs from experiments which use wind data (Figure 14). The added wind information from combined observations of temperature and moisture (**Tq**) is around 60 % of that from winds alone (**uv**). Further, we checked wind tracing as a function of assimilation window length. For 6-hour window (Figure 13g), the wind adjustment is worse, while the 24-hour window (**Tq24**, Figure 13i) generally improves the analysis. In this case, the wind RMSE is almost as low as for the **Tuv** case. However, lengthening the assimilation window can degrade the analysis near the precipitation areas as seen around $\lambda = 208^\circ$, $\phi = 9^\circ\text{S}$, where some spurious wind increments arise as the TL hypothesis is locally violated.

When observations of all model variables are assimilated (**Tuvq**), the analysis RMSEs are smallest (approximately 0.6 m s^{-1}) and wind analysis increments are closest to the true increments (Figure 13f). Figure 14 also suggests that adding moisture data on top of wind and temperature observations (**uv** and **Tuv** versus **Tuvq**) greatly improves the wind analysis. However, this applies to our perfect-model 4D-Var and to the case in which moist processes in the non-linear and TL model are very similar. An interesting feature of RMSEs (not shown) is that adding moisture observations to winds leads to better temperature analysis than if only wind data is assimilated. Similarly, humidity on top of temperature data results in better temperature analysis than temperature alone. Additional ensemble experiments are needed to establish the significance of this result.

5. Conclusions and Outlook

We presented MADDAM, a new 4D-Var data assimilation system based on a spectral numerical prediction model of moist dynamics in the Tropics. MADDAM is a follow-on of previous studies of tropical data assimilation and the role of equatorial waves with dry prognostic equations (Žagar *et al.* 2008, and references therein). The additions to the system include a new treatment of moisture in the forward model, a new background-error model for moisture, TL and AD code of discretized equations and the incremental approach to 4D-Var to account for the nonlinearities introduced by moist processes. The formulation of the moisture control variable resembles the NWP approach with the variable transform which gaussianizes the humidity background errors.

The core purpose of the MADDAM system is to study the interactions between humidity and winds in the process of 4D-Var internal adjustment. Furthermore, MADDAM is applied for identical twin OSSEs to study the extraction of wind information from time series of moisture data in the perfect-model 4D-Var assimilation. Following the NWP practice, the experiments mostly apply the 12-hour assimilation window. The complex 4D-Var dynamics in a saturated atmosphere is illustrated by a single humidity observation experiment which revealed the flow-dependent aspects of the moisture assimilation and the impact of moisture observation on other variables. In comparison to

wind extraction from passive tracers, which has been extensively studied (e.g. Daley 1996; Allen *et al.* 2013), moisture cannot be regarded as a pure tracer as it actively impacts dynamics. The wind tracing from moisture data in 4D-Var strongly depends on the spatial density and accuracy of the observations as well as the frequency of observation update and assimilation window length. The first two are needed to describe the spatial gradients of moisture and the last two provide information about the advection. This behaviour is illustrated by the assimilation of moisture observations in a homogeneous wind environment. In the case with linear flow, the spatial density of observations is more important than their update frequency. The opposite holds in nonlinear flow. The dynamical details of the benefits of longer assimilation windows on wind tracing were also discussed.

We have shown that in an atmosphere near saturation, the moisture data alone do not constrain the flow well. Thus, it is essential to have a good information about the temperature field along with humidity observations. The purely dynamical reason is that the initiation and intensity of condensation process are susceptible to slightest changes in saturation humidity which is temperature dependent. The released latent heating immediately impacts both large-scale and gravity wave dynamics as seen from the experiment describing adjustment to temperature perturbation. Thus, if the humidity observations initiate too strong condensation due to inaccurate temperature and saturation humidity, the 4D-Var internal adjustment acts in a way to compensate for the humidity sink by imposing convergent flow in order to constrain the 4D-Var trajectory closer to humidity observations.

The comparison between wind analyses produced by the assimilation of various data types and their combinations shows that the assimilation of moisture observations can retrieve winds in areas with significant humidity gradients even if the flow is highly nonlinear, e.g. in the presence of precipitation. The combination of moisture and temperature observations leads to much better analyses than if each is assimilated separately. However, this result is obtained by a balanced tropical data assimilation based on the mass-wind coupling characteristic of the equatorial eigenmodes (Žagar *et al.* 2005), the approach which is expected to provide the most beneficial use of mass-field data in the Tropics. The result is thus not entirely representative of the current operational NWP data assimilation practices (e.g. Bonavita and Holm 2016). Nevertheless, the operational NWP has reported a significant impact of moisture data in 4D-Var (Geer *et al.* 2008). Simplified experiments made possible using our MADDAM system can be utilized to explain the associated 4D-Var processes and to diagnose the mutual impact of different observational data.

The described modelling framework can be ported to the mid-latitudes by replacing the background error covariance model for dynamical variables with one similar to the spectral limited-area models HIRLAM and ALADIN; that is, the 2D Fourier representation of the background errors for winds and temperature variables. In this way, balance issues in presence of moisture and the role of inertio-gravity waves in convective-scale data assimilation can be addressed.

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Table 1. Model parameters used for the experiments.

Reference depth	H	4.5 km
Lapse rate	$d\theta_0/dz$	4 K km ⁻¹
Mid-level potential temp.	θ_0	333 K
Horizontal phase speed	$c = NH$	48 m s ⁻¹
Deformation radius	$R_e = \sqrt{c/(2\beta)}$	1030 km
Saturation spec. humidity	q_s	~10.5 g kg ⁻¹

Table 2. Background error standard deviations used in data assimilation experiments.

$\sigma_b^{(u)}$	$\sigma_b^{(v)}$	$\sigma_b^{(T)}$	$\sigma_b^{(\text{RH})}$
3.2 m s ⁻¹	2.95 m s ⁻¹	1.1 K	5 - 15 % RH

Figure 1. Analytical model of the relative humidity background error standard deviations as a function of the background relative humidity (full line) and background relative humidity increased by half of the relative humidity increment (dashed line).

Figure 2. Probability distributions of the normalized relative humidity background error as a function of background relative humidity increased by a half of the increment. Vertical lines denote the physically possible limits (0 and 100 % relative humidity in the model).

Figure 3. Spectral variance density model for relative humidity background errors based on the anisotropic SOAR correlation model (12) and the comparison with the Gaussian correlation spectrum.

Figure 4. 3D-Var analysis increment for a single humidity observation with $\Delta RH = 5\%$ higher relative humidity than the background and observation standard deviation $\sigma_o = 10\%$. The background relative humidity is a) 50 %, and b) is 95 %. Solid isolines for the relative humidity increment are shown every 1%. The dashed lines indicate the background relative humidity contours.

Figure 5. Spectral variance density of the short range forecast errors derived from the ECMWF ensemble data assimilation. Different equatorial mode types are shown separately as indicated in the legend: equatorial Rossby mode (Rossby), Kelvin mode, eastward- and westward-moving inertia-gravity wave modes (EIG and WIG), eastward- and westward-moving mixed Rossby-gravity modes (EMRG and WMRG).

Figure 6. Adjustment to the potential temperature perturbation in (a-c) unsaturated atmosphere, (d-h) saturated atmosphere at (a,d) 3 hours, (b,e) 6 hours, (c,f) 12 hours, (g) 24 hours and (h) 36 hours. Blue-red colourbar denote negative and positive potential temperature perturbations with spacing 0.05 K. Three-hour cumulative precipitation (mm) is shown by the rainbow colourbar.

Figure 7. Analysis solution obtained by the 4D-Var assimilation of a single temperature observation at 8°N supplied at the end of the 12-hour assimilation window in (a-c) unsaturated atmosphere and (d-f) saturated atmosphere. The observation is 2 K warmer than the background. Increments are shown at (a,d) initial time, (b,e) 6 hours and (c,f) 12 hours. Contours denote potential temperature increment, while rainbow colours show 3-hour cumulative precipitation.

Figure 8. Background and analysis increments at (a,d,g) the beginning, (b,e,h) in the middle and (c,f,i) at the end of 12-hour assimilation window for a single humidity observation experiment at the end of the assimilation window. Observation location is denoted by a red dot. (a-c) background wind, specific humidity and 6-hour cumulative precipitation[mm]. (d-f) moisture analysis increments and cumulative 6-hour analysis precipitation. (g-i) wind and potential temperature analysis increments.

Figure 9. Wind and specific humidity fields at +6 hours in (a) nature run and (b-f) analysis solutions at hour 6 within the 12-hour 4D-Var assimilation; for specific humidity observations available (b) every hour at every second grid point in each direction, (c) every hour at every fourth grid point, (d) at the beginning and at the end of the assimilation window at every second grid point, (e) same as (b) but for background and observation errors reduced by half, (f) same as (b) but for 24-hour window. Black lines are isotaches with spacing 1 m s^{-1} .

Figure 10. (a-c) Nature run and (d-f) analysis solutions for the wind tracing experiment. Winds, specific humidity and 6-hour cumulative precipitation are shown (a,d) at the beginning, (b,e) in the middle and (c,f) at the end of 12-hour assimilation window. Black line in (a) is 5.25 m s^{-1} isotach, indicating wind perturbation.

Figure 11. Analysis solution at the end of 12-hour 4D-Var window (same as Figure 10f), but for (a) background potential temperature equal 333 K, (b) 337 K, and (c) 333 K and humidity observations combined with temperature observations.

Figure 12. Background for wind retrieval experiments at (a) time 0, (b) 6 hours and (c) 12 hours. Colours denote specific humidity and 6-hour cumulative precipitation (mm).

Figure 13. (a) Differences between nature and background winds at +6 hour. 4D-Var wind analysis increments at the same time from (b) specific humidity (q) observations, (c) temperature (T) observations, (d) zonal wind observations (u), (e) total wind observations (uv), (f) temperature, wind and humidity ($Tuvq$) observations. (g-i) show analysis increments for combined observations of temperature and specific humidity (Tq) for (g) 6-hour assimilation window, (h) 12-hour and (i) 24-hour window. The observations were updated every hour and available at every second grid point in each direction in subregion $(\lambda, \phi) \in [160^\circ, 236^\circ] \times [-18^\circ, 18^\circ]$. Wind isotachs are shown every 2 m s^{-1} .

Figure 14. Root mean square errors of the 4D-Var analysis in the middle of the 12-hour window for zonal wind, meridional wind and potential temperature for experiments with various observations: humidity (**q**), temperature (**T**), temperature and humidity (**Tq**), zonal wind (**u**), both wind components (**uv**), observations of all three dynamical variables (**Tuv**) and observations of all four model variables (**Tuvq**). Experiment with 24-hour window and observations of temperature and humidity is denoted **Tq24**.