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CONTRIBUTIONS TO A NONCOMMUTATIVE REAL ALGEBRAIC GEOMETRY

Doctoral thesis

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PRISPEVKI K NEKOMUTATIVNI REALNI ALGEBRAIČNI GEOMETRIJI

Doktorska disertacija

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Zahvale

Zahvaljujem se mentorju Jaku in somentorju Igorju. Jaka me je v diplomskem delu navdušil nad področjem realne algebraične geometrije, v magistrskem in doktorskem delu pa mi je pomagal poglobiti poznavanje tega področja. Igor je bil moj mentor med večmesečnim gostovanjem na Univerzi v Aucklandu in mi prek zanimivih raziskovalnih vprašanj predstavil področje proste realne algebraične geometrije.

Hvala Tadeji, družini in vsem, ki ste me podpirali tekom celotnega študija.

Abstract

We study Positivstellensätze from noncommutative real algebraic geometry. Of these, we focus on two specific ones. A version of the matrix Fejér-Riesz theorem characterizes positive semidefinite matrix polynomials on the real line. This characterization has already been extended from the real line to a disjoint union of finitely many closed intervals in the case of scalar polynomials and to a single closed interval in the case of matrix polynomials. Our first interest in this thesis is to figure out what can be said in the case of matrix polynomials and a disjoint union of finitely many closed intervals. Algebraic certificates of positivity for noncommutative matrix polynomials on matrix convex sets, such as the solution set of a linear matrix inequality (LMI), have recently attracted much attention among real algebraic geometers. In the case of LMIs many certificates are known. Since every closed matrix convex set containing the origin is the solution set of a linear operator inequality (LOI), this attracts the second interest of this thesis which is to extend the certificates from matrix to operator polynomials.

Our main result referring to the first problem is a denominator-free characterization in the case of a compact union, called a Compact Positivstellensatz. The technique in the proof is the adaptation of Schur complements and eliminating the denominators with the help of known results for scalar polynomials. We also construct counterexamples for the extension of the characterization to almost all non-compact unions. By developing the connections between matrix polynomials and Laurent matrix polynomials we obtain the matrix Positivstellensatz on a disjoint union of finitely many closed arcs in the unit complex circle and finally, using this result we come to a Non-compact Positivstellensatz for a non-compact union of finitely many closed intervals in the real line using only simple denominators.

Referring to the second problem our first result is an algebraic characterization for the domination of the solution sets of monic LOIs, called a Linear Positivstellensatz. The techniques used are complete positivity and the theory of operator algebras. We provide examples which show that the monicity assumption is necessary. As a consequence we also obtain the description of the polar dual of the LOI. Next we focus on the question of the equality of the solution sets of two LOIs which turns out to be a harder one. We present the answer for LOIs with compact operator coefficients, called a Linear Gleichstellensatz. Namely, under some minimality assumption, the LOIs are unitarily equivalent. The idea is to understand the unital C^*-algebras generated by the coefficients and $*$-isomorphisms between them. We show by examples that the answer does not extend to arbitrary LOIs. Finally, we establish a Convex Positivstellensatz which characterizes matrix polynomials positive semidefinite on the solution set of a LOI and show that in the univariate case it extends to operator polynomials.

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Keywords: positive polynomials; quadratic module; Positivstellensatz; free positivity; linear operator inequality; spectrahedron; noncommutative polynomial; real algebraic geometry.

Povzetek

Študiramo Positivstellensatze iz nekomutativne realne algebraične geometrije. Med njimi se osredotočimo na posebna primera. Verzija matričnega Fejér-Rieszovega izreka karakterizira pozitivno semidefinitne matrične polinome na realni osi. Ta karakterizacija je že bila razširjena iz realne osi na disjunktno unijo končno mnogo zaprtih intervalov v primeru skalarnih polinomov in na primer enega samega zaprtega intervala v primeru matričnih polinomov. Prvi cilj tega dela je ugotoviti, kaj se da povedati za matrične polinome in disjunktno unijo končno mnogo zaprtih intervalov. Algebraična zagotovila za pozitivnost nekomutativnih matričnih polinomov na matrično konveksnih množicah, kot je množica rešitev linearne matrične neenakosti, so nedavno pritegnila pozornost med realnimi algebraičnimi geometri in veliko je že bilo narejenega. Ker je vsaka zaprta matrična konveksna množica, ki vsebuje izhodišče, množica rešitev linearne operatorske neenakosti (LON), to motivira drugi cilj tega dela, ki je razširitev zagotovil za pozitivnost iz matričnih na operatorske polinome.

Naš glavni rezultat pri študiju prvega problema je karakterizacija brez imenovalcev v primeru kompaktnih unij, ki se imenuje Kompaktni Positivstellensatz. Tehnika v dokazu je predelava Schurovih komplementov in odprava imenovalcev z uporabo znanih rezultatov za skalarne polinome. Konstruiramo tudi protiprimere za razširitev te karakterizacije na skoraj vse nekompaktne unije. S študijem povezav med matričnimi polinomi in Laurentovimi matričnimi polinomi izpeljemo matrični Positivstellensatz na disjunktni uniji končno mnogo zaprtih lokov na enotski kompleksni krožnici. Z uporabo tega rezultata nato izpeljemo Nekompaktni Positivstellensatz za nekompaktno unijo zaprtih intervalov na realni osi, v katerem nastopajo le enostavni imenovalci.

Naš prvi rezultat pri študiju drugega problema je algebraično zagotovilo za dominacijo množic rešitev eničnih LONov, ki se imenuje Linearni Positivstellensatz. Glavni uporabljeni tehniki sta popolna pozitivnost in teorija operatorskih algeber. Predstavimo tudi primere, ki pokažejo, da je predpostavka eničnosti potrebna. Opišemo tudi polaro LONa. Nato se osredotočimo na vprašanje enakosti množic rešitev dveh LONov, kar se izkaže za težji problem. Predstavimo odgovor za LONe s kompaktnimi operatorskimi koeficienti, ki se imenuje Linearni Gleichstellensatz. Ta pove, da sta pri predpostavki minimialnosti LONa unitarno ekvivalentna. Ideja je razumeti unitalne C^*-algebre, generirane s koeficienti LONa, in $*$-homomorfizme med njimi. S primeri pokažemo, da se izrek ne razširi na poljubne LONe. Na koncu izpeljemo Konveksni Positivstellensatz, ki karakterizira nekomutativne matrične polinome, pozitivno semidefinitne na množici rešitev LONa. V primeru ene spremenljivke pa ga razširimo na operatorske polinome.

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Ključne besede: pozitivni polinomi; kvadratni moduli; Positivstellensatz; prosta pozitivnost; linearna operatorska neenakost; spektraeder; nekomutativni polinom; realna algebraična geometrija.

Contents

Chapter 1

Introduction

This thesis studies Positivstellensätze for matrix and operator polynomials. The name Positivstellensatz refers to an algebraic certificate for a given polynomial p to have a positivity property on a given closed semialgebraic set K. The problems of finding such certificates belong to the field of real algebraic geometry. Given an arbitrary K it can be very hard to find an optimal certificate, especially if p is not strictly positive on K but is only nonnegative. In the thesis we study two different types of certificates regarding the set K and a noncommutative polynomial p which we would like to represent, both belonging to noncommutative real algebraic geometry.

Two equivalent versions of the matrix Fejér-Riesz theorem characterize positive semidefinite $n \times n$ matrix polynomials on the real line $\mathbb{R}$ and on the complex unit circle $\mathbb{T}$. In the case $n = 1$ the extensions of this result to arbitrary closed semialgebraic sets $K \subseteq \mathbb{R}$ are well-understood by the works of Kuhlmann, Marshall [KM02] and Scheiderer [Sch03], while for arbitrary $n \in \mathbb{N}$ and K being a closed interval by the work of Dette and Studden [DS02]. The first main problem of the thesis is to study the extension of these results to matrix polynomials for arbitrary $n \in \mathbb{N}$ and arbitrary closed semialgebraic set $K \subseteq \mathbb{R}$.

Establishing Positivstellensätze for polynomials, positive semidefinite on matrix convex sets, such as the solution set of a linear matrix inequality (LMI), is the domain of free real algebraic geometry. A polynomial is evaluated on a tuple of matrices and the evaluation is a matrix or an operator. For LMIs and polynomials whose coefficients are self-adjoint matrices, various Positivstellenstellensätze were established by Helton, Klep and McCullough in a series of papers, e.g., [HKM12, HKM13b, HKM16b]. Since by [EW97] every closed matrix convex set containing the origin is a matrix solution set of a linear operator inequality (LOI), this motivates the second main problem of the thesis which is to study the extensions of their results to LOIs and polynomials with operator coefficients.

The thesis is based on the results presented in [Zal15 arxiv, Zal16, Zal17].

Positivstellensätze for univariate matrix polynomials

In Chapter 2 we study univariate matrix polynomials positive semidefinite on semialgebraic sets.

We start Section 2.1 with a well-known characterization of nonnegative polynomials on the real line $\mathbb{R}$ (see Theorem 2.1.1) and its equivalent version for Laurent polynomials nonnegative on the unit complex circle $\mathbb{T}$, called the *Fejér-Riesz theorem* (see Theorem 2.1.2). On one side both theorems have extensions to arbitrary closed semialgebraic sets K by the results of Kuhlmann and Marshall from 2002 [KM02] and Scheiderer from 2003 [Sch03]. Namely, the appropriate algebraic structures in the characterizations are a *quadratic module* M_S and a *preordering* T_S from real algebraic geometry (RAG), generated by a special finite set S of polynomials called the *natural description* of a semialgebraic set K. In the case of a compact semialgebraic set K one can replace S by any set $\widetilde{S}$ satisfying the conditions of a *saturated description* of K. Quadratic modules containing all nonnegative polynomials on K are called *saturated*. On the other hand it has been well-known from the middle of the 20th century that Theorems 2.1.1 and 2.1.2 can be generalized to operator polynomials by replacing squares of polynomials by hermitian squares of operator polynomials [Ros58]. These two aspects motivate our research of the extensions of Theorems 2.1.1 and 2.1.2 from usual polynomials to matrix polynomial positive semidefinite (psd) on arbitrary closed semialgebraic sets. By the results of Dette and Studden from 2002 [DS02], the results of Kuhlmann, Marshall and Scheiderer extend to matrix polynomials in the case K is a single interval (bounded or unbounded) which can be derived from the matrix version of Theorem 2.1.1. However, this technique does not work in the case K is a disjoint union of several intervals. The question what is true for such K is the content of the remaining sections in Chapter 2.

In Section 2.2 we prove the main result of this chapter which characterizes matrix polynomials positive semidefinite on compact semialgebraic sets in the real line. Let $\mathbb{C}[x]$ (resp. $\mathbb{R}[x]$) be the set of complex (resp. real) polynomials and $M_n(\mathbb{C}[x])$ the set of all $n \times n$ matrices over $\mathbb{C}[x]$ with conjugated transpose as the involution $*$. Given a finite set $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ the *closed semialgebraic set associated to S in $\mathbb{R}$* is defined by

$$K_S = \{x \in \mathbb{R}\colon g_j(x) \geq 0,\ j = 1, \ldots, s\},$$

and the *n-th matrix quadratic module generated by S* in $M_n(\mathbb{C}[x])$ by

$$M_S^n := \{G_0^* G_0 + G_1^* G_1 \cdot g_1 + \ldots + G_s^* G_s \cdot g_s\colon G_j \in M_n(\mathbb{C}[x]),\ j = 0, \ldots, s\}.$$

We denote by $\mathrm{Pos}^n_{\succeq 0}(K_S)$ the set of all $n \times n$ matrix polynomials F, such that $F = F^*$ and $F(x_0)$ is a positive semidefinite matrix for every $x_0 \in K_S$. The set M_S^n is called *saturated* if $M_S^n = \mathrm{Pos}^n_{\succeq 0}(K_S)$.

Theorem A (Compact Positivstellensatz; see Theorem 2.2.1)**.** *Suppose $S \in \mathbb{R}[x]$ is a finite set of real polynomials such that the semialgebraic set $K_S \subseteq \mathbb{R}$ is compact. The n-th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ if and only if S satisfies the following two conditions:*

(a) For every left endpoint x_j of K_S there exists $g \in S$ such that $g(x_j) = 0$ and $g'(x_j) > 0$.

(b) For every right endpoint y_j of K_S there exists $h \in S$ such that $h(y_j) = 0$ and $h'(y_j) < 0$.

The proof of Theorem A has three main ingredients, i.e., the $n = 1$ case which is the result of Scheiderer (see Corollary 2.1.13), the so called *"hF-proposition"* (see Proposition 2.2.2) and eliminating h in "hF-proposition". The"hF-proposition" is established in Subsection 2.2.1 by the adaptation of Schur complements. Finally, to eliminate h from "hF-proposition" one needs the $n = 1$ case of Theorem A and is presented in Subsection 2.2.2.

In Section 2.3 we show that Theorem A does not extend from compact semialgebraic sets to unbounded closed semialgebraic sets since there exist counterexamples. Given a finite set $S \subset \mathbb{R}[x]$ we define the *n-th matrix preordering generated by S* by $T_S^n := M^n_{\prod S} \subseteq M_n(\mathbb{C}[x])$ where $\prod S$ stands for the set of all finite products of different elements from S.

Theorem B (see Theorem 2.3.1)**.** *Suppose $K \subseteq \mathbb{R}$ is an unbounded closed semialgebraic set and $K_1, \dots, K_r$ its connected components. Let K satisfy either of the following cases:*

Case 1: *There are $i, j \in \{1, \dots, r\}$ such that K_i and K_j have a non-empty interior, K_i is bounded and K_j is unbounded.*

Case 2: *There are $r \geq 3$ components where exactly one of them is an unbounded interval and all the others are points.*

Case 3: *There are $r \geq 4$ components where exactly two of them are unbounded intervals and all the others are points.*

If $S \subset \mathbb{R}[x]$ is a finite set such that $K_S = K$, then the 2-nd matrix preordering T_S^2 is not saturated.

In Section 2.4 we focus on the question of degree bounds in a saturated matrix preordering T_S generated by the natural description S of a semialgebraic set K. We prove that if K is a disjoint union of two unbounded intervals, then the degree bounds are optimal (see Theorem 2.4.2), i.e., the degrees of the summands can be bounded by the degree of the matrix polynomial they represent. We also prove that if K is finite, then the degrees of the summands can be bounded by the maximum between the degree of the matrix polynomial they represent and one less than the number of points in K; moreover, if there are at least four points in K, then the degree of the matrix polynomial is not sufficient (see Corollary 2.4.4 and Example 2.4.5). However, for other sets K the question of degree bounds remains open.

In Section 2.5 the focus is on complex Laurent matrix polynomials positive semidefinite on a finite disjoint union of closed arcs in the unit complex circle $\mathbb{T}$. We establish an analogous result to Theorem A for such polynomials which generalizes the matrix Fejér-Riesz theorem; see Theorem 2.5.4. To prove Theorem 2.5.4 we first establish connections between semialgebraic sets in $\mathbb{T}$ and $\mathbb{R}$, their descriptions and the corresponding matrix polynomials. This is done in Subsection 2.5.1 with Möbius transformations being the main tool. Then, in Subsection 2.5.2, these connections are used to establish an analog of the "hF-proposition" for matrix Laurent polynomials (see Proposition 2.5.5). Finally, to eliminate the denominator we use a result of Scheiderer (see Proposition 2.5.6).

In Section 2.6 we prove Positivstellensatz for matrix polynomials positive semidefinite on unbounded closed semialgebraic sets.

Theorem C (Non-compact Positivstellensatz; see Theorem 2.6.1). *Suppose $K \subset \mathbb{R}$ is a proper unbounded closed semialgebraic set and S the natural description of K. Then, for any matrix polynomial $F \in M_n(\mathbb{C}[x])$ such that $F = F^*$, the following are equivalent:*

(1) F is positive semidefinite in every $x_0 \in K$.

(2) For every point $w \in \mathbb{C} \setminus K$ there exists $k_w \in \mathbb{N} \cup \{0\}$ such that $|x-w|^{2k_w} \cdot F \in M_S^n$.

(3) There exists $k \in \mathbb{N} \cup \{0\}$ such that $(1+x^2)^k \cdot F \in M_S^n$.

In Subsection 2.6.1 we establish inverse connections to the connections established in Subsection 2.5.1. This enables us to use Theorem 2.5.4 in the proof of Theorem C. The denominators in Theorem C appear since there do not exist optimal degree bounds in Theorem 2.5.4. An interesting result of Theorem C is the fact that there exist uniform denominators up to the exponent for all sets K and all matrix polynomials F.

Finally, in Section 2.7 we explain briefly how Theorem A extends to curves in $\mathbb{R}^d$ (see Theorem 2.7.5).

Positivstellensätze on matrix convex sets

In Chapter 3 we study algebraic certificates of positivity for noncommutative (nc) operator polynomials on matrix convex sets.

In Section 3.1 we introduce the definitions and present known results for linear matrix pencils and matrix polynomials. These results have been established by Helton, Klep and McCullough in a series of papers. Our motivation is to generalize their results to linear operator pencils and operator polynomials.

The main result of Section 3.2 is the algebraic characterization of the inclusion of the solution sets of linear operator inequalities (see Theorem D below). Let $\mathscr{H}$ be a real Hilbert space and $B(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$. We say $A \in B(\mathscr{H})$ is *positive semidefinite* and write $A \succeq 0$ if A is self-adjoint and $\langle Ah, h\rangle_{\mathscr{H}} \geq 0$ for every $h \in \mathscr{H}$ where $\langle \cdot,\cdot\rangle_{\mathscr{H}}$ stands for the inner product on $\mathscr{H}$. Given real Hilbert spaces $\mathscr{H}, \mathscr{K}$, setting

$$\langle h_1 \otimes k_1, h_2 \otimes k_2\rangle_{\mathscr{H}\otimes\mathscr{K}} := \langle h_1, h_2\rangle_{\mathscr{H}}\langle k_1, k_2\rangle_{\mathscr{K}}$$

and extending by linearity, we obtain an inner product on the vector space $\mathscr{H} \otimes \mathscr{K}$. The completion of $\mathscr{H} \otimes \mathscr{K}$ with respect to this inner product is a Hilbert space, which we still denote by $\mathscr{H} \otimes \mathscr{K}$. Given operators $A \in B(\mathscr{H})$ and $B \in B(\mathscr{K})$, setting

$$(A \otimes B)(h \otimes k) := (Ah) \otimes (Bk),$$

and extending by linearity, one obtains an operator $A \otimes B \in B(\mathscr{H} \otimes \mathscr{K})$.

Theorem D (Linear Positivstellensatz; see Theorem 3.2.13)**.** *Let $\mathscr{H}_j$, $j = 1, 2$, be separable real Hilbert spaces and $L_j(x) = I_{\mathscr{H}_j} + \sum_{k=1}^{g} A_{j,k} x_k$, $j = 1, 2$, linear operator pencils where $I_{\mathscr{H}_j}$ is the identity operators on $\mathscr{H}_j$ and each $A_{j,k}$ is a self-adjoint operator on $\mathscr{H}_j$. The following statements are equivalent:*

(1) For every $n \in \mathbb{N}$ and every tuple $X := (X_1, \ldots, X_g)$ of self-adjoint $n \times n$ matrices such that

$$L_1(X) := I_{\mathscr{H}_1} \otimes I_n + \sum_{k=1}^{g} A_{1,k} \otimes X_k \succeq 0,$$

we have that

$$L_2(X) := I_{\mathscr{H}_1} \otimes I_n + \sum_{k=1}^{g} A_{2,k} \otimes X_k \succeq 0.$$

Here I_n stands for the identity matrix of size n.

(2) Let $\widetilde{\mathcal{C}}$ stand for the smallest unital C^*-algebra in $B(\mathscr{H}_1 \oplus \mathbb{R})$ containing

$$\begin{bmatrix} A_k & 0 \\ 0 & 1 \end{bmatrix}, \quad k = 1, \ldots, g.$$

There exist a separable real Hilbert space $\mathscr{K}$, an isometry $V : \mathscr{H}_2 \to \mathscr{K}$ and a unital $*$-homomorphism $\pi : \widetilde{\mathcal{C}} \to B(\mathscr{K})$ such that

$$L_2 = V^* \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) V + V^* \pi\left(\begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}\right) V.$$

(3) Let $\mathcal{C}$ stand for the smallest unital C^*-algebra in $B(\mathscr{H}_1)$ containing A_k, $k = 1, \ldots, g$. There exist a separable real Hilbert space $\mathscr{K}_0$, a contraction $V_0 : \mathscr{H}_2 \to \mathscr{K}_0$, a unital $*$-homomorphism $\pi_0 : \mathcal{C} \to B(\mathscr{K}_0)$ and a positive semidefinite operator $S \in B(\mathscr{H}_2)$ such that

$$L_2 = S + V_0^* \pi_0(L_1) V_0.$$

Theorem D is the extension of [HKM13b, Theorem 1.1] from matrix pencils to operator pencils. The proof is presented in Subsection 3.2.1. The main techniques used are similar to those in [HKM13b], i.e., complete positivity and the theory of operator algebras. We define the unital $*$-linear map τ between the linear spans of the coefficients of the given linear pencils and connect Theorem D (1) with the complete positivity of τ (see Theorem 3.2.5). Then Theorem D follows by invoking the real version of Arveson's extension theorem and the Stinespring dilation theorem. As a consequence of Theorem D we describe in Subsection 3.2.2, assuming the notation of Theorem D, the set of all tuples $(A_{2,1}, \ldots, A_{2,g})$ that satisfy (1) for a given tuple $(A_{1,1}, \ldots, A_{1,g})$.

In Section 3.3 we consider the algebraic characterization of the equality of the solution sets of linear operator inequalities. The main result is the following.

Theorem E (Linear Gleichstellensatz; see Theorem 3.3.1)**.** *Let $\mathscr{H}_j$, $j = 1, 2$, be separable real Hilbert spaces and $L_j(x) = I_{\mathscr{H}_j} + \sum_{k=1}^{g} A_{j,k} x_k$, $j = 1, 2$, linear operator pencils where $I_{\mathscr{H}_j}$ is the identity operators on $\mathscr{H}_j$ and each $A_{j,k}$ is a self-adjoint compact operator on $\mathscr{H}_j$. The following statements are equivalent:*

(1) For every $n \in \mathbb{N}$ and every tuple $X := (X_1, \ldots, X_g)$ of self-adjoint $n \times n$ matrices we have that

$$I_{\mathscr{H}_1} \otimes I_n + \sum_{k=1}^{g} A_{1,k} \otimes X_k \succeq 0 \quad \Leftrightarrow \quad I_{\mathscr{H}_2} \otimes I_n + \sum_{k=1}^{g} A_{2,k} \otimes X_k \succeq 0.$$

Here I_n stands for the identity matrix of size n.

(2) Let $H_j \subseteq \mathscr{H}_j$, $j = 1, 2$, be closed subspaces satisfying the following:

(a) H_j is invariant under each $A_{j,k}$, $k = 1, \ldots, g$.

(b) For every $n \in \mathbb{N}$ and every tuple $X := (X_1, \ldots, X_g)$ of self-adjoint $n \times n$ matrices we have that

$$I_{H_j} \otimes I_n + \sum_{k=1}^{g} A_{j,k}|_{H_j} \otimes X_j \succeq 0 \quad \Leftrightarrow \quad I_{\mathscr{H}_j} \otimes I_n + \sum_{k=1}^{g} A_{j,k} \otimes X_j \succeq 0.$$

Here I_{H_j} stands for the identity operator on H_j and $A_{j,k}|_{H_j}$ for the restriction of $A_{j,k}$ to H_j.

(c) There is no proper closed subspace $H_j' \subset H_j$ satisfying (2a) and (2b).

There is a unitary operator $U : H_2 \to H_1$ such that for $k = 1, \ldots, g$ we have that

$$A_{2,k}|_{H_2} = U^* A_{1,k}|_{H_1} U.$$

Theorem E extends [HKM13b, Theorem 1.2] from monic linear matrix pencils with bounded *spectrahedra* $\{x \in \mathbb{R}^g \colon L_j(x) \succeq 0\}$, $j = 1, 2$, to monic linear operator pencils whose coefficients are compact operators and arbitrary spectrahedra (not necessarily bounded ones). The proof is presented in Subsection 3.3.1. The main technique is to understand the C^*-algebra generated by the unital operator system spanned by the coefficients $A_{j,k}|_{H_j}$, $k = 1, \ldots, g$, in the notation of Theorem E (2) and the isomorphisms between such C^*-algebras. The crucial observation in the extension from bounded to unbounded spectrahedra is Proposition 3.3.7 which connects a tuple $(A_{j,1}|_{H_j}, \ldots, A_{j,g}|_{H_j})$ with a tuple $(A_{j,1}|_{H_j} \oplus \mathbf{0}_{\mathbb{R}}, \ldots, A_{j,g}|_{H_j} \oplus \mathbf{0}_{\mathbb{R}})$ where $\mathbf{0}_{\mathbb{R}}$ stands for the zero operator on $\mathbb{R}$. The existence and characterization of subspaces H_j satisfying the assumptions (2a)-(2c) in Theorem E are proved in Subsection 3.3.2 (see Corollaries 3.3.16 and 3.3.20). Further on, in Subsections 3.3.3

and 3.3.4 we study if Theorem E extends to linear operator pencils whose coefficients are not compact operators. In Subsection 3.3.3 we show, that the subspaces H_j satisfying the assumptions (2a)-(2c) do not necessarily exist (see Example 3.3.21). But even if such subspaces H_j exist, then the conclusion of Theorem E (2) does not extend to pencils with noncompact coefficients by Example 3.3.22 presented in Subsection 3.3.4.

The main result of Section 3.4 is the extension of Theorem D in the case of finite dimensional Hilbert space $\mathscr{H}_2$ from a linear matrix polynomial L_2 to a general noncommutative (nc) matrix polynomial F. Let $\mathbb{R}\langle x\rangle := \mathbb{R}\langle x_1, \ldots, x_g\rangle$ stand for the set of polynomials in the noncommuting variables $x_1, \ldots, x_g$ with coefficients in $\mathbb{R}$. Let $\mathscr{H}_j$, $j = 1, 2$, be separable real Hilbert spaces and $B(\mathscr{H}_1, \mathscr{H}_2)$ the set of all bounded linear operators form $\mathscr{H}_1$ to $\mathscr{H}_2$. In particular, $\mathbb{R}\langle x\rangle$ and $B(\mathscr{H}_1, \mathscr{H}_2)$ are $\mathbb{R}$-modules. *Nc operator polynomials* are the elements of the $\mathbb{R}$-module $B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x\rangle$ which admits an *involution* $*$, being trivial on $\mathbb{R}$, reversing variables and being the usual adjoint on $B(\mathscr{H}_1, \mathscr{H}_2)$. If $\dim \mathscr{H}_j = \nu_j \in \mathbb{N}$, $j = 1, 2$, then we can identify the elements of $B(\mathscr{H}_1, \mathscr{H}_2)$ with the set of real $\nu_2 \times \nu_1$ matrices $M_{\nu_2,\nu_1}(\mathbb{R})$ and the elements of $M_{\nu_2,\nu_1}(\mathbb{R}) \otimes \mathbb{R}\langle x\rangle$ are called *nc matrix polynomials.*

Theorem F (Convex Positivstellensatz; see Theorem 3.4.1)**.** *Let $\mathscr{H}$ be a separable real Hilbert space and $L(x) = I_{\mathscr{H}} + \sum_{k=1}^{g} A_k x_k$ a monic linear operator pencil where $I_{\mathscr{H}}$ is the identity operator on $\mathscr{H}$ and each A_k is a self-adjoint operator on $\mathscr{H}$. Let $F \in M_\nu(\mathbb{R}) \otimes \mathbb{R}\langle x\rangle$ be a nc matrix polynomial such that $F = F^*$. The following statements are equivalent:*

(1) For every $n \in \mathbb{N}$ and every tuple $X := (X_1, \ldots, X_g)$ of self-adjoint $n \times n$ matrices we have that

$$L(X) := I_{\mathscr{H}} \otimes I_n + \sum_{k=1}^{g} A_k \otimes X_k \succeq 0 \quad \Rightarrow \quad F(X) \succeq 0,$$

where $F(X)$ is defined by replacing x_i with X_i and sending the constant term $F_0 \otimes 1$ to $F_0 \otimes I_n$. Here I_n stands for the identity matrix of size n.

(2) Let $\mathcal{C}$ stand for the smallest unital C^*-algebra in $B(\mathscr{H})$ containing A_k, $k = 1, \ldots, g$. There exist a separable real Hilbert space $\mathscr{K}$, a $*$-homomorphism $\pi : \mathcal{C} \to B(\mathscr{K})$, finitely many nc matrix polynomials $R_j \in M_\nu(\mathbb{R}) \otimes \mathbb{R}\langle x\rangle$ and operator polynomials $Q_k \in B(\mathbb{R}^\nu, \mathscr{K}) \otimes \mathbb{R}\langle x\rangle$ all of degree at most $\frac{1}{2} \cdot \deg(F)$, $j_0, k_0 \in \mathbb{N}$, such that

$$F = \sum_{j=1}^{j_0} R_j^* R_j + \sum_{k=1}^{k_0} Q_k^* \pi(L) Q_k.$$

Theorem F for a matrix pencil L was proved in [HKM12] by modifying the classical Putinar-type separation argument. By essentially using Theorem D and a version of the Hahn-Banach theorem [HKM16b, Theorem 2.2] we are able to apply the separation argument from [HKM12] also in the case of an operator pencil L.

Finally, in Section 3.5 we extend Theorem F in the univariate case from a nc matrix polynomial F to a nc operator polynomial, see Theorem 3.5.1. The main step of the proof is the reduction to Theorem D by the use of variants of the operator Fejér-Riesz theorem [Ros68]. By Examples 3.2.16 and 3.5.2, Theorem 3.5.1 does not extend to the non-monic case.

Chapter 2

Positivstellensätze for univariate matrix polynomials

Two equivalent versions of the matrix Fejér-Riesz theorem characterize Laurent matrix polynomials positive semidefinite (psd) on the complex unit circle $\mathbb{T}$ and complex matrix polynomials psd on the real line $\mathbb{R}$. In this chapter we extend both characterizations to arbitrary closed semialgebraic sets $K \subseteq \mathbb{R}$ and $\mathscr{K} \subseteq \mathbb{T}$ by the use of matrix quadratic modules from real algebraic geometry. In the $\mathbb{T}$-case there is a denominator-free characterization for all sets $\mathscr{K}$, while in the $\mathbb{R}$-case, a denominator-free characterization exists for compact sets K, but there are counterexamples for such characterization for non-compact sets K. However, there is a weaker characterization with denominators in the non-compact case. Furthermore, we study a complexity of the characterizations in terms of a bound on the degrees of the summands needed. We provide examples of sets $K \subseteq \mathbb{R}$ where the degrees can be bounded by the degree of the matrix polynomial, as well as counterexamples for this statement. At the end we extend the results to algebraic curves.

This chapter is based on [Zal15 arxiv, Zal16].

2.1 Notations and known results

Let $\mathbb{C}[x]$ be the set of complex univariate polynomials equipped with the involution which is conjugation on the coefficients and is trivial on x, i.e., $x^* = x$. By the fundamental theorem of algebra, a polynomial $f \in \mathbb{C}[x]$ non-negative on the real line is a hermitian square of a complex polynomial.

Theorem 2.1.1. *Let* $f(x) = \sum_{m=0}^{2N} f_m x^m \in \mathbb{C}[x]$ *be a complex polynomial which is non-negative on* $\mathbb{R}$. *Then there exists a complex polynomial* $g(x) = \sum_{m=0}^{N} g_m x^m \in \mathbb{C}[x]$ *such that* $f(x) = g(x)^* g(x)$.

Let $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ be the complex unit circle and $\mathbb{C}\big[z, \frac{1}{z}\big]$ the set of complex Laurent polynomials equipped with the involution which is conjugation on the coefficients and $z^* = \frac{1}{z}$. In 1916, Fejér [Fej16, p. 55] conjectured that a similar result is true when one exchanges the real line by the unit complex circle and complex polynomials by complex Laurent polynomials. In the same article he confirmed his conjecture by a proof of Riesz which, similarly as in the real line case, essentially uses the fundamental theorem of algebra. This result is now known as the Fejér-Riesz theorem.

Theorem 2.1.2 (Fejér, Riesz)**.** *Let* $a(z) = \sum_{m=-N}^{N} a_m z^m \in \mathbb{C}\left[z, \frac{1}{z}\right]$ *be a Laurent polynomial which is non-negative on the unit complex circle* $\mathbb{T}$*. Then there exists a polynomial* $b(z) = \sum_{m=0}^{N} b_m z^m \in \mathbb{C}[z]$ *such that* $a(z) = b(z)^* b(z)$*.*

Let $M_n\big(\mathbb{C}\big[z, \frac{1}{z}\big]\big)$ be the set of $n \times n$ matrices over $\mathbb{C}\big[z, \frac{1}{z}\big]$ with conjugated transpose as the involution. Elements of $M_n\big(\mathbb{C}\big[z, \frac{1}{z}\big]\big)$ are called *matrix Laurent polynomials*. We say $A(z) \in M_n\big(\mathbb{C}\big[z, \frac{1}{z}\big]\big)$ is *hermitian*, if $A(z) = A(z)^*$. We write $H_n\big(\mathbb{C}\big[z, \frac{1}{z}\big]\big)$ for the set of all hermitian matrix Laurent polynomials. We call $A(z) \in H_n\big(\mathbb{C}\big[z, \frac{1}{z}\big]\big)$ *positive definite* (resp. *positive semidefinite*) in $z_0 \in \mathbb{T}$ if $v^* A(z_0) v > 0$ (resp. $v^* A(z_0) v \geq 0$) for every non-zero $v \in \mathbb{C}^n$.

In 1958, Rosenblatt [Ros58] generalized Theorem 2.1.2 to matrix polynomials positive definite on $\mathbb{T}$, while in 1964 Helson [Hel64] relaxed the assumption of positive definiteness to positive semidefiniteness.

Theorem 2.1.3. *Let* $A(z) = \sum_{m=-N}^{N} A_m z^m \in H_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ *be a* $n \times n$ *hermitian matrix Laurent polynomial which is positive semidefinite on* $\mathbb{T}$*. Then there exists a matrix polynomial* $B(z) = \sum_{m=0}^{N} B_m z^m \in M_n(\mathbb{C}[z])$ *such that* $A(z) = B(z)^* B(z)$*, where* $B(z)^* = \sum_{m=0}^{N} \overline{B_m}^T z^{-m}$*.*

In 1966, Popov [Pop66] deduced from Theorem 2.1.3 an analogous generalization of Theorem 2.1.1 from the case of polynomials to the case of matrix polynomials. Before stating his result we need some definitions. Let $M_n(\mathbb{C}[x])$ be a set of all $n \times n$ matrices over $\mathbb{C}[x]$ with conjugated transpose as the involution. Elements of $M_n(\mathbb{C}[x])$ are called *matrix polynomials*. We say $F(x) \in M_n(\mathbb{C}[x])$ is *hermitian*, if $F(x) = F(x)^*$. We write $H_n(\mathbb{C}[x])$ for the set of all hermitian matrix polynomials. We call $F(x) \in H_n(\mathbb{C}[x])$ *positive definite* (resp. *positive semidefinite*) in $x_0 \in \mathbb{C}$ if $v^* F(x_0) v > 0$ (resp. $v^* F(x_0) v \geq 0$) for every non-zero $v \in \mathbb{C}^n$.

Theorem 2.1.4. *Let* $F(x) = \sum_{m=0}^{2N} F_m x^m \in H_n(\mathbb{C}[x])$ *be a* $n \times n$ *hermitian matrix polynomial which is positive semidefinite on* $\mathbb{R}$*. Then there exists a matrix polynomial* $G(x) = \sum_{m=0}^{N} G_m x^m \in M_n(\mathbb{C}[x])$ *such that* $F(x) = G(x)^* G(x)$*, where*

$$G(x)^* = \sum_{m=0}^{N} \overline{G_m}^T x^m.$$

The first direct proof of Theorem 2.1.4 has been given in 1970 by Jakubović [Jak70]. Using Popov's [Pop66] reasoning one can see that Theorems 2.1.3 and 2.1.4 are in fact equivalent. The factorizations from Theorems 2.1.3 and 2.1.4 are very important in control theory [Pop66, KSH00]. Due to this importance many proofs of either of them have appeared by now by different authors, e.g., [GLR82, Dri04, EJL09, SS12, Eph14, HS+]. Moreover, in 1968, Rosenblum [Ros68, Theorem 7] generalized Theorem 2.1.3 from matrix to operator polynomials.

In the rest of this subsection we will focus on a generalization of Theorem 2.1.1 to semialgebraic sets in $\mathbb{R}$. The *closed semialgebraic set* associated to a finite subset $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is given by

$$K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\}.$$

The *n-th matrix quadratic module generated by S* in $H_n(\mathbb{C}[x])$ is defined by

$$M_S^n := \{G_0^* G_0 + G_1^* G_1 \cdot g_1 + \ldots + G_s^* G_s \cdot g_s \colon G_j \in M_n(\mathbb{C}[x]),\ j = 0, \ldots, s\},$$

and the *n-th matrix preordering generated by S* in $H_n(\mathbb{C}[x])$ by

$$T_S^n := \left\{ \sum_{e \in \{0,1\}^s} G_e^* G_e \cdot \underline{g}^e \colon G_e \in M_n(\mathbb{C}[x]) \text{ for all } e \in \{0,1\}^s \right\},$$

where $e := (e_1, \ldots, e_s)$ and $\underline{g}^e$ stands for $g_1^{e_1} \cdots g_s^{e_s}$.

Remark 2.1.5. Note that T_S^n is the quadratic module generated by all products $\underline{g}^e$, $e \in \{0,1\}^s$.

Let $\text{Pos}_{\succeq 0}^n(K_S)$ stand for the set of all $n \times n$ hermitian matrix polynomials which are positive semidefinite in every point of K_S. We call the set M_S^n (resp. T_S^n) *saturated* if $M_S^n = \text{Pos}_{\succeq 0}^n(K_S)$ (resp. $T_S^n = \text{Pos}_{\succeq 0}^n(K_S)$). The *degree* of a matrix polynomial $F(x) = \sum_{m=0}^{N} F_m x^m \in M_n(\mathbb{C}[x])$ is N if $F_N \neq 0$. If every $F \in T_S^n$ from a saturated preordering T_S^n has a representation of the form $\sum_{e \in \{0,1\}^s} G_e^* G_e \cdot \underline{g}^e$ with

- $\deg(G_e^* G_e \cdot \underline{g}^e) \leq \deg(F)$ for every $e \in \{0,1\}^s$, then T_S^n is called *very strongly boundedly saturated*,
- $\deg(G_e^* G_e \cdot \underline{g}^e) \leq f(\deg(F))$ for every $e \in \{0,1\}^s$ where $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ is some map, then T_S^n is called *boundedly saturated.*

Theorem 2.1.4 can be restated in the following form.

Theorem 2.1.6. *The set $M_\emptyset^n = T_\emptyset^n$ is very strongly boundedly saturated for every $n \in \mathbb{N}$.*

Given a closed semialgebraic set $K \subseteq \mathbb{R}$, there is a natural choice of the set $S \subset \mathbb{R}[x]$ such that $K = K_S$. The set $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ is the *natural description* of K if it satisfies the following conditions:

(a) If K has the least element a, then $x - a \in S$.

(b) If K has the greatest element a, then $a - x \in S$.

(c) For every $a \neq b \in K$, if $(a, b) \cap K = \emptyset$, then $(x - a)(x - b) \in S$.

(d) These are the only elements of S.

The extension of Theorem 2.1.1 to an arbitrary closed semialgebraic set $K \subseteq \mathbb{R}$ was proved in 2002 by Kuhlmann and Marshall [KM02, Theorem 2.2].

Theorem 2.1.7 (Kuhlmann, Marshall)**.** *Suppose $K \subset \mathbb{R}$ is a non-empty closed semialgebraic set. If $S \subseteq \mathbb{R}[x]$ is the natural description of K, then the preordering T_S^1 is saturated. Moreover, if K is not compact, then T_S^1 is saturated if and only if $S \subseteq \mathbb{R}[x]$ contains each of the polynomials in the natural description of K up to scaling by positive constants.*

Remark 2.1.8. Kuhlmann and Marshall in fact work with polynomials from $\mathbb{R}[x]$ and their squares are usual ones, i.e., p^2 where $p \in \mathbb{R}[x]$, while we work with $\mathbb{C}[x]$ and hermitian squares, i.e., p^*p where $p \in \mathbb{C}[x]$. However, since a hermitian $f \in \mathbb{C}[x]$ belongs to $\mathbb{R}[x]$ and since every sum of squares of real polynomials is a single hermitian square by Theorem 2.1.1, Theorem 2.1.7 follows from [KM02, Theorem 2.2].

Taking care of the degrees in the proof of Theorem 2.1.7, one notices that the preordering T_S^1 from Theorem 2.1.7 is in fact very strongly boundedly saturated. This was first noticed by Kuhlmann, Marshall and Schwartz [KMS05, Theorem 4.1].

Theorem 2.1.9 (Kuhlmann, Marshall, Schwartz)**.** *Suppose $K \subseteq \mathbb{R}$ is a non-empty closed semialgebraic set with the natural description $S \subseteq \mathbb{R}[x]$. Then T_S^1 is very strongly boundedly saturated.*

Theorem 2.1.7 characterizes saturated preorderings T_S^1 in case K_S is not compact. It is possible to do the same for compact sets K_S. For this aim another definition has to be introduced. Let $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ be a finite subset with a compact K_S. A set S is a *saturated description* of K_S if and only if the following two conditions hold:

(a) For every left endpoint x_j there exists $k \in \{1, \ldots, s\}$ such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$.

(b) For every right endpoint y_j there exists $\ell \in \{1, \ldots, s\}$ such that $g_\ell(y_j) = 0$ and $g_\ell'(y_j) < 0$.

In 2003, Scheiderer [Sch03, Theorem 5.17] proved that given a compact semialgebraic set K, finite sets $S \subset \mathbb{R}[x]$ with $K_S = K$ such that the preordering T_S^1 is saturated are exactly saturated descriptions of K.

Theorem 2.1.10 (Scheiderer). *Suppose $K \subset \mathbb{R}$ is a non-empty compact semialgebraic set and $S \subset \mathbb{R}[x]$ a finite set such that $K_S = K$. Then the preordering T_S^1 is saturated if and only if S is a saturated description of K_S.*

Remark 2.1.11. Scheiderer works with polynomials from $\mathbb{R}[x]$ where squares are usual ones, i.e., p^2 where $p \in \mathbb{R}[x]$. By the same reasoning as in Remark 2.1.8, Theorem 2.1.10 follows from [Sch03, Theorem 5.17].

For an alternative proof of Theorem 2.1.10 see [KMS05, Theorem 3.2]. Moreover, we can replace the preordering T_S^1 in Theorem 2.1.10 by the quadratic module M_S^1 by another result of Scheiderer [Sch05, Corollary 4.4].

Theorem 2.1.12 (Scheiderer). *Suppose $S \subset \mathbb{R}[x]$ is a finite set such that K_S is non-empty and compact. Then $M_S^1 = T_S^1$.*

Corollary 2.1.13 (Scheiderer). *Suppose K is a non-empty compact semialgebraic set and $S \subset \mathbb{R}[x]$ a finite set such that $K_S = K$. Then the quadratic module M_S^1 is saturated if and only if S is a saturated description of K_S.*

Finally, we focus on matrix generalizations of Theorem 2.1.7. In 2002, Dette and Studden [DS02] extended Theorem 2.1.9 from polynomials to matrix polynomials for $K = [0, 1]$ (see [DS02, Theorem 2.5]) and $K = [0, \infty)$ (see [DS02, Theorem 5.1]).

Theorem 2.1.14 (Dette, Studden). *The quadratic modules $M^n_{\{x,1-x\}}$ and $M^n_{\{x\}}$ are very strongly boundedly saturated for every $n \in \mathbb{N}$.*

For alternative proofs of Theorem 2.1.14 see [SS12, Theorems 7, 8] and [CZ13, Proposition 3].

2.2 Compact Positivstellensatz

The main result of this section, Theorem 2.2.1 below, is the extension of Corollary 2.1.13 from polynomials to matrix polynomials.

Theorem 2.2.1 (Compact Positivstellensatz). *Suppose $K \subseteq \mathbb{R}$ is a non-empty compact semialgebraic set. The n-th matrix quadratic module M_S^n is saturated for every $n \in \mathbb{N}$ if and only if S a saturated description of K.*

The main ingredients in the proof of Theorem 2.2.1 are:

(1) Corollary 2.1.13 as the $n = 1$ case.

(2) The "hF-proposition", i.e., Proposition 2.2.2 below. The proof consists of the use of Schur complements and multiplying with the denominators (see [Scm09, §4.3]). However, one has to ensure that the denominators do not vanish in the specified complex point x_0. This is done by factoring out the highest possible power of the minimal polynomial m_{x_0} of x_0 over $\mathbb{R}[x]$.

(3) Eliminating h in "hF-proposition". There are two ways to establish this. One way is presented in [Zal16, §2.2] and is based on the use of another result of Scheiderer [Sch06, Proposition 2.7]. The other way, which will be presented below, is based only on the use of points (1) and (2).

2.2.1 "hF-proposition"

We call the following result "hF-proposition".

Proposition 2.2.2 ("hF-proposition"). *Suppose K is a non-empty compact semi-algebraic set in $\mathbb{R}$ with a saturated description S. Then, for any hermitian polynomial $F \in H_n(\mathbb{C}[x])$ such that $F \succeq 0$ on K and every point $x_0 \in \mathbb{C}$, there exists a polynomial $h \in \mathbb{R}[x]$ such that $h \geq 0$ on K, $h(x_0) \neq 0$ and $hF \in M_S^n$.*

To prove Proposition 2.2.2 we need Lemmas 2.2.3 and 2.2.4 below.

Lemma 2.2.3. *Let $G = [g_{k\ell}]_{k\ell} \in M_n(\mathbb{C}[x])$ be a matrix polynomial. For every $k, \ell \in \mathbb{N}$ satisfying $1 \leq k \leq \ell \leq n$ there exist unitary matrices $U_{k\ell} \in M_n(\mathbb{R})$ and $V_{k\ell} \in M_n(\mathbb{C})$ such that*

$$U_{k\ell} G U_{k\ell}^* = \begin{bmatrix} p_{k\ell} & * \\ * & * \end{bmatrix}, \quad V_{k\ell} G V_{k\ell}^* = \begin{bmatrix} r_{k\ell} & * \\ * & * \end{bmatrix},$$

where

$$p_{k\ell} = \begin{cases} g_{k\ell}, & \text{for } 1 \leq k = \ell \leq n \\ \frac{1}{2}(g_{k\ell} + g_{\ell k} + g_{kk} + g_{\ell\ell}), & \text{for } 1 \leq k < \ell \leq n \end{cases},$$

$$r_{k\ell} = \begin{cases} g_{k\ell}, & \text{for } 1 \leq k = \ell \leq n \\ \frac{i}{2}(-g_{k\ell} + g_{\ell k}) + \frac{1}{2}(g_{kk} + g_{\ell\ell}), & \text{for } 1 \leq k < \ell \leq n \end{cases}.$$

Proof. We define $U_{11} = V_{11} := I_n$, $U_{kk} = V_{kk} := P_k$ for $k = 2, \ldots, n$, where P_k denotes the permutation matrix which permutes the first row and the k-th row.

For $1 \leq k < \ell \leq n$, define $U_{k\ell} := P_k S_{k\ell}$ where $S_{k\ell} = \left(s_{pr}^{(k\ell)}\right)_{pr} \in M_n(\mathbb{R})$ is the matrix with $s_{kk}^{(k\ell)} = s_{k\ell}^{(k\ell)} = s_{\ell k}^{(k\ell)} = \frac{1}{\sqrt{2}}$, $s_{\ell\ell}^{(k\ell)} = -\frac{1}{\sqrt{2}}$, $s_{pp}^{(k\ell)} = 1$ if $p \notin \{k, \ell\}$ and $s_{pr}^{(k\ell)} = 0$ otherwise.

For $1 \leq k < \ell \leq n$, define $V_{k\ell} := P_k \widetilde{S}_{k\ell}$ where $\widetilde{S}_{k\ell} = \left(\widetilde{s}_{pr}^{(k\ell)}\right)_{pr} \in M_n(\mathbb{C})$ is the matrix with $\widetilde{s}_{kk}^{(k\ell)} = \widetilde{s}_{\ell k}^{(k\ell)} = \frac{1}{\sqrt{2}}$, $\widetilde{s}_{k\ell}^{(k\ell)} = \frac{i}{\sqrt{2}}$, $\widetilde{s}_{\ell\ell}^{(k\ell)} = -\frac{i}{\sqrt{2}}$, $\widetilde{s}_{pp}^{(k\ell)} = 1$ if $p \notin \{k, \ell\}$ and $\widetilde{s}_{pr}^{(k\ell)} = 0$ otherwise. □

Lemma 2.2.4. *For a hermitian matrix polynomial*

$$\begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \in H_n(\mathbb{C}[x]),$$

where $a = a^* \in \mathbb{R}[x]$, $\beta \in M_{1,n-1}(\mathbb{C}[x])$ and $C \in H_{n-1}(\mathbb{C}[x])$, it holds that

$$a \cdot \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} = \begin{bmatrix} a & 0 \\ \beta^* & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & aC - \beta^*\beta \end{bmatrix} \cdot \begin{bmatrix} a & \beta \\ 0 & I_{n-1} \end{bmatrix}.$$

Proof. Easy computation. □

Now we are ready to prove Proposition 2.2.2.

Proof of Proposition 2.2.2. The proof is by induction on the size n of matrix polynomials. For $n = 1$, we can take $h = 1$ by Corollary 2.1.13. Suppose the proposition holds for $n-1$. We will prove that it holds for n. Let us take $F := [f_{k\ell}]_{k\ell} \in H_n(\mathbb{C}[x])$ where $F \succeq 0$ on K and let m_{x_0} be the minimal polynomial of x_0 over $\mathbb{R}[x]$, i.e.,

$$m_{x_0}(x) = \begin{cases} x - x_0, & \text{if } x_0 \in \mathbb{R} \\ (x-x_0)(x-\overline{x_0}), & \text{if } x_0 \in \mathbb{C} \setminus \mathbb{R} \end{cases}.$$

If $F \equiv 0$, then $F \in M_S^n$ and we can take $h = 1$. Otherwise $F \not\equiv 0$. Let us factor out of F the polynomial $m_{x_0}^q$ to the highest possible power $q \in \mathbb{N} \cup \{0\}$, i.e.,

$$F = m_{x_0}^q \cdot G,$$

where $G = [g_{k\ell}]_{k\ell} \in H_n(\mathbb{C}[x])$ and

$$G(x_0) = [g_{k\ell}(x_0)]_{k\ell} \neq 0. \tag{2.2.1}$$

Claim 1. One of the following two cases applies:

Case 1: There exists $k_0 \in \{1, \dots, n\}$ such that $g_{k_0k_0}(x_0) \neq 0$.

Case 2: For all $k \in \{1, \dots, n\}$ we have $g_{kk}(x_0) = 0$ and there exist $k_0, \ell_0 \in \{1, \dots, n\}$ such that $k_0 < \ell_0$ and

$$\operatorname{Re} g_{k_0\ell_0}(x_0) \neq 0 \quad \text{or} \quad \operatorname{Im} g_{k_0\ell_0}(x_0) \neq 0,$$

where $\operatorname{Re} g_{k_0\ell_0} := \frac{g_{k_0\ell_0} + \overline{g_{k_0\ell_0}}}{2} \in \mathbb{R}[x]$ and $\operatorname{Im} g_{k_0\ell_0} := \frac{g_{k_0\ell_0} - \overline{g_{k_0\ell_0}}}{2i} \in \mathbb{R}[x]$ are the real and the imaginary part of $g_{k_0\ell_0}$.

Proof of Claim 1. Let us assume that none of the two cases applies. Then we have $\operatorname{Re} g_{k\ell}(x_0) = \operatorname{Im} g_{k\ell}(x_0) = 0$ for all $k, \ell \in \{1, \dots, n\}$ satisfying $k \le \ell$. Let us take $\ell < k$. Since G is hermitian, it follows that $g_{\ell k} = \overline{g_{k\ell}} = \operatorname{Re} g_{k\ell} - i \cdot \operatorname{Im} g_{k\ell}$. Therefore $g_{\ell k}(x_0) = \operatorname{Re} g_{k\ell}(x_0) - i \cdot \operatorname{Im} g_{k\ell}(x_0) = 0$. Hence $g_{k\ell}(x_0) = 0$ for all $k, \ell \in \{1, \dots, n\}$. This is a contradiction with (2.2.1) and proves Claim 1.

Claim 2. There exists a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$G = U \cdot \begin{bmatrix} a & \beta \\ \beta^* & C \end{bmatrix} \cdot U^* \in H_n(\mathbb{C}[x]),$$

where $a = a^* \in \mathbb{R}[x]$, $\beta \in M_{1,n-1}(\mathbb{C}[x])$, $C \in H_{n-1}(\mathbb{C}[x])$ and $a(x_0) \neq 0$.

Proof of Claim 2. Let $U_{k\ell}$, $V_{k\ell}$, $p_{k\ell}$, $r_{k\ell}$ be as in Lemma 2.2.3. If we are in Case 1 of Claim 1, then take $U = U_{k_0k_0}$ and notice that $p_{k_0k_0}(x_0) = g_{k_0k_0}(x_0) \neq 0$, which proves Claim 2. Otherwise we are in Case 2 of Claim 1. If $p_{k_0\ell_0}(x_0) \neq 0$, then we take $U = U_{k_0\ell_0}$ and Claim 2 follows. Similarly, if $r_{k_0\ell_0}(x_0) \neq 0$, then we take $U = V_{k_0\ell_0}$ and Claim 2 follows. Otherwise we must have $p_{k_0\ell_0}(x_0) = r_{k_0\ell_0}(x_0) = 0$.

However, we will prove that this cannot happen. By definition and assumptions it holds that

$$\begin{aligned} p_{k_0\ell_0}(x_0) &= \frac{1}{2}(g_{k_0\ell_0}+g_{\ell_0k_0}+g_{k_0k_0}+g_{\ell_0\ell_0})(x_0) = \frac{1}{2}(g_{k_0\ell_0}+g_{\ell_0k_0})(x_0) = \\ &= (\operatorname{Re} g_{k_0\ell_0})(x_0) = 0, \\ r_{k_0\ell_0}(x_0) &= \frac{i}{2}(-g_{k_0\ell_0}+g_{\ell_0k_0})(x_0) + \frac{1}{2}(g_{k_0k_0}+g_{\ell_0\ell_0})(x_0) = \frac{i}{2}(-g_{k_0\ell_0}+g_{\ell_0k_0})(x_0) = \\ &= (\operatorname{Im} g_{k_0\ell_0})(x_0) = 0. \end{aligned}$$

Since we are in Case 2, $(\operatorname{Re} g_{k_0\ell_0})(x_0) \neq 0$ or $(\operatorname{Im} g_{k_0\ell_0})(x_0) \neq 0$ which is a contradiction. This proves Claim 2.

Using Lemma 2.2.4 it follows that

$$a \cdot G = U \cdot \begin{bmatrix} a & 0 \\ \beta^* & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & aC - \beta^*\beta \end{bmatrix} \cdot \begin{bmatrix} a & \beta \\ 0 & I_{n-1} \end{bmatrix} \cdot U^*. \tag{2.2.2}$$

Multiplying (2.2.2) by $m^q_{x_0} a$ we get

$$a^2 \cdot F = U \cdot \begin{bmatrix} a & 0 \\ \beta^* & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} m^q_{x_0} a & 0 \\ 0 & D \end{bmatrix} \cdot \begin{bmatrix} a & \beta \\ 0 & I_{n-1} \end{bmatrix} \cdot U^*, \tag{2.2.3}$$

where

$$D := m^q_{x_0} a(aC - \beta^*\beta) \in H_{n-1}(\mathbb{C}[x]).$$

Claim 3. $D \succeq 0$ on K.

Proof of Claim 3. For $x_1 \in K$, the inequality

$$F(x_1) = U \cdot \begin{bmatrix} (m^q_{x_0} a)(x_1) & (m^q_{x_0}\beta)(x_1) \\ (m^q_{x_0}\beta^*)(x_1) & (m^q_{x_0} C)(x_1) \end{bmatrix} \cdot U^* \succeq 0,$$

implies that $(m^q_{x_0} a)(x_1) \geq 0$. To prove Claim 3 we have to separate two cases.

Case 1: $a(x_1) \neq 0$. Then $\begin{bmatrix} a(x_1) & \beta(x_1) \\ 0 & I_{n-1} \end{bmatrix} =: V$ is invertible and by (2.2.3) we have

$$\begin{bmatrix} (m^q_{x_0} a)(x_1) & 0 \\ 0 & D(x_1) \end{bmatrix} = (V^{-1})^* U^* \cdot \big(a^2 \cdot F\big)(x_1) \cdot U V^{-1} \succeq 0.$$

Thus $D(x_1) \succeq 0$.

Case 2: $a(x_1) = 0$. From $a(x_0) \neq 0$ it follows that $x_1 \neq x_0$. Since $x_1 \in \mathbb{R}$, this also implies that $x_1 \neq \overline{x_0}$. Hence $m^q_{x_0}(x_1) \neq 0$. From

$$0 \preceq U^* F(x_1) U = \begin{bmatrix} 0 & m^q_{x_0}(x_1)\beta(x_1) \\ m^q_{x_0}(x_1)\beta^*(x_1) & m^q_{x_0}(x_1) C(x_1) \end{bmatrix},$$

it follows that $\beta(x_1) \equiv 0$. Therefore $D(x_1) = 0$.

By the induction hypothesis used for the polynomial $D \in H_{n-1}(\mathbb{C}[x])$, there exists $h_1 \in \mathbb{R}[x]$ such that $h_1 \geq 0$ on K, $h_1(x_0) \neq 0$ and $h_1 D \in M_S^{n-1}$. By Corollary 2.1.13, $h_1 m_{x_0}^q a \in M_S^1$. Hence $hF \in M_S^n$ where $h := h_1 a^2 \in \mathbb{R}[x]$ and $h(x_0) \neq 0$. This concludes the proof of Proposition 2.2.2. $\square$

Remark 2.2.5. By keeping track on the degree of h and using Theorem 2.1.9, we can prove more in Proposion 2.2.2 above. Namely, if $S = \{g_1, \ldots, g_s\}$ is the natural description of K, then h can be chosen of degree at most $\deg(F)(3^n - 1)$ and $hF = \sum_{e \in \{0,1\}^s} A_e^* A_e \cdot \underline{g}^e \in T_S^n$ for some $A_e \in M_n(\mathbb{C}[x])$ with $\deg(A_e^* A_e \cdot \underline{g}^e) \leq \deg(hF)$.

2.2.2 Eliminating h

To eliminate h in "hF-proposition", one can use a result of Scheiderer [Sch06, Proposition 2.7] in the way presented in [Zal16, §2.2]. Here we will see another approach which is based only on the use of Corollary 2.1.13.

Proof of Theorem 2.2.1. By Corollary 2.1.13, if M_S^1 is saturated, then S is a saturated description of K. It remains to proof the implication ($\Leftarrow$). Take $F \in \mathrm{Pos}_{\succeq 0}^n(K)$. We have to prove that $F \in M_S^n$. We write $S = \{g_1, \ldots, g_s\}$. Let ∂K be the set of boundary points of K. Since K is non-empty and compact, the set ∂K is non-empty. By Proposition 2.2.2, for every $x_0 \in \partial K$, there exists $h_{x_0} \in \mathbb{R}[x]$, such that $h_{x_0} \geq 0$ on K, $h_{x_0}(x_0) \neq 0$ and $h_{x_0} F \in M_S^n$. Therefore, defining the polynomial

$$h := \sum_{x_0 \in \partial K} h_{x_0} \in \mathbb{R}[x],$$

there is a representation

$$hF = A_0^* A_0 + A_1^* A_1 \cdot g_1 + \ldots + A_s^* A_s \cdot g_s, \tag{2.2.4}$$

where $A_i \in M_n(\mathbb{C}[x])$ for $i = 0, \ldots, s$. Multiplying (2.2.4) with h we get

$$h^2 F = A_0^* A_0 \cdot h + A_1^* A_1 \cdot h g_1 + \ldots + A_s^* A_s \cdot h g_s. \tag{2.2.5}$$

Notice that $h^2 \in \mathrm{Pos}_{\succeq 0}^1(\mathbb{R})$ and $h^2(x_0) > 0$ for every $x_0 \in \partial K$.

Claim. The set $S_1 := \{h^2 g_1, \ldots, h^2 g_s\}$ is a saturated description of K.

Proof of Claim. We have to check both conditions (a) and (b) in the definition of a saturated description of K. Let us take a left endpoint x_j of K. Since S is a saturated description of K, there exists $k \in \{1, \ldots, s\}$ such that $g_k(x_j) = 0$ and $g_k'(x_j) > 0$. Hence $(h^2 g_k)(x_j) = 0$ and

$$(h^2 g_k)'(x_j) = (h^2)'(x_j) g_k(x_j) + h^2(x_j) g_k'(x_j) = h^2(x_j) g_k'(x_j) > 0.$$

This proves the condition (a). Similarly, for every right endpoint y_j of K one can find $\ell \in \{1, \ldots, s\}$ such that $(h^2 g_\ell)(y_j) = 0$ and $(h^2 g_\ell)'(y_j) < 0$. This proves the condition (b) and establishes Claim.

Using Corollary 2.1.13 subsequently for $h, hg_1, \ldots, hg_s$ which are all non-negative on K, we conclude that $h, hg_1, \ldots, hg_s \in M_{S_1}^1$. Using this in (2.2.5) together with Theorem 2.1.4 to replace sums of the form $\sum_i G_i^* G_i$ where $G_i \in M_n(\mathbb{C}[x])$ with G^*G for some $G \in M_n(\mathbb{C}[x])$, we get

$$h^2 F = B_0^* B_0 + B_1^* B_1 \cdot h^2 g_1 + \ldots + B_s^* B_s \cdot h^2 g_s, \tag{2.2.6}$$

where $B_i \in M_n(\mathbb{C}[x])$ for $i = 0, \ldots, s$. Rearanging (2.2.6) one obtains

$$h^2 \underbrace{(F - B_1^* B_1 \cdot g_1 - \ldots - B_s^* B_s \cdot g_s)}_{=:\widetilde{F}} = B_0^* B_0. \tag{2.2.7}$$

(2.2.7) implies that $\widetilde{F}$ is positive semidefinite on $\mathbb{R} \setminus \{x \in \mathbb{R} \colon h^2(x) = 0\}$. Since the set $\{x \in \mathbb{R} \colon h^2(x) = 0\}$ is finite, $\widetilde{F}$ is positive semidefinite on $\mathbb{R}$. By Theorem 2.1.4 there is $B \in M_n(\mathbb{C}[x])$ such that $\widetilde{F} = B^*B$ and hence

$$F = B^*B + B_1^* B_1 \cdot g_1 + \ldots + B_s^* B_s \cdot g_s \in M_S^n.$$

This concludes the proof of Theorem 2.2.1. □

2.3 Non-saturated preorderings

In this section we study if Theorem 2.1.7 in the case $K \subseteq \mathbb{R}$ is an *unbounded* closed semialgebraic set, extends from polynomials to matrix polynomials. By Theorems 2.1.6 and 2.1.14, the extension exists in the case K is an unbounded interval. However, the main result of this section Theorem 2.3.1 states that for most unbounded sets K the extension does not exist.

Theorem 2.3.1. *Suppose $K \subseteq \mathbb{R}$ is an unbounded closed semialgebraic set and $K_1, \ldots, K_r$ its connected components. Let K satisfy either of the following cases:*

> **Case 1:** *There are $i, j \in \{1, \ldots, r\}$ such that K_i and K_j have a non-empty interior, K_i is bounded and K_j is unbounded.*
>
> **Case 2:** *There are $r \geq 3$ components where exactly one of them is an unbounded interval and all the others are points.*
>
> **Case 3:** *There are $r \geq 4$ components where exactly two of them are unbounded intervals and all the others are points.*

If $S \subset \mathbb{R}[x]$ is a finite set such that $K_S = K$, then the 2-nd matrix preordering T_S^2 is not saturated.

It is sufficient to prove Theorem 2.3.1 in the case S is the natural description of K by the following lemma.

Lemma 2.3.2. *Suppose $K \subseteq \mathbb{R}$ is an unbounded closed semialgebraic set with the natural description S. Let $S_1 \subset \mathbb{R}[x]$ be a finite set such that $K_{S_1} = K$. Then for every $n \in \mathbb{N}$ it holds that $T^n_{S_1} \subseteq T^n_S$.*

Proof. We write $S := \{g_1, \dots, g_s\}$ and $S_1 := \{f_1, \dots, f_t\}$. We will prove that every matrix polynomial F from $T^n_{S_1}$ belongs to T^n_S. By definition of $T^n_{S_1}$, F is of the form

$$F = \sum_{e' \in \{0,1\}^t} G^*_{e'} G_{e'} \cdot f_1^{e'_1} \cdots f_t^{e'_t}, \tag{2.3.1}$$

where $e' := (e'_1, \dots, e'_t)$ and $G_{e'} \in M_n(\mathbb{C}[x])$. By Theorem 2.1.7, we have

$$f_1^{e'_1} \dots f_t^{e'_t} \in T^1_S$$

for every $e' \in \{0,1\}^t$. Using this fact together with Theorem 2.1.4 to replace sums of the form $\sum_i G_i^* G_i$ where $G_i \in M_n(\mathbb{C}[x])$ with H^*H for some $H \in M_n(\mathbb{C}[x])$, (2.3.1) can be rewritten in the form

$$F = \sum_{e \in \{0,1\}^s} H^*_e H_e \cdot g_1^{e_1} \cdots g_s^{e_s}, \tag{2.3.2}$$

where $e := (e_1, \dots, e_s)$ and $H_e \in M_n(\mathbb{C}[x])$. Hence $F \in T^n_S$, which concludes the proof of the lemma. □

Let K be a closed semialgebraic set with a natural description $S = \{g_1, \dots, g_s\}$. For $n \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{0\}$ we define the set

$$T^n_{S,d} := \left\{ \sum_{e \in \{0,1\}^s} G^*_e G_e \cdot \underline{g}^e \colon G_e \in M_n(\mathbb{C}[x]) \text{ and } \deg(G^*_e G_e \cdot \underline{g}^e) \le d \;\; \forall e \in \{0,1\}^s \right\},$$

where $e := (e_1, \dots, e_s)$ and $\underline{g}^e$ stands for $g_1^{e_1} \cdots g_s^{e_s}$. We call $T^n_{S,d}$ the *degree d part* of the n-th matrix preordering T^n_S. The following proposition will be the crucial part in the proof of Theorem 2.3.1.

Proposition 2.3.3. *Suppose $K = [x_1, x_2] \cup [x_3, \infty) \subset \mathbb{R}$ is a disjoint union of a bounded and an unbounded interval, where $x_1 < x_2 < x_3$. For $k \in \mathbb{R}$, let $F_k(x) \in H_2(\mathbb{C}[x])$ be a matrix polynomial defined by*

$$F_k(x) := \begin{bmatrix} x + A(k) & D(k) \\ \overline{D(k)} & x^2 + B(k)x + C(k) \end{bmatrix},$$

where

$$A(k) := k - x_1, \qquad C(k) := k^2 + k(-x_1 + x_2 + x_3) + x_2 x_3,$$
$$B(k) := -k - x_2 - x_3, \qquad D(k) := \sqrt{A(k)C(k) + x_1 x_2 x_3}.$$

Let $p_k(x) \in \mathbb{R}[x]$ be a polynomial defined by

$$p_k(x) := x^2 + B(k)x + C(k).$$

The following statements are true:

Statement 1. *For every $k > 0$ which satisfies*

$$D(k)^2 = k^3 + k^2(-2x_1 + x_2 + x_3) + k\big(x_2x_3 + x_1^2 - x_1x_2 - x_1x_3\big) > 0, \tag{2.3.3}$$

$$p_k\left(-\frac{B(k)}{2}\right) = \frac{3}{4}k^2 + k\left(-x_1 + \frac{x_2 + x_3}{2}\right) - \left(\frac{x_2 - x_3}{2}\right)^2 > 0, \tag{2.3.4}$$

it holds that

$$F_k(x) \in \mathrm{Pos}^2_{\succeq 0}(K).$$

Statement 2. *Suppose $k > 0$ satisfies the conditions (2.3.3), (2.3.4) and K_1 is a set of the form*

$$[x_1, x_2] \bigcup \bigcup_{j=1}^{m} [x_{2j+1}, x_{2j+2}] \bigcup [x_{2m+3}, \infty) \subseteq K,$$

where $m \in \mathbb{N} \cup \{0\}$ and

$$x_1 < x_2 < x_3 \leq x_4 < x_5 \leq x_6 < \ldots < x_{2j+1} \leq x_{2j+2} < x_{2j+3} \leq \ldots < x_{2m+3}.$$

Then

$$F_k \in \mathrm{Pos}^2_{\succeq 0}(K_1) \setminus T^2_{S_1},$$

where S_1 is the natural description of K_1. In particular,

$$F_k(x) \in \mathrm{Pos}^2_{\succeq 0}(K) \setminus T^2_S,$$

where S is the natural description of K.

Statement 3. *Suppose $k > 0$ satisfies the conditions (2.3.3), (2.3.4) and K_2 is a set of the form*

$$\bigcup_{\ell=1}^{m'} \{x_{-i}\} \bigcup [x_1, x_2] \bigcup \bigcup_{j=1}^{m} [x_{2j+1}, x_{2j+2}] \bigcup [x_{2m+3}, \infty),$$

where $m', m \in \mathbb{N} \cup \{0\}$ and

$$\begin{aligned} x_{-m'} < \ldots < x_{-1} < x_1 < x_2 < x_3 \leq x_4 < x_5 \leq x_6 < \\ < \ldots < x_{2j+1} \leq x_{2j+2} < x_{2j+3} \leq \ldots < x_{2m+3}. \end{aligned}$$

Then

$$\left(\prod_{i=1}^{m'} (x - x_{-i})\right) \cdot F_k \in \mathrm{Pos}^2_{\succeq 0}(K_2) \setminus T^2_{S_2},$$

where S_2 is the natural description of K_2.

Statement 4. *Suppose $k > 0$ satisfies the conditions (2.3.3), (2.3.4) and K_3 is a set of the form*

$$[x_1, x_2] \bigcup \bigcup_{j=3}^{m+3} \{x_j\} \subset K,$$

where $m \in \mathbb{N}$ and $x_1 < x_2 < x_3 < x_4 < \ldots < x_{m+3}$. Then

$$F_k \in \mathrm{Pos}^2_{\succeq 0}(K_3) \setminus T^2_{S_3,2},$$

where S_3 is the natural description of K_3.

Proof. First we will prove Statement 1. Let us choose $k > 0$ which satisfies the conditions (2.3.3), (2.3.4). (Note that every sufficiently large k is a good choice.) The determinant of $F_k(x)$ is

$$(x - x_1)(x - x_2)(x - x_3) \in \mathrm{Pos}^1_{\succeq 0}(K).$$

The (1,1)-minor of F_k, which is $x + A(k)$, is non-negative for $x \geq x_1 - k$ and since $k > 0$, it follows that $x + A(k) \in \mathrm{Pos}^1_{\succeq 0}(K)$. The (2,2)-minor of F_k, which is a quadratic polynomial $p_k(x)$ with a minimum in the vertex $x = \frac{-B(k)}{2}$, is positive on $\mathbb{R}$, since k satisfies (2.3.4). In particular, $p_k \in \mathrm{Pos}^1_{\succeq 0}(K)$. Since all principal minors of $F_k(x)$ are non-negative on K, it follows that $F_k(x) \in \mathrm{Pos}^2_{\succeq 0}(K)$. This proves Statement 1.

Now we will prove Statement 2. By definition the natural description S_1 of K_1 is the set

$$S_1 = \{\underbrace{x - x_1}_{g_1(x)}, \underbrace{(x - x_2)(x - x_3)}_{g_2(x)}, \ldots, \underbrace{(x - x_{2m+2})(x - x_{2m+3})}_{g_{m+2}(x)}\}.$$

Since $K_1 \subseteq K$ and $F_k \in \mathrm{Pos}^2_{\succeq 0}(K)$, it follows that $F_k \in \mathrm{Pos}^2_{\succeq 0}(K_1)$. We will prove that $F_k(x) \notin T^2_{S_1}$ by contradiction. Let us assume that $F_k \in T^2_{S_1}$. By definition of $T^2_{S_1}$, F_k is of the form

$$F_k = \sum_{e \in \{0,1\}^{m+2}} G_e^* G_e \cdot g_1^{e_1} \cdots g_{m+2}^{e_{m+2}}, \tag{2.3.5}$$

where $e := (e_1, \ldots, e_{m+2})$ and $G_e \in M_2(\mathbb{C}[x])$ for every $e \in \{0,1\}^{m+2}$. By the degree comparison of both sides of (2.3.5), there exist matrices $G_j \in M_2(\mathbb{C})$ for $j = 0, \ldots, m+3$, such that

$$F_k(x) = (G_1 x + G_0)^*(G_1 x + G_0) + G_2^* G_2 \cdot (x - x_1) + \\ + \sum_{j=1}^{m+1} G_{j+2}^* G_{j+2} \cdot (x - x_{2j})(x - x_{2j+1}). \tag{2.3.6}$$

By observing the monomial x^2 on both sides of (2.3.6), it follows in particular that

$$G_3^* G_3 = \begin{bmatrix} 0 & 0 \\ 0 & k_0 \end{bmatrix}$$

for some $k_0 \in [0, 1]$. Now, (2.3.6) can be rewritten as

$$F_k(x) - \begin{bmatrix} 0 & 0 \\ 0 & k_0 \end{bmatrix} \cdot (x - x_2)(x - x_3) = (G_1 x + G_0)^*(G_1 x + G_0) + \\ + G_2^* G_2 \cdot (x - x_1) + \sum_{j=2}^{m+1} G_{j+2}^* G_{j+2} \cdot (x - x_{2j})(x - x_{2j+1}). \tag{2.3.7}$$

The right-hand side of (2.3.7) belongs to $\mathrm{Pos}^2_{\succeq 0}(K_1 \cup [x_2, x_3])$. We will prove that the left-hand side of (2.3.7), with the determinant

$$q(x) := (x - x_2)(x - x_3)((x - x_1)(1 - k_0) - k k_0)),$$

does not belong to $\mathrm{Pos}^2_{\succeq 0}(K_1 \cup [x_2, x_3])$, which is a contradiction. There are two cases to consider.

Case 1: $k_0 = 0$. Then

$$q(x) = (x - x_1)(x - x_2)(x - x_3) \notin \mathrm{Pos}^1_{\succeq 0}(K_1 \cup [x_2, x_3]).$$

Case 2: $k_0 > 0$. Then

$$q(x_1) = (x_1 - x_2)(x_1 - x_3)(-kk_0) < 0.$$

In both cases we conclude that

$$F_k(x) - \begin{bmatrix} 0 & 0 \\ 0 & k_0 \end{bmatrix} \cdot (x - x_2)(x - x_3) \notin \mathrm{Pos}^2_{\succeq 0}(K_1 \cup [x_2, x_3]).$$

Therefore F_k cannot be expressed in the form (2.3.5) and $F_k \notin T^2_{S_1}$.

Next we will prove Statement 3. Note that

$$\underbrace{\left(\prod_{i=1}^{m'}(x - x_{-i})\right)}_{g(x)} \cdot F_k \in \mathrm{Pos}^2_{\succeq 0}(K_2),$$

since $K_2 \subseteq K \bigcup \bigcup_{\ell=1}^{m'}\{x_{-i}\}$ and $F_k \in \mathrm{Pos}^2_{\succeq 0}(K)$. We will prove that $g \cdot F_k \notin T^2_{S_2}$ by contradiction. Let us assume that $g \cdot F_k \in T^2_{S_2}$. We write $S_2 := \{h_1, \ldots, h_s\}$. By definition of $T^2_{S_1}$, $g \cdot F_k$ is of the form

$$g \cdot F_k = \sum_{f \in \{0,1\}^s} H_f^* H_f \cdot h_1^{f_1} \cdots h_s^{f_s}, \tag{2.3.8}$$

where $f := (f_1, \ldots, f_s) \in \{0, 1\}^s$ and $H_f \in M_2(\mathbb{C}[x])$ for each f. Note that for each $i \in \{1, \ldots, m'\}$ and each $f \in \{0, 1\}^s$ we have

$$(g \cdot F_k)(x_{-i}) = 0 \quad \text{and} \quad (H_f^* H_f \cdot h_1^{f_1} \cdots h_s^{f_s})(x_{-i}) \succeq 0. \tag{2.3.9}$$

From (2.3.8), (2.3.9) it follows that

$$(H_f^* H_f \cdot h_1^{f_1} \cdots h_s^{f_s})(x_{-i}) = 0 \tag{2.3.10}$$

for each $i \in \{1, \ldots, m'\}$ and each $f \in \{0, 1\}^s$.

Claim 1. For each $f \in \{0, 1\}^s$ there exist $H'_f \in M_2(\mathbb{C}[x])$ and

$$h'_f \in \mathrm{Pos}^1_{\succeq 0}\left(K_2 \setminus \bigcup_{i=1}^{m'}\{x_{-i}\}\right),$$

such that

$$H_f^* H_f \cdot h_1^{f_1} \cdots h_s^{f_s} = \left(\prod_{i=1}^{m'}(x - x_{-i})\right) \cdot (H'_f)^* H'_f \cdot h'_f.$$

Proof of Claim 1. By (2.3.10) we know that for each $i \in \{1, \dots, m'\}$ and each $f \in \{0,1\}^s$, $x - x_{-i}$ divides $H_f^* H_f \cdot h_1^{f_1} \cdots h_s^{f_s}$. Therefore $x - x_{-i}$ divides $H_f^* H_f$ or $h_1^{f_1} \cdots h_s^{f_s}$. Note that if $x - x_{-i}$ divides $H_f^* H_f$, then $x - x_{-i}$ divides already H_f and hence $(x - x_{-i})^2$ divides $H_f^* H_f$. Therefore

$$H_f^* H_f = \Big(\prod_{\ell=1}^{j} (x - x_{-i_\ell})^2\Big) \cdot (H_f')^* H_f', \qquad h_1^{f_1} \cdots h_s^{f_s} = \Big(\prod_{\ell=j+1}^{m'} (x - x_{-i_\ell})\Big) \cdot \widehat{h}_f,$$

where $j \in \mathbb{N} \cup \{0\}$, $(i_1, \dots, i_{m'})$ is some permutation of $(1, \dots, m')$, $\widehat{H}_f \in M_2(\mathbb{C}[x])$ and $\widehat{h}_f \in \mathrm{Pos}^1_{\succeq 0}\Big(K_2 \setminus \bigcup_{i=1}^{m'} \{x_{-i}\}\Big)$. Hence

$$H_f^* H_f \cdot h_1^{f_1} \cdots h_s^{f_s} = \Big(\prod_{i=1}^{m'} (x - x_{-i})\Big) \cdot (H_f')^* H_f' \cdot \Big(\prod_{\ell=1}^{j} (x - x_{-i_\ell})\Big) \cdot \widehat{h}_f.$$

Then defining a polynomial

$$h_f' := \Big(\prod_{\ell=1}^{j} (x - x_{-i_\ell})\Big) \cdot \widehat{h}_f \in \mathrm{Pos}^1_{\succeq 0}\left(K_2 \setminus \bigcup_{i=1}^{m'} \{x_{-i}\}\right),$$

proves Claim 1.

Using Claim 1 in (2.3.8), we get

$$F_k = \sum_{f \in \{0,1\}^s} (H_f')^* H_f' \cdot h_f'.$$

By Theorem 2.1.7, $h_f' \in T^1_{\widetilde{S}_2}$ where $\widetilde{S}_2$ is the natural description of $K_2 \setminus \bigcup_{i=1}^{m'} \{x_{-i}\}$. Therefore $F_k \in T^2_{\widetilde{S}_2}$, which is a contradiction with Statement 2. This proves Statement 3.

It remains to prove Statement 4. By definition the natural description S_3 of K_3 is the set

$$S_3 = \{x - x_1, (x - x_2)(x - x_3), \dots, (x - x_{m+2})(x - x_{m+3}), x_{m+3} - x\}.$$

Since $K_3 \subseteq K$ and $F_k \in \mathrm{Pos}^2_{\succeq 0}(K)$, it follows that $F_k \in \mathrm{Pos}^2_{\succeq 0}(K_3)$. We will prove that $F_k \notin T^2_{S_3,2}$ by contradiction. Let us assume that $F_k \notin T^2_{S_3,2}$. By definition of $T^2_{S_3,2}$, there are matrices $H_j \in M_2(\mathbb{C})$ for $j = 0, \dots, m+2$, such that

$$\begin{aligned} F_k(x) = (H_1 x + H_0)^*(H_1 x + H_0) + H_2^* H_2 \cdot (x - x_1) + \sum_{j=2}^{m+2} H_{j+1}^* H_{j+1} \cdot \\ \cdot (x - x_j)(x - x_{j+1}) + H_{m+4}^* H_{m+4} \cdot (x_{m+3} - x) + H_{m+5}^* H_{m+5} \cdot (x - x_1)(x_{m+3} - x), \end{aligned} \tag{2.3.11}$$

From (2.3.11) it follows that

$$\Big(\underbrace{F_k(x) - \sum_{j=3}^{m+2} H_{j+1}^* H_{j+1} \cdot (x - x_j)(x - x_{j+1})}_{G(x)}\Big) \in \mathrm{Pos}^2_{\succeq 0}([x_1, x_2] \cup [x_3, x_{m+3}]). \tag{2.3.12}$$

Since $G(x_1) \succeq 0$ and $G(x_2) \succeq 0$, it follows that for $j = 3, \ldots, m+2$ we have

$$\ker F_k(x_1) \subseteq \ker(H_{j+1}^* H_{j+1}) \quad \text{and} \quad \ker F_k(x_2) \subseteq \ker(H_{j+1}^* H_{j+1}). \tag{2.3.13}$$

Claim 2. $\ker F_k(x_1) + \ker F_k(x_2) = \mathbb{C}^2$.

Proof of Claim 2. Note that

$$F_k(x_1) = \begin{bmatrix} k & D(k) \\ D(k) & p(x_1) \end{bmatrix} \quad \text{and} \quad F_k(x_2) = \begin{bmatrix} x_2 - x_1 + k & D(k) \\ D(k) & p(x_2) \end{bmatrix}.$$

By the proof of Statement 1, we know that $F_k(x_i)$ is singular for $i = 1, 2$. Since $F_k(x_i) \neq 0$ for $i = 1, 2$, it follows that $\dim(\ker F_k(x_i)) = 1$ for $i = 1, 2$. If $[a, b]^T \in \ker F_k(x_1) \cap \ker F_k(x_2)$, then in particular

$$ak = -D(k)b = a(x_2 - x_1 + k).$$

Since $k > 0$ and $x_2 - x_1 + k > 0$, it follows that $a = 0$. Since $D(k) > 0$, this also implies $b = 0$. Hence $\ker F_k(x_1) \cap \ker F_k(x_2) = \{0\}$. Therefore $\dim(\ker F_k(x_1) + \ker F_k(x_2)) = 2$ and Claim 2 follows.

Using (2.3.13) and Claim 2, we conclude that $H_{j+1}^* H_{j+1} = 0$ for $j = 3, \ldots, m+2$. Hence (2.3.11) becomes

$$\begin{aligned} F_k(x) = &(H_1 x + H_0)^*(H_1 x + H_0) + H_2^* H_2 \cdot (x - x_1) + H_3^* H_3 \cdot \\ &\cdot (x - x_2)(x - x_3) + H_{m+4}^* H_{m+4} \cdot (x_{m+3} - x) + H_{m+5}^* H_{m+5} \cdot (x - x_1)(x_{m+3} - x). \end{aligned}$$

Therefore

$$\Big(\underbrace{F_k(x) - H_3^* H_3 \cdot (x - x_2)(x - x_3)}_{H(x)}\Big) \in \mathrm{Pos}^2_{\succeq 0}([x_1, x_{m+3}]).$$

For $i = 1, 2, 3$, $F_k(x_i)$ being singular and

$$(H_3^* H_3 \cdot (x - x_2)(x - x_3))(x_i) \succeq 0,$$

implies that $H(x_i)$ is also singular. Therefore $\det H(x)$ is divisible by

$$(x - x_1)(x - x_2)(x - x_3).$$

Since $\det H(x)$ is non-negative on the interval $[x_1, x_{m+3}]$, which contains x_2 and x_3 in the interior, it follows that x_2 and x_3 must be zeroes of even degree. Since $\det H(x)$ is a polynomial of degree at most 4 divisible by $\prod_{i=1}^{3}(x - x_i)$ this is possible only if $\det H(x) \equiv 0$. But this is not the case by Claim 3 below, which leads to a contradiction. Therefore $F_k \notin T^2_{S_3,2}$.

Claim 3. $\det H(x) \not\equiv 0$.

Proof of Claim 3. Since $H(x_1) \succeq 0$, it follows that

$$\{0\} \neq \ker F_k(x_1) \subseteq \ker(H_3^* H_3).$$

Since the matrices $F_k(x_1) \neq 0$ and $H_3^* H_3$ are hermitian of size 2, we conclude that

$$F_k(x_1) = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} U^* \quad \text{and} \quad H_3^* H_3 = U \begin{bmatrix} \lambda_2 & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where $U \in M_2(\mathbb{C})$ is a unitary matrix and $\lambda_1 > 0$, $\lambda_2 \geq 0$. Therefore

$$H_3^* H_3 = \frac{\lambda_2}{\lambda_1} \cdot F_k(x_1). \tag{2.3.14}$$

Let us write $H(x) = Ax^2 + Bx + C$, where $A, B, C \in M_2(\mathbb{C})$. Using (2.3.14) we get that

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\lambda_2}{\lambda_1} \cdot F_k(x_1).$$

If $\lambda_2 = 0$, then $H(x) = F_k(x)$, which is not possible since $H(x) \in \mathrm{Pos}^2_{\succeq 0}([x_1, x_{m+3}])$ and $F_k(x) \notin \mathrm{Pos}^2_{\succeq 0}([x_1, x_{m+3}])$. Otherwise $\lambda_2 > 0$ and

$$\det A = \frac{\lambda_2}{\lambda_1} \cdot \left(\frac{\lambda_2}{\lambda_1} \cdot \det F_k(x_1) - k \right) = -\frac{\lambda_2}{\lambda_1} \cdot k \neq 0.$$

Writing $\det H(x) = \sum_{i=0}^4 a_i x^i$ for some $a_i \in \mathbb{R}$, we observe that $a_4 = \det A \neq 0$, which proves Claim 3. □

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. By Lemma 2.3.2, we may assume that S is the natural description of K. We will construct a polynomial $F \in \mathrm{Pos}^2_{\succeq 0}(K)$ such that $F \notin T^2_S$. We will do this for each case of Theorem 2.3.1 separately.

First let us assume that K satisfies Case 1 of Theorem 2.3.1. We separate three subcases.

Case 1.1: *K is bounded from below and unbounded from above.* Note that K is of the form of some K_2 in the Statement 3 of Proposition 2.3.3. Hence there exists $F \in \mathrm{Pos}^2_{\succeq 0}(K) \setminus T^2_S$.

Case 1.2: *K is unbounded from below and bounded from above.* Applying a substitution $x \mapsto -x$, observe that the set $-K$ is of the form of Case 1.1 and that the natural description of K maps to the natural description of $-K$. Hence Case 1.2 follows from Case 1.1.

Case 1.3: *K is unbounded from below and above.* Let $d \in \mathbb{R}$ be the smallest endpoint of K. Define the map $\lambda_d : \mathbb{R} \setminus \{d\} \to \mathbb{R}$ by $\lambda_d(x) := \frac{1}{d-x}$. Observe that $\lambda_d(K) =: K'$ is the set of the form

$$\bigcup_{\ell=1}^{m'} \{x_{-i}\} \bigcup [x_1, x_2] \bigcup \bigcup_{j=1}^{m} [x_{2j+1}, x_{2j+2}] \bigcup [x_{2m+3}, \infty),$$

where $m', m \in \mathbb{N} \cup \{0\}$ and

$$x_{-m'} < \ldots < x_{-1} < x_1 < x_2 < x_3 \leq x_4 < x_5 \leq x_6 <$$
$$< \ldots < x_{2j+1} \leq x_{2j+2} < x_{2j+3} \leq \ldots < x_{2m+3}.$$

Let S' be the natural description of $\lambda_d(K)$. By the Statement 3 of Proposition 2.3.3 there exists a polynomial $G \in \mathrm{Pos}^2_{\succeq 0}(\lambda_d(K)) \setminus T^2_{S'}$. Then

$$F(x) := x^{2\lceil \frac{\deg(G)}{2} \rceil} \cdot G\left(d - \frac{1}{x}\right) \in \mathrm{Pos}^2_{\succeq 0}(K),$$

and $F(x) \notin T^2_S$, since otherwise $G(x) = (d-x)^{2\lceil \frac{\deg(G)}{2} \rceil} \cdot F\left(\frac{1}{d-x}\right) \in T^2_{S'}$.

It remains to prove Cases 2 and 3 of Theorem 2.3.1. The proof of both cases is the same. Suppose K satisfies Case 2 or 3. Let $d' \in \mathbb{R} \setminus K$ be an arbitrary point. Define the map $\lambda_{d'} : \mathbb{R} \setminus \{d\} \to \mathbb{R}$ by $\lambda_{d'}(x) := \frac{1}{d'-x}$. Observe that $\lambda_{d'}(K)$ is the set of the form

$$[x_1, x_2] \bigcup \bigcup_{j=3}^{m} \{x_j\}$$

where $m \geq 4$ and the points x_j are pairwise different. Further on, we may choose $d \in \mathbb{R}$ such that

$$x_1 < x_2 < x_3 < \ldots < x_m \quad \text{or} \quad x_m < x_{m-1} < \ldots < x_3 < x_1 < x_2.$$

By substitution $x \mapsto -x$, we may assume that $x_1 < x_2 < x_3 < \ldots < x_m$. Let S'' be the natural description of $\lambda_{d'}(K)$. By the Statement 4 of Proposition 2.3.3 there exists a polynomial $G' \in \mathrm{Pos}^2_{\succeq 0}(\lambda_{d'}(K)) \setminus T^2_{S'',2}$. Then

$$F'(x) := x^2 \cdot G'\left(d' - \frac{1}{x}\right) \in \mathrm{Pos}^2_{\succeq 0}(K),$$

and $F'(x) \notin T^2_S$, since otherwise $G'(x) = (d'-x)^2 \cdot F\left(\frac{1}{d'-x}\right) \in T^2_{S'',2}$. This concludes the proof. □

2.4 Boundedly saturated preorderings

In this section we study when a saturated matrix preordering is (very strongly) boundedly saturated. We prove that if a semialgebraic set is a disjoint union of two unbounded intervals, then the matrix preordering generated by the natural description is very strongly boundedly saturated; see Theorem 2.4.2. We also prove that for a finite semialgebraic set K, the matrix preordering generated by the natural description is boundedly saturated with the degrees bounded by the maximum between the degree of the matrix polynomial and one less than the number of points in K, but if there are at least 4 points in K, then the matrix preordering is not very strongly boundedly saturated; see Corollary 2.4.4 and Example 2.4.5.

By the following lemma, it suffices to assume that S is the natural description of a given closed semialgebraic set $K \subseteq \mathbb{R}$ when studying if the matrix preordering T^n_S is (very strongly) boundedly saturated.

Lemma 2.4.1. *Suppose $K \subseteq \mathbb{R}$ is a non-empty closed semialgebraic set with the natural description S. Let $S_1 \subset \mathbb{R}[x]$ be a finite set such that $K_{S_1} = K$ and $n \in \mathbb{N}$. If T^n_S is not (very strongly) boundedly saturated, then $T^n_{S_1}$ is not (very strongly) boundedly saturated.*

Proof. Assume the notation from the proof of Lemma 2.3.2. Suppose $F \in T^n_{S_1}$ has a representation (2.3.1) with

$$m := \max_{e' \in \{0,1\}^t} \left(\deg(G^*_{e'} G_{e'} \cdot f_1^{e'_1} \cdots f_t^{e'_t}) \right)$$

Since the preordering T^1_S is very strongly boundedly saturated by Theorem 2.1.9, there is a representation (2.3.2) such that

$$m = \max_{e \in \{0,1\}^s} \left(\deg(H^*_e H_e \cdot g_1^{e_1} \cdots g_t^{e_s}) \right).$$

This concludes the proof of the lemma. □

If a semialgebraic set is a disjoint union of two unbounded intervals, then the matrix preordering generated by the natural description is very strongly boundedly saturated by the following.

Theorem 2.4.2. *Suppose $K = (-\infty, a] \cup [b, \infty)$ is a disjoint union of two unbounded intervals where $a, b \in \mathbb{R}$ and $a < b$. The preordering $T^n_{\{(x-a)(x-b)\}}$ is very strongly boundedly saturated for every $n \in \mathbb{N}$.*

Proof. By a linear change of variables we may assume that $a = -1$ and $b = 1$. Suppose $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. We will prove that there is a representation of F as an element of $T^n_{\{x^2-1\}}$ with the degrees of the summands bounded by $\deg(F)$. Observe that $\deg(F)$ is even. Let $F_1 \in H_n(\mathbb{C}[x])$ be the matrix polynomial defined by

$$F_1(x) = x^{\deg(F)} \cdot F\left(\frac{1}{x}\right) \in H_n(\mathbb{C}[x]).$$

We have that $F_1 \succeq 0$ on $[-1, 1]$. By Theorem 2.1.14 there exist matrix polynomials $G_0, G_1, G_2, G_3 \in M_n(\mathbb{C}[x])$ such that

$$F_1(x) = G_0(x)^* G_0(x) + G_1(x)^* G_1(x) \cdot (x+1) + G_2(x)^* G_2(x) \cdot (1-x) + \\ + G_3(x)^* G_3(x) \cdot (1-x^2),$$

and their degrees are bounded by

$$\begin{aligned}
\deg(G_0) &\leq \left\lfloor \frac{\deg(F_1)}{2} \right\rfloor \leq \frac{\deg(F)}{2}, \\
\deg(G_i) &\leq \left\lfloor \frac{\deg(F_1) - 1}{2} \right\rfloor \leq \left\lfloor \frac{\deg(F) - 1}{2} \right\rfloor = \frac{\deg(F)}{2} - 1 \quad \text{for } i = 1, 2, \\
\deg(G_3) &\leq \left\lfloor \frac{\deg(F_1)}{2} \right\rfloor - 1 \leq \left\lfloor \frac{\deg(F)}{2} \right\rfloor - 1 = \frac{\deg(F)}{2} - 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
F(x) &= x^{\deg(F)} \cdot F_1\left(\frac{1}{x}\right) \\
&= x^{\deg(F)} \left(G_0\left(\frac{1}{x}\right)^* G_0\left(\frac{1}{x}\right) + G_1\left(\frac{1}{x}\right)^* G_1\left(\frac{1}{x}\right)\left(\frac{1}{x} + 1\right) + \right. \\
&\quad \left. + G_2\left(\frac{1}{x}\right)^* G_2\left(\frac{1}{x}\right)\left(1 - \frac{1}{x}\right) + G_3\left(\frac{1}{x}\right)^* G_3\left(\frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) \right) \\
&=: \widetilde{G}_0(x)^* \widetilde{G}_0(x) + \widetilde{G}_1(x)^* \widetilde{G}_1(x) \cdot (1+x)x + \widetilde{G}_2(x)^* \widetilde{G}_2(x)(x-1)x + \\
&\quad + \widetilde{G}_3(x)^* \widetilde{G}_3(x)(x^2 - 1),
\end{aligned} \tag{2.4.1}$$

where the matrix polynomials $\widetilde{G}_0, \widetilde{G}_1, \widetilde{G}_2, \widetilde{G}_3 \in M_n(\mathbb{C}[x])$ are defined by

$$\widetilde{G}_0 := x^{\frac{\deg(F)}{2}} \cdot G_0\left(\frac{1}{x}\right) \quad \text{and} \quad \widetilde{G}_i := x^{\frac{\deg(F)}{2}-1} \cdot G_i\left(\frac{1}{x}\right) \quad \text{for } i = 1, 2, 3,$$

and their degrees are bounded by

$$\deg(\widetilde{G}_0) \leq \frac{\deg(F)}{2},$$
$$\deg(\widetilde{G}_i) \leq \frac{\deg(F)}{2} - 1 \quad \text{for } i = 1, 2, 3.$$

By the identity

$$x^2 \pm x = \frac{(x^2 - 1) + (x \pm 1)^2}{2},$$

the equality (2.4.1) can be rewritten as

$$\begin{aligned} F(x) = \Big(\widetilde{G}_0(x)^*\widetilde{G}_0(x) + \widetilde{G}_1(x)^*\widetilde{G}_1(x)\frac{(x+1)^2}{2} + \widetilde{G}_2(x)^*\widetilde{G}_2(x)\frac{(x-1)^2}{2}\Big) + \\ + \Big(\widetilde{G}_3(x)^*\widetilde{G}_3(x) + \widetilde{G}_1(x)^*\widetilde{G}_1(x)\frac{1}{2} + \widetilde{G}_2(x)^*\widetilde{G}_2(x)\frac{1}{2}\Big)(x^2 - 1), \\ =: H_0(x) + H_1(x)(x^2 - 1), \qquad (2.4.2) \end{aligned}$$

where $H_0, H_1 \in H_n(\mathbb{C}[x])$ are positive semidefinite on $\mathbb{R}$ and

$$\deg(H_0) \leq \deg(F) \quad \text{and} \quad \deg(H_1) \leq \deg(F) - 2.$$

By Theorem 2.1.4 there are matrix polynomials $\widetilde{H}_0, \widetilde{H}_1 \in M_n(\mathbb{C}[x])$ such that

$$H_i = \widetilde{H}_i^* \widetilde{H}_i \quad \text{and} \quad \deg(\widetilde{H}_i) = \frac{\deg(H_i)}{2} \quad \text{for } i = 0, 1.$$

Plugging this into (2.4.2) we get the representation of F as an element of $T^n_{\{x^2-1\}}$ with the degrees of the summands bounded by $\deg(F)$. □

Let $K \subseteq \mathbb{R}$ be a semialgebraic set and $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ a finite set with $K = K_S$. Recall that for $d \in \mathbb{N} \cup \{0\}$ the degree d part of the n-th preordering T^n_S is defined by

$$T^n_{S,d} := \left\{ \sum_{e \in \{0,1\}^s} G_e^* G_e \cdot \underline{g}^e : G_e \in M_n(\mathbb{C}[x]) \text{ and } \deg(G_e^* G_e \cdot \underline{g}^e) \leq d \;\; \forall e \in \{0,1\}^s \right\},$$

where $e := (e_1, \ldots, e_s)$ and $\underline{g}$ stand for $\underline{g} = g_1^{e_1} \cdots g_s^{e_s}$.

Proposition 2.4.3. *Suppose $K = \bigcup_{j=1}^{m} \{x_j\} \subseteq \mathbb{R}$ is a disjoint union of points with the natural description S where $m \in \mathbb{N}$. For every $n \in \mathbb{N}$, if $F \in \mathrm{Pos}^n_{\succeq 0}(K)$, then*

$$F \in T^n_{S,k},$$

where $k := \max(\deg(F), m - 1)$.

Proof. Assume that $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. If we divide every entry of F with $\prod_{j=1}^m (x-x_j)$, we get

$$F(x) = \prod_{j=1}^m (x - x_j) \cdot G(x) + R(x),$$

where $G(x), R(x) \in M_n(\mathbb{C}[x])$ and $\deg(R) < m$. Expanding every entry of $R(x)$ in the basis

$$\left\{ f_k(x) := \frac{\prod_{j=1}^m (x - x_j)}{x - k} \in \mathbb{R}[x] \colon k = 1, \ldots, m \right\},$$

leads to

$$F(x) = \prod_{j=1}^m (x - x_j) \cdot G(x) + \sum_{k=1}^m (-1)^{\ell_k} f_k(x) \cdot F_k, \tag{2.4.3}$$

where $\ell_k \in \{0, 1\}$ is such that $(-1)^{\ell_k} f_k \in \mathrm{Pos}^1_{\succeq 0}(K)$, and $F_k \in M_n(\mathbb{C})$. Since $F \in \mathrm{Pos}^n_{\succeq 0}(K)$, it follows that $(-1)^{\ell_k} f_k(x_k) \cdot F_k = F(x_k) \succeq 0$ for each k. In particular, $F_k \succeq 0$. Write $G(x)$ in the form

$$G(x) := G_1\big(x^2\big) + x \cdot G_2\big(x^2\big), \tag{2.4.4}$$

where $G_1, G_2 \in M_n(\mathbb{C}[x])$ and

$$2\deg(G_1) \leq \deg(G) \quad \text{and} \quad 2\deg(G_2) \leq \deg(G) - 1. \tag{2.4.5}$$

Using the identity

$$A = \frac{(A+1)^*(A+1)}{4} - \frac{(A-1)^*(A-1)}{4}$$

for the matrix coefficients of G_j, $j = 1, 2$, note that $G_j(x^2)$ can be written as

$$G_j\big(x^2\big) = \sum_{\ell} G_{j\ell}(x)^* G_{j\ell}(x) - \sum_{\ell'} G_{j\ell'}(x)^* G_{j\ell'}(x), \tag{2.4.6}$$

where $G_{j\ell}, G_{j\ell'} \in M_n(\mathbb{C}[x])$ and

$$\deg(G_{j\ell}), \deg(G_{j\ell'}) \leq \deg(G_j). \tag{2.4.7}$$

By Theorem 2.1.9, we have that

$$(-1)^{\ell_k} \cdot f_k \in T^1_{S,m-1}, \quad \pm \prod_{j=1}^m (x - x_j) \in T^1_{S,m}, \quad \pm x \cdot \prod_{j=1}^m (x - x_j) \in T^1_{S,m+1}. \tag{2.4.8}$$

Using (2.4.3)-(2.4.8), we conclude that $F \in T^n_{S,k}$ where $k := \max(\deg(F), m-1)$. $\square$

Corollary 2.4.4. *Suppose $K \subset \mathbb{R}$ is a disjoint union of m points with the natural description S. For every $n \in \mathbb{N}$ the following are true:*

(1) The preordering T^n_S is boundedly saturated.

(2) If $m \leq 3$, then the preordering T^n_S is very strongly boundedly saturated.

Proof. Part 1 follows directly from Proposition 2.4.3. Let us prove part 2. We separate three cases.

Case 1: $m = 1$. The statement follows by Proposition 2.4.3.

Case 2: $m = 2$. Let $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. If $\deg(F) = 0$, then $F = G^*G$ for some $G \in M_n(\mathbb{C})$. Hence, $F \in T^n_{S,0}$. Otherwise, $\deg(F) \geq 1$ and $F \in T^n_{S,\deg(F)}$ by Proposition 2.4.3. Hence, T^n_S is very strongly boundedly saturated for every $n \in \mathbb{N}$.

Case 3: $m = 3$. Write $K = \{x_1, x_2, x_3\}$ with $x_1, x_2, x_3 \in \mathbb{R}$ and $x_1 < x_2 < x_3$. Let $F \in \mathrm{Pos}^n_{\succeq 0}(K)$. If $\deg(F) = 0$, then $F = G^*G$ for some $G \in M_n(\mathbb{C})$. Hence, $F \in T^n_{S,0}$. If $\deg(F) = 1$, then by the convexity of the set $\{x \in \mathbb{R} \colon F(x) \succeq 0\}$, it follows that $F \in \mathrm{Pos}^n_{\succeq 0}([x_1, x_3])$. By Theorem 2.1.14, F is of the form

$$F_0^*F_0 + F_1^*F_1 \cdot (x - x_1) + F_2^*F_2 \cdot (x_3 - x),$$

where $F_0, F_1, F_2 \in M_n(\mathbb{C})$. Hence, $F \in T^n_{S,1}$. Finally, if $\deg(F) \geq 2$, then $F \in T^n_{S,\deg(F)}$ by Proposition 2.4.3. Hence T^n_S is very strongly boundedly saturated for every $n \in \mathbb{N}$. □

Part 2 of Corollary 2.4.4 does not extend to $m > 3$ by the following example.

Example 2.4.5. Suppose $K = \{x_1, x_2, x_3, x_4\}$ is a four element set, where $x_1 < x_2 < x_3 < x_4$. For $j, \ell = 1, 2, 3, 4$, we define the polynomials $e_{j\ell}(x) := (x - x_j)(x - x_\ell)$. Fix $k \in \mathbb{R}$ and define the matrix polynomial

$$F_k(x) = A_k \cdot e_{12}(x) + B_k \cdot e_{23}(x) + C_k \cdot e_{34}(x) \in H_2(\mathbb{C}[x]),$$

where the matrices A_k, B_k, C_k are the following

$$A_k = (x_4 - x_3) \cdot \begin{bmatrix} 1 & k \\ k & k^2 \end{bmatrix}, \quad B_k = (x_1 - x_4) \cdot \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}, \quad C_k = (x_2 - x_1) \cdot \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix}.$$

The following statements are true:

(1) $F_k \in \mathrm{Pos}^2_{\succeq 0}(K)$ for $k > 1$.

(2) $F_k \in T^2_S$ for $k > 1$.

(3) $F_k \notin T^2_{S,2}$ for $k \in \left(1, \sqrt{\frac{x_3 - x_1}{x_2 - x_1}}\right)$.

Proof. First we calculate

$$F_k(x_1) = (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{bmatrix} k^2 - 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_k(x_2) = (x_2 - x_1)(x_3 - x_2)(x_4 - x_2) \begin{bmatrix} k^2 & k \\ k & 1 \end{bmatrix},$$

$$F_k(x_3) = (x_3 - x_1)(x_3 - x_2)(x_4 - x_3) \begin{bmatrix} 1 & k \\ k & k^2 \end{bmatrix},$$

$$F_k(x_4) = (x_4 - x_1)(x_4 - x_2)(x_4 - x_3) \begin{bmatrix} 0 & 0 \\ 0 & k^2 - 1 \end{bmatrix}.$$

Note that for $k > 1$, $F_k(x_j) \succeq 0$ for $j = 1, 2, 3, 4$. This proves that $F_k \in \text{Pos}^2_{\succeq 0}(K)$. By Theorem 2.2.1 we conclude that $F_k \in T^2_S$ for $k > 1$. It remains to prove that $F_k \notin T^2_{S,2}$ for every $k \in \big(1, \sqrt{\frac{x_3-x_1}{x_2-x_1}}\big)$. We assume on contrary that $F_k \in T^2_{S_2}$ for some $k \in \big(1, \sqrt{\frac{x_3-x_1}{x_2-x_1}}\big)$. Then we can write F_k in the form

$$F_k(x) = A^*A \cdot e_{12}(x) + B^*B \cdot e_{23}(x) + C^*C \cdot e_{34}(x) + D^*D \cdot (-e_{14}(x)) + G(x)^*G(x), \tag{2.4.9}$$

where $A, B, C, D \in M_2(\mathbb{C})$ and $G(x) = G_1x + G_0$ with $G_1, G_0 \in M_2(\mathbb{C})$. From

$$F(x_2) = C^*C \cdot e_{34}(x_2) + D^*D \cdot (-e_{14}(x_2)) + G(x_2)^*G(x_2),$$
$$F(x_3) = A^*A \cdot e_{12}(x_3) + D^*D \cdot (-e_{14}(x_3)) + G(x_3)^*G(x_3),$$

it follows that

$$\ker F(x_2) \subseteq \ker D^*D \quad \text{and} \quad \ker F(x_3) \subseteq \ker D^*D. \tag{2.4.10}$$

Claim. $\ker F(x_2) + \ker F(x_3) = \mathbb{C}^2$.

Since the matrix

$$\frac{F(x_2)}{(x_2-x_1)(x_3-x_2)(x_4-x_2)} + \frac{F(x_3)}{(x_3-x_1)(x_3-x_2)(x_4-x_3)} = \begin{bmatrix} k^2+1 & 2k \\ 2k & k^2+1 \end{bmatrix}$$

is invertible for $k > 1$ (the determinant is $(k-1)^2(k+1)^2$), it follows that

$$\text{Ran}\, F(x_2) + \text{Ran}\, F(x_3) = \mathbb{C}^2,$$

and hence also $\ker F(x_2) + \ker F(x_3) = \mathbb{C}^2$.

From (2.4.10) and Claim it follows that $D^*D = 0$. Comparing the leading coefficients of both sides of (2.4.9) we get

$$A_k + B_k + C_k = A^*A + B^*B + C^*C + G_1^*G_1. \tag{2.4.11}$$

The right hand side of (2.4.11) is positive semidefinite, while the left hand side is not, since the (11)-minor equals to

$$k^2(x_2 - x_1) + x_1 - x_3 < \frac{x_3 - x_1}{x_2 - x_1}(x_2 - x_1) + x_1 - x_3 = 0.$$

This is a contradiction, which proves that $F_k \notin T^2_{S,2}$ for every $k \in \big(1, \sqrt{\frac{x_3-x_1}{x_2-x_1}}\big)$. $\square$

2.5 Matrix Laurent polynomials

The main result of this section, Theorem 2.5.4 below, is the extension of Theorem 2.1.3 to an arbitrary closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$. By connecting semialgebraic sets in $\mathbb{T}$ with compact semialgebraic sets in $\mathbb{R}$ and matrix Laurent polynomials $M_n(\mathbb{C}[z, \frac{1}{z}])$ with complex matrix polynomials $M_n(\mathbb{C}[x])$ (see Subsection 2.5.1 below), we are able to use Theorem 2.2.1 in the proof of Theorem 2.5.4. However, since matrix quadratic modules in Theorem 2.2.1 are not very strongly boundedly saturated, we can only establish the analog of "hF-proposition" (recall Proposition 2.2.2) for matrix Laurent polynomials (see Proposition 2.5.5 below). To eliminate the denominators we use a result of Scheiderer [Sch06, Proposition 2.7].

The closed semialgebraic set associated to a finite subset $\mathcal{S} = \{b_1, \dots, b_s\}$ of hermitian Laurent polynomials $H_1(\mathbb{C}[z, \frac{1}{z}])$ is given by

$$\mathscr{K}_{\mathcal{S}} = \{z \in \mathbb{T} \colon b_j(z) \geq 0,\ j = 1, \dots, s\}.$$

The *n-th matrix quadratic module generated by $\mathcal{S}$* in $H_n(\mathbb{C}[z, \frac{1}{z}])$ is defined by

$$\mathcal{M}^n_{\mathcal{S}} := \{A_0^* A_0 + A_1^* A_1 \cdot b_1 + \ldots + A_s^* A_s \cdot b_s \colon A_j \in M_n(\mathbb{C}[z]) \text{ for } j = 0, \dots, s\}.$$

Let $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{\mathcal{S}})$ stand for the set of all $n \times n$ hermitian matrix polynomials from $\mathbb{H}_n(\mathbb{C}[z, \frac{1}{z}])$ which are positive semidefinite in every point of $\mathscr{K}_{\mathcal{S}}$. We call the set $\mathcal{M}^n_{\mathcal{S}}$ *saturated* if $\mathcal{M}^n_{\mathcal{S}} = \mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{\mathcal{S}})$. The *degree* of a hermitian matrix Laurent polynomial $A(z) = \sum_{m=-N}^{N} A_m z^m \in H_n\big(\mathbb{C}[z, \frac{1}{z}]\big)$ is N if $A_N \neq 0$. If every $A \in \mathcal{M}^n_{\mathcal{S}}$ from a saturated quadratic module $\mathcal{M}^n_{\mathcal{S}}$ has a representation of the form $\sum_{i=0}^{s} A_i^* A_i \cdot b_i$ with

- $\deg(A_i^* A_i \cdot b_i) \leq \deg(A)$ for every $i = 0, \dots, s$, then $\mathcal{M}^n_{\mathcal{S}}$ is called *very strongly boundedly saturated*,
- $\deg(A_i^* A_i \cdot b_i) \leq f(\deg(A))$ for every $i = 0, \dots, s$, where $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ is some map, then $\mathcal{M}^n_{\mathcal{S}}$ is called *boundedly saturated*.

The analog of the saturated description of a compact semialgebraic set $K \subseteq \mathbb{R}$ for semialgebraic sets in $\mathbb{T}$ is the following. Let $\mathscr{K} \subseteq \mathbb{T}$ be a semialgebraic set. Then the set $\mathcal{S} := \{b_1, \dots, b_s\} \subset H_1\big(\mathbb{C}\big[z, \frac{1}{z}\big]\big)$ is a *saturated description* of $\mathscr{K}$, if the following conditions hold:

(a) $\mathscr{K} = \mathscr{K}_{\mathcal{S}}$.

(b) For every boundary point $a \in \mathscr{K}$, which is not isolated, there exists $k \in \{1, \dots, s\}$, such that $b_k(a) = 0$ and $\frac{db_k}{dz}(a) \neq 0$.

(c) For every isolated point $a \in \mathscr{K}$, there exist $k, \ell \in \{1, \dots, s\}$, such that $b_k(a) = b_\ell(a) = 0$, $\frac{db_k}{dz}(a) \neq 0$, $\frac{db_\ell}{dz}(a) \neq 0$ and $b_k b_\ell \leq 0$ on some neighborhood of a.

2.5.1 Connections

In this subsection we connect closed semialgebraic sets in $\mathbb{T}$ with compact semialgebraic sets in $\mathbb{R}$ and matrix Laurent polynomials $M_n(\mathbb{C}[z, \frac{1}{z}])$ with complex matrix polynomials $M_n(\mathbb{C}[x])$. Every closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$ maps under some Möbius transformations λ to a compact semialgebraic set $\lambda(\mathscr{K}) \subseteq \mathbb{R}$ and every Laurent matrix polynomial $A \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$ maps under the substitution of variables to a complex matrix polynomial $F \in \mathrm{Pos}^n_{\succeq 0}(\lambda(\mathscr{K}))$. Below we will make this precise.

Möbius transformations

Restrictions to $\mathbb{R}\cup\{\infty\}$ of Möbius transformations that map $\mathbb{R}\cup\{\infty\}$ bijectively into $\mathbb{T}$ are exactly the maps of the form

$$\lambda_{z_0,w_0}:\mathbb{R}\cup\{\infty\}\to\mathbb{T},\quad \lambda_{z_0,w_0}(x):=z_0\frac{x-w_0}{x-\overline{w_0}},$$

where $z_0\in\mathbb{T}$ and $w_0\in\mathbb{C}\setminus\mathbb{R}$ are arbitrary by Lemma 2.5.1 below. It is easy to check that their inverses are

$$\lambda^{-1}_{z_0,w_0}:\mathbb{T}\to\mathbb{R}\cup\{\infty\},\quad \lambda^{-1}_{z_0,w_0}(z)=\frac{z\overline{w_0}-z_0w_0}{z-z_0}.$$

Lemma 2.5.1. *Let λ be a Möbius transformation that maps $\mathbb{R}\cup\{\infty\}$ bijectively into $\mathbb{T}$. Then $\lambda|_{\mathbb{R}\cup\{\infty\}}$ is of the form*

$$x\mapsto z_0\frac{x-w_0}{x-\overline{w_0}}\quad \textit{where}\quad z_0\in\mathbb{T}\quad \textit{and}\quad w_0\in\mathbb{C}\setminus\mathbb{R}. \tag{2.5.1}$$

Conversely, every map of the form (2.5.1) maps $\mathbb{R}\cup\{\infty\}$ bijectively into $\mathbb{T}$.

Proof. We know that every Möbius transformation λ is of the form

$$\lambda(z)=\frac{az+b}{cz+d}=\frac{a}{c}\frac{z+\frac{b}{a}}{z+\frac{d}{c}},$$

where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. Assume that $\lambda(\mathbb{R}\cup\{\infty\})=\mathbb{T}$. From $\lambda(0)\in\mathbb{T}$ it follows that $\frac{b}{d}\in\mathbb{T}$ and from $\lambda(\infty)\in\mathbb{T}$ it follows that $\frac{a}{c}\in\mathbb{T}$. Hence also $\frac{b}{d}\frac{c}{a}\in\mathbb{T}$ which implies that $r_1:=\left|\frac{b}{a}\right|=\left|\frac{d}{c}\right|$. From $\lambda(1)\in\mathbb{T}$ and $\frac{a}{c}\in\mathbb{T}$ it follows that $r_2:=\left|1+\frac{b}{a}\right|=\left|1+\frac{d}{c}\right|$. Let $\mathcal{K}(a+ib,R)$ stand for the circle in $\mathbb{C}$ with the center in $a+ib\in\mathbb{C}$, $a,b\in\mathbb{R}$, and radius R. By the above we have

$$\left\{1+\frac{b}{a},1+\frac{d}{c}\right\}\subset\mathcal{K}(1+i\cdot 0,r_1)\bigcap\mathcal{K}(0+i\cdot 0,r_2)=\{w,\overline{w}\}$$

where $w\in\mathbb{C}$. Therefore $\frac{b}{a}=\frac{d}{c}$ or $\frac{b}{a}=\overline{\frac{d}{c}}$. If $\frac{b}{a}=\frac{d}{c}$, then $ad-bc=0$ which is not true. Hence $\frac{b}{a}=\overline{\frac{d}{c}}$. Defining $w_0:=-\frac{b}{a}$, proves that $\lambda|_{\mathbb{R}\cup\{\infty\}}$ is of the form (2.5.1).

The converse is clear by checking that $0,1,\infty$ map to $\mathbb{T}$ under every map of the form (2.5.1) and this precisely determines a Möbius transformation. □

Associating a closed semialgebraic set $\mathscr{K}\subset\mathbb{T}$ with a compact semialgebraic set $K_{z_0,w_0}\subset\mathbb{R}$

Let $\mathscr{K}\subset\mathbb{T}$ be a proper non-empty closed semialgebraic set. Fix $z_0\in\mathbb{T}\setminus\mathscr{K}$ and $w_0\in\mathbb{C}\setminus\mathbb{R}$. Mapping $\mathscr{K}$ with $\lambda^{-1}_{z_0,w_0}$ we get a compact semialgebraic set

$$K_{z_0,w_0}:=\lambda^{-1}_{z_0,w_0}(\mathscr{K})\subseteq\mathbb{R}.$$

Associating a matrix Laurent polynomial $A(z) \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$ with a matrix polynomial $\Gamma_{z_0,w_0,A}(x) \in \mathrm{Pos}^n_{\succeq 0}(K_{z_0,w_0})$

Let $A(z)$ be a matrix Laurent polynomial from the set $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$. Note that a matrix polynomial $\Gamma_{z_0,w_0,A}(x)$ defined by

$$\begin{aligned}\Gamma_{z_0,w_0,A}(x) &= \left(\frac{(x-\overline{w_0})(x-w_0)}{4\cdot \mathrm{Im}(w_0)^2}\right)^{\deg(A)} \cdot A(\lambda_{z_0,w_0}(x)) \\ &= \left(\frac{|x-w_0|}{2\cdot \mathrm{Im}(w_0)}\right)^{2\cdot\deg(A)} \cdot A(\lambda_{z_0,w_0}(x)),\end{aligned}$$

where $\mathrm{Im}(w_0)$ is the imaginary part of w_0, belongs to $\mathrm{Pos}^n_{\succeq 0}(K_{z_0,w_0})$. Note that the degree of $\Gamma_{z_0,w_0,A}(x)$ is at most $2\deg(A)$. We also have

$$\begin{aligned}A(z) &= ((z-z_0)^*(z-z_0))^{\deg(A)} \cdot \Gamma_{z_0,w_0,A}(\lambda^{-1}_{z_0,w_0}(z)) \\ &= |z-z_0|^{2\cdot\deg(A)} \cdot \Gamma_{z_0,w_0,A}(\lambda^{-1}_{z_0,w_0}(z)).\end{aligned}$$

Connection between saturated descriptions of $\mathscr{K} \subseteq \mathbb{T}$ and $K_{z_0,w_0} \subset \mathbb{R}$

Proposition 2.5.2. *Suppose $\mathscr{K} \subseteq \mathbb{T}$ is a proper non-empty semialgebraic set with a saturated description $\mathcal{S} = \{b_1, \dots, b_s\}$. Fix $z_0 \in \mathbb{T} \setminus \mathscr{K}$. The set*

$$S := \{\Gamma_{z_0,w_0,b_1}, \dots, \Gamma_{z_0,w_0,b_s}\}$$

is a saturated description of the set $K_{z_0,w_0} := \lambda^{-1}_{z_0,w_0}(\mathscr{K})$.

To prove Proposition 2.5.2 we need the following lemma.

Lemma 2.5.3. *Let $b(z) \in H_1(\mathbb{C}[z, \frac{1}{z}])$ be a hermitian Laurent polynomial with zeroes $z_1, \dots, z_{2\deg(g)} \in \mathbb{C}$. Fix $z_0 \in \mathbb{T}$ such that $b(z_0) \neq 0$ and $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Then $\Gamma_{z_0,w_0,b}(x) \in \mathbb{R}[x]$ is a polynomial with zeroes*

$$\lambda^{-1}_{z_0,w_0}(z_1), \dots, \lambda^{-1}_{z_0,w_0}(z_{2\deg(g)})$$

and the associated semialgebraic set

$$K_{\Gamma_{z_0,w_0,b}} = \lambda^{-1}_{z_0,w_0}(\mathscr{K}_{\{b\}}).$$

Proof. First we will prove that $\Gamma_{z_0,w_0,b}(x)$ is hermitian, i.e., real. By definition,

$$\begin{aligned}\Gamma_{z_0,w_0,b}(x)^* &= \left(\left(\frac{|x-w_0|}{2\cdot \mathrm{Im}(w_0)}\right)^{2\deg(b)} \cdot b(\lambda_{z_0,w_0}(x))\right)^* \\ &= \left(\frac{|x-w_0|}{2\cdot \mathrm{Im}(w_0)}\right)^{2\deg(b)} \cdot b(\lambda_{z_0,w_0}(x))^* \\ &= \left(\frac{|x-w_0|}{2\cdot \mathrm{Im}(w_0)}\right)^{2\deg(b)} \cdot b(\lambda_{z_0,w_0}(x)) = \Gamma_{z_0,w_0,b}(x),\end{aligned}$$

where we used that b is a hermitian Laurent polynomial in the third equality. Write $k = \deg(b)$. The polynomial b factors as

$$b(z) = c \cdot \frac{\prod_{i=1}^{2k}(z-z_i)}{z^k},$$

where $c \in \mathbb{C}$. Then

$$\Gamma_{z_0,w_0,b}(x) = \left(\frac{(x-w_0)^*(x-w_0)}{4\cdot \operatorname{Im}(w_0)^2}\right)^k \cdot b(\lambda_{z_0,w_0}(x)) \tag{2.5.2}$$
$$= \left(\frac{(x-w_0)^*(x-w_0)}{4\cdot \operatorname{Im}(w_0)^2}\right)^k \cdot c \cdot \frac{\prod_{i=1}^{2k}(z_0 \cdot \frac{x-w_0}{x-\overline{w_0}} - z_i)}{\left(z_0 \cdot \frac{x-w_0}{x-\overline{w_0}}\right)^k}$$
$$= \frac{c\cdot \prod_{i=1}^{2k}(z_0 - z_i)}{(4\cdot \operatorname{Im}(w_0)^2 \cdot z_0)^k} \cdot \prod_{i=1}^{2k}(x - \lambda_{z_0,w_0}^{-1}(z_i)).$$

Since by assumption $b(z_0) \neq 0$, it follows that $\lambda_{z_0,w_0}^{-1}(z_i) \in \mathbb{R}$ and $\Gamma_{z_0,w_0,b}(x)$ is a well-defined real polynomial of degree $2k$ with zeroes $\lambda_{z_0,w_0}^{-1}(z_i)$ for $i = 1,\ldots,2k$. Equality (2.5.2) implies that

$$\begin{aligned} \Gamma_{z_0,w_0,b}(x) \geq 0 \quad &\Leftrightarrow \quad b(\lambda_{z_0,w_0}(x)) \geq 0 \quad \Leftrightarrow \quad \lambda_{z_0,w_0}(x) \in \mathscr{K}_{\{b\}} \\ &\Leftrightarrow \quad x \in \lambda_{z_0,w_0}^{-1}(\mathscr{K}_{\{b\}}) \end{aligned}$$

This proves Lemma 2.5.3. □

Now we will prove Proposition 2.5.2.

Proof of Proposition 2.5.2. We have to prove that $K_{z_0,w_0} = \lambda_{z_0,w_0}^{-1}(K)$ is compact and check all conditions in the definition of a saturated description of K_{z_0,w_0}.

(a) *K_{z_0,w_0} is compact.*
Since $\lambda_{z_0,w_0}^{-1}(z_0) = \infty$, (a) follows from $z_0 \notin \mathscr{K}$.

(b) *For every left endpoint $a \in K_{z_0,w_0}$ there exist $g \in S$ such that $g(a) = 0$ and $g'(a) > 0$.*
We separate two subcases.

Subcase 1: *a is not isolated.* The point $\lambda_{z_0,w_0}(a) \in \mathscr{K}$ is a boundary point, which is not isolated, Since $\mathcal{S}$ is a saturated description of $\mathscr{K}$, there is $b \in \mathcal{S}$ such that $b(\lambda_{z_0,w_0}(a)) = 0$ and $\frac{db}{dz}(\lambda_{z_0,w_0}(a)) \neq 0$. In particular, $\lambda_{z_0,w_0}(a)$ is a simple zero of b. By Lemma 2.5.3, a is a simple zero of $\Gamma_{z_0,w_0,b} \in S$. Since a is a left endpoint of K_{z_0,w_0}, which is not isolated, $\Gamma'_{z_0,w_0,b}(a) > 0$.

Subcase 2: *a is isolated.* The point $\lambda_{z_0,w_0}(a) \in \mathscr{K}$ is isolated. Since $\mathcal{S}$ is a saturated description of $\mathscr{K}$, there are $b, c \in \mathcal{S}$ such that

$$b(\lambda_{z_0,w_0}(a)) = c(\lambda_{z_0,w_0}(a)) = 0, \quad \frac{db}{dz}(\lambda_{z_0,w_0}(a)) \neq 0, \quad \frac{dc}{dz}(\lambda_{z_0,w_0}(a)) \neq 0$$

and

$$bc \leq 0 \quad \text{on some neighbourhood of } \lambda_{z_0,w_0}(a).$$

In particular, $\lambda_{z_0,w_0}(a)$ is a simple zero of both b and c. By Lemma 2.5.3, a is a simple zero of $\Gamma_{z_0,w_0,b} \in S$ and $\Gamma_{z_0,w_0,c} \in S$, and $\Gamma_{z_0,w_0,b} \cdot \Gamma_{z_0,w_0,c} \leq 0$ on some neighbourhood of a. Hence one of $\Gamma'_{z_0,w_0,b}(a)$ and $\Gamma'_{z_0,w_0,c}(a)$ is positive.

(c) *For every right endpoint $a \in K_{z_0,w_0}$ there exist $g \in S$ such that $g(a) = 0$ and $g'(a) < 0$.*

The proof is analogous to the proof of (b).

This concludes the proof of Proposition 2.5.2. $\square$

2.5.2 Saturated matrix quadratic modules

In this subsection we extend Theorem 2.1.3 to an arbitrary closed semialgebraic set $\mathscr{K} \subseteq \mathbb{T}$.

Theorem 2.5.4. *Suppose $\mathscr{K} \subseteq \mathbb{T}$ is a non-empty closed semialgebraic set. If $\mathcal{S}$ is a saturated description of $\mathscr{K}$, then the n-th matrix quadratic module $\mathcal{M}^n_{\mathcal{S}}$ is saturated for every $n \in \mathbb{N}$.*

Let $b_0 := 1$. For a finite set $\mathcal{S} = \{b_1, \ldots, b_s\} \subset H_1(\mathbb{C}[z, \frac{1}{z}])$ we define the *bounded part* $\mathcal{M}^n_{\mathcal{S},b}$ of the n-th matrix quadratic module $\mathcal{M}^n_{\mathcal{S}}$ by

$$\mathcal{M}^n_{\mathcal{S},b} = \left\{ A := \sum_{i=0}^{s} B_i^* B_i \cdot b_i \colon B_i \in M_n(\mathbb{C}[x]) \text{ and } \deg(B_i^* B_i \cdot b_i) \leq \deg(A) \ \forall i \right\}.$$

In the proof of Theorem 2.5.4 we need the following analog of "hF-proposition" for matrix Laurent polynomials.

Proposition 2.5.5. *Suppose $\mathscr{K}$ is a proper non-empty basic closed semialgebraic set in $\mathbb{T}$ with a saturated description of $\mathcal{S} = \{b_1, \ldots, b_s\}$. Then, for any hermitian matrix Laurent polynomial $A \in H_n(\mathbb{C}[z, \frac{1}{z}])$ such that $A \succeq 0$ on $\mathscr{K}$ and every point $z_0 \in \mathbb{T} \setminus \mathscr{K}$, there exists $k_{z_0} \in \mathbb{N} \cup \{0\}$ such that $|z - z_0|^{2k_{z_0}} \cdot A \in \mathcal{M}^n_{\mathcal{S},b}$.*

Proof. Let us take $A \succeq 0$ on $\mathscr{K}$ and a point $z_0 \in \mathbb{T} \setminus \mathscr{K}$. Fix $w_0 \in \mathbb{C} \setminus \mathbb{R}$. Let $K_{z_0,w_0} \subset \mathbb{R}$ and $\Gamma_{z_0,w_0,A}(x) \in \mathrm{Pos}^n_{\succeq 0}(K_{z_0,w_0})$ be defined as in Subsection 2.5.1. Since $z_0 \notin \mathscr{K}$, it follows that K_{z_0,w_0} is compact. By Proposition 2.5.2, the set

$$S := \{\Gamma_{z_0,w_0,b_1}, \ldots, \Gamma_{z_0,w_0,b_s}\}$$

is a saturated descripiton of K_{z_0,w_0}. Let us define $b_0 = 1 \in \mathbb{R}[x]$. Then $\Gamma_{z_0,w_0,b_0} = 1$. By Theorem 2.2.1,

$$\Gamma_{z_0,w_0,A}(x) = \sum_{i=0}^{s} G_i^* G_i \cdot \Gamma_{z_0,w_0,b_i} \in M^n_S,$$

where $G_i \in M_n(\mathbb{C}[x])$ for each i. Then for

$$k = \max_{i=0,\ldots,s} \left\{ \frac{\deg(G_i^* G_i)}{2} + \deg(b_i) - \deg(A) \right\},$$

we have

$$\begin{aligned}
|z-z_0|^{2k}\cdot A(z) &= |z-z_0|^{2\cdot\deg(A)+2k}\cdot\Gamma_{z_0,w_0,A}(\lambda_{z_0,w_0}^{-1}(z))\\
&= |z-z_0|^{2\cdot\deg(A)+2k}\cdot\left(\sum_{i=0}^{s} G_i^*G_i\cdot\Gamma_{z_0,w_0,b_i}\right)(\lambda_{z_0,w_0}^{-1}(z))\\
&= \sum_{i=0}^{s}\Big(|z-z_0|^{2\cdot\deg(A)+2k}\cdot(G_i^*G_i\cdot\Gamma_{z_0,w_0,b_i})(\lambda_{z_0,w_0}^{-1}(z))\Big)\\
&= \sum_{i=0}^{s}|z-z_0|^{2\cdot\deg(A)+2k-2\deg(b_i)}\cdot(G_i^*G_i)(\lambda_{z_0,w_0}^{-1}(z))\cdot b_i(z)\in\mathcal{M}_{\mathcal{S},b}^n,
\end{aligned}$$

where we used the estimate

$$2\cdot\deg(A)+2k-2\deg(b_i)\geq\deg(G_i^*G_i)$$

in the last equality. This concludes the proof of the proposition. □

The following result of Scheiderer [Sch06, Proposition 2.7] will be essentially used in the proof of Theorem 2.5.4 to eliminate denominators from Proposition 2.5.5.

Proposition 2.5.6 (Scheiderer)**.** *Suppose R is a commutative ring with 1 and $\mathbb{Q}\subseteq R$. Let $\Phi: R\to C(K,\mathbb{R})$ be a ring homomorphism, where K is a topological space which is compact and Hausdorff. Suppose $\Phi(R)$ separates points in K. Suppose $f_1,\dots,f_k\in R$ are such that $\Phi(f_j)\geq 0$ on K for each j and the ideal $I=\langle f_1,\dots,f_k\rangle$ equals to R. Then there exist $s_1,\dots,s_k\in R$ such that $s_1f_1+\ldots+s_kf_k=1$ and $\Phi(s_j)>0$ on K for each j.*

Now we are ready to prove Theorem 2.5.4.

Proof of Theorem 2.5.4. Suppose $\mathcal{S}$ is a saturated description of $\mathscr{K}$. We will prove that the set $\mathcal{M}_{\mathcal{S}}^n$ is saturated for every $n\in\mathbb{N}$. Let $\Phi: H_1(\mathbb{C}[z,\frac{1}{z}])\to C(\mathbb{T},\mathbb{R})$ be the natural map, i.e., $\Phi(a)=a|_{\mathbb{T}}$. The map Φ is a ring homomorphism and $\mathbb{T}$ is a compact Hausdorff topological space.

Claim 1. $\Phi(H_1(\mathbb{C}[z,\frac{1}{z}]))$ separates points in $\mathbb{T}$.

Proof of Claim 1. Define Laurent polynomials $a_1(z)=\frac{1}{z}+z\in H_1(\mathbb{C}[z,\frac{1}{z}])$ and $a_2(z)=i(\frac{1}{z}-z)\in H_1(\mathbb{C}[z,\frac{1}{z}])$. For $z_1\in\mathbb{T}$, $z_2\in\mathbb{T}$ we have

$$\begin{aligned}
a_1(z_1)=a_2(z_2)\quad &\Leftrightarrow\quad \frac{1}{z_1}+z_1=\frac{1}{z_2}+z_2\quad\Leftrightarrow\quad\frac{1+z_1^2}{z_1}=\frac{1+z_2^2}{z_2}\\
&\Leftrightarrow\quad (1+z_1^2)z_2=(1+z_2^2)z_1\quad\Leftrightarrow\quad z_2-z_1=z_1z_2(z_2-z_1)\\
&\Leftrightarrow\quad z_2\in\{z_1,\overline{z_1}\}.
\end{aligned}$$

So a_1 separates all non-conjugate pairs z_1,z_2. Similarly,

$$a_2(z_1)=a_2(z_2)\quad\Leftrightarrow\quad z_1z_2(z_2-z_1)=z_1-z_2\quad\Leftrightarrow\quad z_2\in\{z_1,-\overline{z_1}\}.$$

So a_2 separates all conjugate pairs z_1,z_2. This proves Claim 1.

Let $A \in \mathrm{Pos}^n_{\succeq 0}(\mathscr{K})$. We will prove that $A \in \mathcal{M}^n_{\mathcal{S}}$. We define the ideal I' in $\mathbb{C}\big[z, \frac{1}{z}\big]$ by

$$I' := \Big\langle b^*b \in H_1\big(\mathbb{C}[z, \frac{1}{z}]\big) : b \in \mathbb{C}[z] \text{ and } b^*b \cdot A \in \mathcal{M}^n_{\mathcal{S}} \Big\rangle \subseteq \mathbb{C}\Big[z, \frac{1}{z}\Big].$$

Claim 2. $I' = \mathbb{C}\big[z, \frac{1}{z}\big]$.

Proof of Claim 2. Maximal ideals in $\mathbb{C}\big[z, \frac{1}{z}\big]$ are precisely $\langle z - w\rangle$, where $w \in \mathbb{C} \setminus \{0\}$. By Proposition 2.5.5, for every $w \in \mathbb{C} \setminus \{0\}$ there exists $b \in \mathbb{C}[z]$, such that $(b^*b)(w) \neq 0$ and $b^*b \cdot A \in \mathcal{M}^n_{\mathcal{S}}$. Therefore $I' \not\subseteq \langle z - w\rangle$ for every $w \in \mathbb{C} \setminus \{0\}$ and hence $I' = \mathbb{C}[z, \frac{1}{z}]$.

Now we define the ideal I in $H_1(\mathbb{C}[z, \frac{1}{z}])$ by

$$I := \Big\langle b^*b \in H_1\big(\mathbb{C}[z, \frac{1}{z}]\big) : b \in \mathbb{C}[z] \text{ and } b^*b \cdot A \in \mathcal{M}^n_{\mathcal{S}} \Big\rangle \subseteq H_1(\mathbb{C}[z, \frac{1}{z}]).$$

Claim 3. $I = H_1(\mathbb{C}\big[z, \frac{1}{z}\big])$.

Proof of Claim 3. By Claim 2, $I' = \mathbb{C}\big[z, \frac{1}{z}\big]$. From this fact and since I' is finitely generated, there exist $b_1, \dots, b_m \in \mathbb{C}[z]$ and $c_1, \dots, c_m \in \mathbb{C}\big[z, \frac{1}{z}\big]$ such that

$$b_j^* b_j \cdot A \in \mathcal{M}^n_{\mathcal{S}} \quad \text{for each } j \quad \text{and} \quad \sum_{j=1}^m c_j(b_j^* b_j) = 1.$$

This implies that $\sum_{j=1}^m c_j^*(b_j^* b_j) = 1$ and hence,

$$\sum_{j=1}^m \frac{c_j^* + c_j}{2}(b_j^* b_j) = 1 \in I.$$

This proves Claim 3.

Now we use Proposition 2.5.6 for the ring $R = H_1(\mathbb{C}[z, \frac{1}{z}])$, the homomorphism $\Phi : H_1(\mathbb{C}[z, \frac{1}{z}]) \to C(\mathbb{T}, \mathbb{R})$ and the ideal I (which is finitely generated) and obtain $d_1, \dots, d_m \in \mathrm{Pos}^1_{\succ 0}(\mathbb{T})$, such that

$$\sum_{j=1}^m d_j(b_j^* b_j) = 1 \quad \text{and} \quad b_j^* b_j A \in \mathcal{M}^n_{\mathcal{S}}.$$

Since each d_j equals to $\tilde{d}_j^* \tilde{d}_j$ for some $\tilde{d}_j \in \mathbb{C}[z]$ by Theorem 2.1.3, it follows that

$$d_j(b_j^* b_j A) = \tilde{d}_j^* \tilde{d}_j (b_j^* b_j A) \in \tilde{d}_j^* \tilde{d}_j \cdot \mathcal{M}^n_{\mathcal{S}} \subseteq \mathcal{M}^n_{\mathcal{S}}.$$

Therefore

$$A = \sum_{j=1}^m (d_j b_j^* b_j) A = \sum_{j=1}^m d_j(b_j^* b_j A) \in \mathcal{M}^n_{\mathcal{S}},$$

which concludes the proof of Theorem 2.5.4. □

2.6 Positivstellensatz on unbounded sets

By the results of Section 2.3, for almost every unbounded set K there are matrix polynomials positive semidefinite on K that do not belong to a matrix preordering generated by any finite set S with $K = K_S$. The main result of this section, Theorem 2.6.1 below, is Positivstellensatz for matrix polynomials positive semidefinite on unbounded closed semialgebraic sets. By connecting semialgebraic sets in K with semialgebraic sets in $\mathbb{T}$ and complex matrix polynomials $M_n(\mathbb{C}[x])$ with matrix Laurent polynomials $M_n(\mathbb{C}[z, \frac{1}{z}])$ (see Subsection 2.6.1 below), we will be able to use Theorem 2.5.4 in the proof of Theorem 2.6.1.

Theorem 2.6.1 (Non-compact Positivstellensatz)**.** *Suppose $K \subset \mathbb{R}$ is a proper unbounded closed semialgebraic set and S the natural description of K. Then, for any hermitian matrix polynomial $F \in H_n(\mathbb{C}[x])$, the following are equivalent:*

(1) $F \in \mathrm{Pos}^n_{\succeq 0}(K)$.

(2) For every point $w \in \mathbb{C} \setminus K$ there exists $k_w \in \mathbb{N} \cup \{0\}$ such that

$$|x - w|^{2k_w} \cdot F \in M^n_S.$$

(3) There exists $k \in \mathbb{N} \cup \{0\}$ such that

$$(1 + x^2)^k \cdot F \in M^n_S.$$

(4) For every natural number $p \in \mathbb{N}$ there exists a polynomial $h \in \mathrm{Pos}^1_{>0}(\mathbb{R})$ and a matrix polynomial $G \in M^n_S$ such that

$$hF = F^{2p} + G \in M^n_S.$$

Remark 2.6.2. The characterization of $\mathrm{Pos}^n_{\succeq 0}(K)$ in the case of multivariate real matrix polynomials is [Cim12, Theorem B]. The improvement of [Cim12, Theorem B] in the univariate case is the fact, that h in Theorem 2.6.1 (4) above can be taken from $\mathbb{R}[x]$ instead of $M_n(\mathbb{R}[x])$ and that we can take M^n_S instead of T^n_S.

To prove Theorem 2.6.1 we first establish the connection between semialgebraic sets in K with semialgebraic sets in $\mathbb{T}$ and complex matrix polynomials $M_n(\mathbb{C}[x])$ with matrix Laurent polynomials $M_n(\mathbb{C}[z, \frac{1}{z}])$, which is the contents of the next subsection. Recall that the reverse direction was given in Subsection 2.5.1 above.

2.6.1 Reverse connections

In this subsection we connect closed semialgebraic sets in $\mathbb{R}$ with closed semialgebraic sets in $\mathbb{T}$ and complex matrix polynomials $M_n(\mathbb{C}[x])$ with matrix Laurent polynomials $M_n(\mathbb{C}[z, \frac{1}{z}])$. Every closed semialgebraic set $K \subseteq \mathbb{R}$ maps under some Möbius transformations λ to a closed semialgebraic set $\lambda(K) \subseteq \mathbb{T}$ and every complex matrix polynomial $F \in \mathrm{Pos}^n_{\succeq 0}(K)$ maps under the substitution of variables to a matrix Laurent polynomial $A \in \mathrm{Pos}^n_{\succeq 0}(\lambda(K))$. Below we will make this precise.

Möbius transformations

Recall from Subsection 2.5.1 that the restriction to $\mathbb{R}\cup\{\infty\}$ of Möbius transformations that map $\mathbb{R}\cup\{\infty\}$ bijectively into $\mathbb{T}$ are exactly the maps of the form

$$\lambda_{z_0,w_0}:\mathbb{R}\cup\{\infty\}\to\mathbb{T},\quad \lambda_{z_0,w_0}(x):=z_0\frac{x-w_0}{x-\overline{w_0}},$$

where $z_0\in\mathbb{T}$ and $w_0\in\mathbb{C}\setminus\mathbb{R}$ are arbitrary. Their inverses are

$$\lambda_{z_0,w_0}^{-1}:\mathbb{T}\to\mathbb{R}\cup\{\infty\},\quad \lambda_{z_0,w_0}^{-1}(z)=\frac{z\overline{w_0}-z_0w_0}{z-z_0}.$$

Associating a closed semialgebraic set $K\subset\mathbb{R}$ with a semialgebraic set $\mathscr{K}_{z_0,w_0}\subset\mathbb{T}$

Let $K\subseteq\mathbb{R}$ be a closed semialgebraic set. Fix $z_0\in\mathbb{T}\setminus\mathscr{K}$ and $w_0\in\mathbb{C}\setminus\mathbb{R}$. Mapping K with λ_{z_0,w_0} and taking a closure we get a closed semialgebraic set

$$\mathscr{K}_{z_0,w_0}:=\overline{\lambda_{z_0,w_0}(K)}\subseteq\mathbb{T}.$$

Associating a matrix polynomial $F(x)\in\mathrm{Pos}^n_{\succeq 0}(K)$ with a matrix Laurent polynomial $\Lambda_{z_0,w_0,F}(z)\in\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{z_0,w_0})$

Let $F(x)$ be a matrix polynomial from the set $\mathrm{Pos}^n_{\succeq 0}(K)$. Note that a matrix Laurent polynomial $\Lambda_{z_0,w_0,F}(z)$ defined by

$$\begin{aligned}\Lambda_{z_0,w_0,F}(z)&:=((z-z_0)^*(z-z_0))^{\lceil\frac{\deg(F)}{2}\rceil}\cdot F\big(\lambda_{z_0,w_0}^{-1}(z)\big)\\ &=|z-z_0|^{2\cdot\lceil\frac{\deg(F)}{2}\rceil}\cdot F\big(\lambda_{z_0,w_0}^{-1}(z)\big),\end{aligned}\tag{2.6.1}$$

where $\lceil\cdot\rceil$ is the ceiling function, belongs to $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{z_0,w_0})$. Note that the degree of $\Lambda_{z_0,w_0,F}(z)$ is at most $\left\lceil\frac{\deg(F)}{2}\right\rceil$. We also have

$$\begin{aligned}F(x)&=\left(\frac{(x-\overline{w_0})(x-w_0)}{4\cdot\mathrm{Im}(w_0)^2}\right)^{\lceil\frac{\deg(F)}{2}\rceil}\cdot\Lambda_{z_0,w_0,F}(\lambda_{z_0,w_0}(x))\\ &=\left(\frac{|x-w_0|}{2\cdot\mathrm{Im}(w_0)}\right)^{2\cdot\lceil\frac{\deg(F)}{2}\rceil}\cdot\Lambda_{z_0,w_0,F}(\lambda_{z_0,w_0}(x)),\end{aligned}$$

where $\mathrm{Im}(w_0)$ is the imaginary part of w_0.

Connection between the natural description of $K\subseteq\mathbb{R}$ and a saturated description of $\mathscr{K}_{z_0,w_0}\subseteq\mathbb{T}$

Proposition 2.6.3. *Suppose $K\subseteq\mathbb{R}$ is an unbounded semialgebraic set with a natural description $S:=\{g_1(x),\dots,g_s(x)\}\subset\mathbb{R}[x]$. Fix $z_0\in\mathbb{T}$ and $w_0\in\mathbb{C}\setminus\mathbb{R}$. Then the set*

$$\mathcal{S}:=\{\Lambda_{z_0,w_0,g_1},\dots,\Lambda_{z_0,w_0,g_s}\}$$

is a saturated description of $\mathscr{K}_{z_0,w_0}$.

To prove Proposition 2.6.3 we need the following lemma.

Lemma 2.6.4. *Let* $g(x) = \sum_{i=0}^{\deg(g)} a_i x^i \in \mathbb{R}[x]$ *be a non-zero real polynomial with zeroes* $x_1, \dots, x_{\deg(g)} \in \mathbb{C}$. *Fix* $z_0 \in \mathbb{T}$ *and* $w_0 \in \mathbb{C} \setminus \mathbb{R}$ *such that* $g(w_0) \neq 0$. *Then* $\Lambda_{z_0,w_0,g}(z) \in H_1\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right)$ *is a hermitian Laurent polynomial and:*

(1) If $\deg(g) \in 2\mathbb{N} \cup \{0\}$, *then* $\Lambda_{z_0,w_0,g}(z)$ *is of degree* $\frac{\deg(g)}{2}$, *has zeroes*

$$\lambda_{z_0,w_0}(x_1), \dots, \lambda_{z_0,w_0}(x_{\deg(g)})$$

and the associated semialgebraic set

$$\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \begin{cases} \lambda_{z_0,w_0}(K_{\{g\}}) \cup \{z_0\}, & \text{if } a_{2k} > 0 \\ \lambda_{z_0,w_0}(K_{\{g\}}), & \text{if } a_{2k} < 0 \end{cases}.$$

(2) If $\deg(g) - 1 \in 2\mathbb{N} \cup \{0\}$, *then* $\Lambda_{z_0,w_0,g}(z)$ *is of degree* $\lceil \frac{\deg(g)}{2} \rceil$, *has zeroes*

$$\lambda_{z_0,w_0}(x_1), \dots, \lambda_{z_0,w_0}(x_{\deg(g)}), z_0$$

and the associated semialgebraic set

$$\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \lambda_{z_0,w_0}(K_{\{g\}}) \cup \{z_0\}.$$

(3) If $\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \overline{\lambda_{z_0,w_0}(K_{\{g\}})}$.

Proof. First we will prove that $\Lambda_{z_0,w_0,g}(z)^*$ is hermitian. By definition,

$$\begin{aligned}\Lambda_{z_0,w_0,g}(z)^* &= \left(|z - z_0|^{2\cdot\lceil \frac{\deg(F)}{2} \rceil} \cdot g\big(\lambda_{z_0,w_0}^{-1}(z)\big)\right)^* = |z - z_0|^{2\cdot\lceil \frac{\deg(F)}{2} \rceil} \cdot g\big(\lambda_{z_0,w_0}^{-1}(z)^*\big) \\ &= |z - z_0|^{2\cdot\lceil \frac{\deg(F)}{2} \rceil} \cdot g\big(\lambda_{z_0,w_0}^{-1}(z)\big) = \Lambda_{z_0,w_0,g}(z),\end{aligned}$$

where we used that g is a real polynomial in the second equality of the first line and $\lambda_{z_0,w_0}^{-1}(z)^* = \lambda_{z_0,w_0}^{-1}(z)$ in the first equality of the second line. Now we separate two cases:

Case 1: $\deg(g) \in 2\mathbb{N} \cup \{0\}$.

Write $\deg(g) = 2k$. The polynomial g factors as

$$g(x) = a_{2k} \cdot \prod_{i=1}^{2k} (x - x_i),$$

where $a_{2k} \in \mathbb{R}$. Then

$$\begin{aligned}\Lambda_{z_0,w_0,g}(z) &= ((z - z_0)^*(z - z_0))^k \cdot g(\lambda_{z_0,w_0}^{-1}(z)) \qquad (2.6.2)\\ &= a_{2k} \cdot (-1)^k \cdot \frac{(z - z_0)^{2k}}{(z_0 z)^k} \cdot \prod_{i=1}^{2k} \left(\frac{z\overline{w_0} - z_0 w_0}{z - z_0} - x_i \right) \\ &= \frac{a_{2k} \cdot (-1)^k \cdot \prod_{i=1}^{2k} (\overline{w_0} - x_i)}{z_0^k} \cdot \frac{\prod_{i=1}^{2k} (z - \lambda_{z_0,w_0}(x_i))}{z^k}.\end{aligned}$$

Since by assumption $g(w_0) \neq 0$, it follows that $0 = \lambda_{z_0,w_0}(w_0) \neq \lambda_{z_0,w_0}(x_i)$ for each i, and hence $\Lambda_{z_0,w_0,g}(z)$ is of degree k and has zeroes $\lambda_{z_0,w_0}(x_i)$ for $i = 1, \ldots, 2k$. Equality (2.6.2) implies that

$$\begin{aligned}
\Lambda_{z_0,w_0,g}(z) \geq 0 \quad &\Leftrightarrow \quad (z \neq z_0 \text{ and } g(\lambda^{-1}_{z_0,w_0}(z)) \geq 0) \text{ or } (z = z_0 \text{ and } a_{2k} > 0) \\
&\Leftrightarrow \quad (\lambda^{-1}_{z_0,w_0}(z) \in K_{\{g\}})) \text{ or } (z = z_0 \text{ and } a_{2k} > 0) \\
&\Leftrightarrow \quad (z \in \lambda_{z_0,w_0}(K_{\{g\}})) \text{ or } (z = z_0 \text{ and } a_{2k} > 0).
\end{aligned}$$

This proves part (1) of Lemma 2.6.4.

Case 2: $\deg(g) - 1 \in 2\mathbb{N} \cup \{0\}$.

Write $\deg(g) = 2k - 1$. The polynomial g factors as

$$g(x) = a_{2k-1} \cdot \prod_{i=1}^{2k-1} (x - x_i),$$

where $a_{2k-1} \in \mathbb{R}$. Then

$$\begin{aligned}
\Lambda_{z_0,w_0,g}(z) &= ((z - z_0)^*(z - z_0))^k \cdot g(\lambda^{-1}_{z_0,w_0}(z)) \qquad (2.6.3) \\
&= a_{2k-1} \cdot (-1)^k \cdot \frac{(z - z_0)^{2k}}{(z_0 z)^k} \cdot \prod_{i=1}^{2k-1} \left(\frac{z\overline{w_0} - z_0 w_0}{z - z_0} - x_i \right) \\
&= \frac{a_{2k-1} \cdot (-1)^k \cdot \prod_{i=1}^{2k-1} (\overline{w_0} - x_i)}{z_0^k} \cdot \frac{(z - z_0) \cdot \prod_{i=1}^{2k-1} (z - \lambda_{z_0,w_0}(x_i))}{z^k}.
\end{aligned}$$

Since by assumption $g(w_0) \neq 0$, it follows that $0 = \lambda_{z_0,w_0}(w_0) \neq \lambda_{z_0,w_0}(x_i)$ for each i, and hence $\Lambda_{z_0,w_0,g}(z)$ is of degree k and has zeroes $\lambda_{z_0,w_0}(x_i)$ for $i = 1, \ldots, 2k - 1$ and z_0. Equality (2.6.3) together with $\Lambda_{z_0,w_0,g}(z_0) = 0$ implies that

$$\begin{aligned}
\Lambda_{z_0,w_0,g}(z) \geq 0 \quad &\Leftrightarrow \quad (z \neq z_0 \text{ and } g(\lambda^{-1}_{z_0,w_0}(z)) \geq 0) \text{ or } (z = z_0) \\
&\Leftrightarrow \quad (\lambda^{-1}_{z_0,w_0}(z) \in K_{\{g\}})) \text{ or } (z = z_0) \\
&\Leftrightarrow \quad (z \in \lambda_{z_0,w_0}(K_{\{g\}})) \text{ or } (z = z_0).
\end{aligned}$$

This proves part (2) of Lemma 2.6.4.

To prove part (3) of Lemma 2.6.4, notice that if $\deg(g)$ is even and $a_{2k} < 0$, then $\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \lambda_{z_0,w_0}(K_{\{g\}})$ and z_0 does not lie on the boundary of $\lambda_{z_0,w_0}(K_{\{g\}})$. Hence $\lambda_{z_0,w_0}(K_{\{g\}})$ is closed and $\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \overline{\lambda_{z_0,w_0}(K_{\{g\}})}$. If $\deg(g)$ is even and $a_{2k} > 0$ or $\deg(g)$ is odd, then $\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \lambda_{z_0,w_0}(K_{\{g\}}) \cup \{z_0\}$ and z_0 lies on the boundary of $\lambda_{z_0,w_0}(K_{\{g\}})$. This again implies that $\mathscr{K}_{\{\Lambda_{z_0,w_0,g}\}} = \overline{\lambda_{z_0,w_0}(K_{\{g\}})}$. □

Now we will prove Proposition 2.6.3.

Proof of Proposition 2.6.3. We have to check all three conditions in the definition of the saturated description of $\mathscr{K}_{z_0,w_0} = \lambda_{z_0,w_0}(K)$.

(a) $\mathscr{K}_{z_0,w_0} = \mathscr{K}_{\mathcal{S}}$.

Notice that

$$\mathscr{K}_{\mathcal{S}} = \bigcap_{i=1}^{s} \mathscr{K}_{\{\Lambda_{z_0,w_0,g_i}\}} = \bigcap_{i=1}^{s} \overline{\lambda_{z_0,w_0}(K_{\{g_i\}})} = \bigcap_{i=1}^{s} \big(\lambda_{z_0,w_0}(K_{\{g_i\}}) \cup \{z_0\}\big)$$
$$= \left(\bigcap_{i=1}^{s} \lambda_{z_0,w_0}(K_{\{g_i\}})\right) \cup \{z_0\} = \lambda_{z_0,w_0}(K) \cup \{z_0\} = \overline{\lambda_{z_0,w_0}(K)} = \mathscr{K}_{z_0,w_0},$$

where the second equality follows by Lemma 2.6.4 (3), the third by Lemma 2.6.4 (1),(2) and the fact that the polynomials g_i in the natural description of K are either of even degree and have a positive leading coefficient or are of odd degree, the fifth equality by the bijectivity of λ_{z_0,w_0} and the sixth equality by the fact that K_S is unbounded.

(b) *For every boundary point $a \in \mathscr{K}_{z_0,w_0}$, which is not isolated, there exists $b \in \mathcal{S}$ such that $b(a) = 0$ and $\frac{db}{dz}(a) \neq 0$.*

We separate two subcases.

Subcase 1: $a \neq z_0$. Notice that $\lambda^{-1}_{z_0,w_0}(a)$ is a non-isolated boundary point of K. Since S is a natural description of K, there is $g \in S$ such that $g(\lambda^{-1}_{z_0,w_0}(a)) = 0$ and $g'(\lambda^{-1}_{z_0,w_0}(a)) \neq 0$. In particular, $\lambda^{-1}_{z_0,w_0}(a)$ is a simple zero of g. By Lemma 2.6.4, a is a simple zero of $\Lambda_{z_0,w_0,g} \in \mathcal{S}$.

Subcase 2: $a = z_0$. Notice that since z_0 is a boundary point of $\mathscr{K}_{z_0,w_0}$, the set K must be bounded from above or from below. In the first case, S contains a polynomial of the form $c - x$, $c \in \mathbb{R}$, and hence a polynomial

$$\Lambda_{z_0,w_0,c-x}(z) = (c - \overline{w_0})\frac{(z_0 - z)(z - \lambda_{z_0,w_0}(c))}{zz_0}$$

has a simple zero in z_0. In the second case, contains a polynomial of the form $x - d$, $d \in \mathbb{R}$, and hence a polynomial

$$\Lambda_{z_0,w_0,x-d}(z) = (c - \overline{w_0})\frac{(z_0 - z)(\lambda_{z_0,w_0}(d) - z)}{zz_0}$$

has a simple zero in z_0.

(c) *For every isolated point $a \in \mathscr{K}_{z_0,w_0}$, there exist $b, c \in \mathcal{S}$ such that $b(a) = c(a) = 0$, $\frac{db}{dz}(a) \neq 0$, $\frac{dc}{dz}(a) \neq 0$ and $bc \leq 0$ on some neighbourhood of a.*

Notice that since K is unbounded, z_0 is not an isolated point of $\mathscr{K}_{z_0,w_0}$. Therefore $\lambda^{-1}_{z_0,w_0}(a)$ is an isolated point of K. Since S is a natural description of K, there are $g, h \in S$ such that $g(\lambda^{-1}_{z_0,w_0}(a)) = h(\lambda^{-1}_{z_0,w_0}(a)) = 0$ and $g'(\lambda^{-1}_{z_0,w_0}(a)) > 0$ and $h'(\lambda^{-1}_{z_0,w_0}(a)) < 0$. In particular, $\lambda^{-1}_{z_0,w_0}(a)$ is a simple zero of both g and h, and $gh \leq 0$ on some neighbourhood of a. By Lemma 2.6.4, a is a simple zero of both $\Lambda_{z_0,w_0,g} \in \mathcal{S}$ and $\Lambda_{z_0,w_0,h} \in \mathcal{S}$, and $\Lambda_{z_0,w_0,g} \cdot \Lambda_{z_0,w_0,h} \leq 0$, which proves (c).

This concludes the proof of Proposition 2.6.3. $\square$

2.6.2 Proof of the Positivstellensatz

In this subsection we prove Theorem 2.6.1.

Proof of the equivalence (1) $\Leftrightarrow$ (2) **of Theorem 2.6.1**

The non-trivial implication is ($\Rightarrow$). First we prove the following claim.

Claim. For every $w \in \mathbb{C} \setminus \mathbb{R}$ the matrix Laurent polynomial $\Lambda_{1,w,F}(z)$ belongs to the set $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{1,w})$.

Proof of Claim. By the equality (2.6.1) it follows that $\Lambda_{1,w,F}(z_0) \succeq 0$ if and only if one of the following is true:

(1) $z_0 \neq 1$ and $F(\lambda_{1,i}^{-1}(z_0)) \succeq 0$,

(2) $z_0 = 1$ and F is of odd degree (in which case $\Lambda_{1,w,F}(1) = 0$),

(3) $z_0 = 1$, $F(x) := \sum_{j=0}^{2k} F_j x^j \in M_n(\mathbb{C})[x]$ is of even degree $2k \in 2\mathbb{N} \cup \{0\}$ and $F_{2k} \succeq 0$.

If F is of even degree $2k$, then since the set K is unbounded, we have $F_{2k} \succeq 0$. Using this together with (1),(2),(3) above we conclude that the matrix Laurent polynomial $\Lambda_{1,w,F}(z)$ belongs to the set $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{1,w})$. This proves Claim.

We write $S = \{g_1, \ldots, g_s\}$. We separate two cases.

Case 1: $w \in \mathbb{C} \setminus \mathbb{R}$. The set

$$\mathcal{S} := \{\Lambda_{1,w,g_1}(z), \ldots, \Lambda_{1,w,g_s}(z)\}$$

is a saturated description of $\mathscr{K}_{1,w}$ by Proposition 2.6.3. By Claim and Theorem 2.5.4,

$$\Lambda_{1,w,F}(z) = \sum_{i=0}^{s} A_i^* A_i \cdot \Lambda_{1,w,g_i} \in \mathcal{M}^n_{\mathcal{S}},$$

where $A_i \in M_n(\mathbb{C}[z, \frac{1}{z}])$ for each i. Then for

$$k_w = \max_{i=0,\ldots,s} \left\{ \deg(A_i^* A_i) + \deg(\Lambda_{1,w,g_i}) - \left\lceil \frac{\deg(F)}{2} \right\rceil \right\},$$

we have

$$
\begin{aligned}
\left(\frac{|x-w|}{2\cdot\operatorname{Im}(w)}\right)^{2k_w}\cdot F(x) &= \left(\frac{|x-w|}{2\cdot\operatorname{Im}(w)}\right)^{2k_w+2\left\lceil\frac{\deg(F)}{2}\right\rceil}\cdot\Lambda_{1,w,F}(\lambda_{1,w}(x))\\
&= \left(\frac{|x-w|}{2\cdot\operatorname{Im}(w)}\right)^{2k_w+2\left\lceil\frac{\deg(F)}{2}\right\rceil}\cdot\left(\sum_{i=0}^{s}A_i^*A_i\cdot\Lambda_{1,w,g_i}\right)(\lambda_{1,w}(x))\\
&= \sum_{i=0}^{s}\left(\left(\frac{|x-w|}{2\cdot\operatorname{Im}(w)}\right)^{2k_w+2\left\lceil\frac{\deg(F)}{2}\right\rceil}\cdot((A_i^*A_i)\cdot\Lambda_{1,w,g_i})(\lambda_{1,w}(x))\right)\\
&= \sum_{i=0}^{s}\left(\frac{|x-w|}{2\cdot\operatorname{Im}(w)}\right)^{2k_w+2\left\lceil\frac{\deg(F)}{2}\right\rceil-2\left\lceil\frac{\deg(g_i)}{2}\right\rceil}\cdot(A_i^*A_i)(\lambda_{1,w}(x))\cdot g_i(x)\in M_S^n,
\end{aligned}
$$

where in the last equality we used the estimate

$$
\begin{aligned}
2k_w+2\left\lceil\frac{\deg(F)}{2}\right\rceil-2\left\lceil\frac{\deg(g_i)}{2}\right\rceil &\geq 2\deg(A_i^*A_i)+2\deg(\Lambda_{1,w,g_i})-2\left\lceil\frac{\deg(g_i)}{2}\right\rceil\\
&\geq 2\deg(A_i^*A_i).
\end{aligned}
$$

Note that the second inequality follows by Lemma 2.6.4.

Case 2: $w\in\mathbb{R}\setminus K$. The set

$$
\mathcal{S}:=\{\Lambda_{1,i,g_1}(z),\dots,\Lambda_{1,i,g_s}(z)\}
$$

is a saturated description of $\mathscr{K}_{1,i}$ by Proposition 2.6.3. Fix $z_0\in\mathbb{T}\setminus\mathscr{K}$. By Proposition 2.5.5, there exists $k_{z_0}\in\mathbb{N}\cup\{0\}$, such that

$$
|z-z_0|^{2k_{z_0}}\cdot\Lambda_{1,i,F}(z)\in\mathcal{M}_{\mathcal{S},b}^n.
$$

Therefore

$$
\begin{aligned}
&\left(\frac{|x-i|}{2}\right)^{2\left\lceil\frac{\deg(F)}{2}\right\rceil+2k_{z_0}}\cdot\left(|z-z_0|^{2k_{z_0}}\cdot\Lambda_{1,i,F}\right)(\lambda_{1,i}(x))=\\
&=\left(\frac{|x-i|}{2}\right)^{2\left\lceil\frac{\deg(F)}{2}\right\rceil+2k_{z_0}}\left(\left(\frac{(\lambda_{1,i}(x)-z_0)^2}{-z_0\lambda_{1,i}(x)}\right)^{k_{z_0}}\cdot\Lambda_{1,i,F}(\lambda_{1,i}(x))\right)=\\
&=\left(\frac{(2-(z_0+\overline{z_0}))}{4}\right)^{k_{z_0}}\cdot(x-\lambda_{1,i}(z_0))^{k_{z_0}}\cdot F(x)\in M_S^n,
\end{aligned}
$$

where we used that $z_0\neq 1$ in the last equality. (This follows by the fact that K is unbounded, and hence $1\in\mathscr{K}_{1,i}$.) With z_0 running over $\mathbb{T}\setminus\mathscr{K}$, $\lambda_{1,i}(z_0)$ runs over $w\in\mathbb{R}\setminus K$. This proves Case 2.

Proof of the equivalence (1) $\Leftrightarrow$ (3) **of Theorem 2.6.1**

The non-trivial implication is ($\Rightarrow$). Since we have (1) $\Leftrightarrow$ (2), this follows from (2) used for $w=i$.

Proof of the equivalence $(1) \Leftrightarrow (4)$ **of Theorem 2.6.1**

The non-trivial implication is $(\Rightarrow)$. We write

$$F^{2p-1} := [f_{j\ell}]_{j\ell} = \sum_{j=1}^{n} f_{jj} E_{jj} + \sum_{1 \le j < \ell \le n} (f_{j\ell} E_{j\ell} + \overline{f_{j\ell}} E_{\ell j}), \tag{2.6.4}$$

where E_{ij} stand for the standard coordinate matrices, i.e., the only non-zero entry of E_{ij} is in the i-th row and j-th column and is equal to 1.

Claim. There exists a polynomial $\widetilde{h} \in \mathrm{Pos}^1_{>0}(\mathbb{R})$ such that $\widetilde{h}F - F^{2p} \in \mathrm{Pos}^n_{\succeq 0}(K)$.

Proof of Claim. For every $j = 1, \ldots, n$ we estimate

$$f_{jj} E_{jj} \preceq \big(1 + f_{jj}^2\big) E_{jj} \preceq \big(1 + f_{jj}^2\big) I_n, \tag{2.6.5}$$

and for every $1 \le j < \ell \le n$ we estimate

$$f_{j\ell} E_{j\ell} + \overline{f_{j\ell}} E_{\ell j} \preceq \big(1 + f_{j\ell}\overline{f_{j\ell}}\big)(E_{jj} + E_{\ell\ell}) \preceq \big(1 + f_{j\ell}\overline{f_{j\ell}}\big) I_n. \tag{2.6.6}$$

Using (2.6.5) and (2.6.6) in (2.6.4) we obtain $F^{2p-1} \preceq \widetilde{h} I_n$, where

$$\widetilde{h} := \sum_{1 \le j \le \ell \le n} \big(1 + f_{j\ell}\overline{f_{j\ell}}\big) \in \mathrm{Pos}^1_{>0}(\mathbb{R}).$$

From the equality

$$\widetilde{h}F - F^{2p} = F(\widetilde{h} I_n - F^{2p-1}),$$

the inclusions $F \in \mathrm{Pos}^n_{\succeq 0}(K)$ and $\widetilde{h} I_n - F^{2p-1} \in \mathrm{Pos}^n_{\succeq 0}(\mathbb{R}) \subseteq \mathrm{Pos}^n_{\succeq 0}(K)$, and the fact that the matrix polynomials F, $\widetilde{h} I_n - F^{2p-1}$ commute, it follows that

$$\widetilde{h}F - F^{2p} \in \mathrm{Pos}^n_{\succeq 0}(K).$$

This proves Claim.

By the equivalence $(1) \Leftrightarrow (3)$ of Theorem 2.6.1 and Claim there exists $k \in \mathbb{N} \cup \{0\}$, such that

$$(1 + x^2)^k \cdot (\widetilde{h}F - F^{2p}) \in M_S^n.$$

It follows that

$$(1 + x^2)^k \cdot \widetilde{h}F = (1 + x^2)^k \cdot F^{2p} + H,$$

where $H \in M_S^n$. We expand $(1 + x^2)^k$ and get

$$(1 + x^2)^k = 1 + \sum_{j=1}^{k} \binom{k}{j} x^{2j} = 1 + \sum_{j=1}^{k} \left(\sqrt{\binom{k}{j}} \cdot x^j \right)^2.$$

Then

$$(1 + x^2)^k \cdot F^{2p} + H = F^{2p} + \underbrace{\sum_{j=1}^{k} \left(\sqrt{\binom{k}{j}} \cdot x^j \right)^2 \cdot F^{2p} + H}_{=:G}.$$

Defining $h := (1 + x^2)^k \widetilde{h} \in \mathrm{Pos}^1_{>0}(\mathbb{R})$ and noticing that $G \in M_S^n$, proves the implication $(\Rightarrow)$.

2.7 Generalizations to curves

The main result of this section, Theorem 2.7.5, generalizes Theorem 2.2.1 to curves in $\mathbb{R}^d$ and extends Scheiderer's [Sch03, Theorem 5.17] and [Sch05, Corollary 4.4] (see Theorems 2.7.2 and 2.7.3 below) from polynomials to matrix polynomials.

We write $\underline{x} := (x_1, \ldots, x_d)$. Let I be an ideal in $\mathbb{R}[\underline{x}]$ and

$$\mathcal{Z}(I) = \left\{\underline{x} \in \mathbb{R}^d : g(\underline{x}) = 0 \text{ for all } g \in I\right\}$$

its vanishing set. Let $I_{\mathbb{C}} := I + iI \subseteq \mathbb{C}[\underline{x}]$ be the complexification of the ideal I. Let $M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ be the set of all $n \times n$ matrix polynomials over $\mathbb{C}[\underline{x}]/I_{\mathbb{C}}$ equipped with conjugated transpose as the involution, where $x_j^* = x_j$ for every $j = 1, \ldots, d$. We say that $F(x) \in M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ is *hermitian*, if $F(x) = F(x)^*$. We write $H_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ for the set of all hermitian matrix polynomials from $M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$. We call $F(x_0)$ *positive definite* (resp. *positive semidefinite*) in $x_0 \in \mathcal{Z}(I)$, if $v^*F(x_0)v > 0$ (resp. $v^*F(x_0)v \geq 0$) for every non-zero $v \in \mathbb{C}^n$. We write $\sum M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})^2$ for the set of all finite sums of the expressions of the form $G(\underline{x})^*G(\underline{x})$ where $G(\underline{x}) \in M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$. We call such expressions *hermitian squares* of matrix polynomals from $M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$. The *closed semialgebraic set* associated to a finite subset $S := \{g_1, \ldots, g_s\} \subset \mathbb{R}[\underline{x}]$ is given by

$$K_S = \left\{\underline{x} \in \mathbb{R}^d : g_i(\underline{x}) \geq 0 \text{ for every } i = 1, \ldots, s\right\}.$$

The *n-th matrix quadratic module generated by S* in $H_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ is defined by

$$M_S^n := \left\{\tau_0 + \tau_1 \cdot g_1 + \ldots + \tau_s \cdot g_s : \tau_j \in \sum M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})^2,\ j = 0, \ldots, s\right\},$$

and the *n-th matrix preordering generated by S* in $H_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ by

$$T_S^n := \left\{\sum_{e \in \{0,1\}^s} \tau_e \cdot \underline{g}^e : \tau_e \in \sum M_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})^2 \text{ for all } e \in \{0,1\}^s\right\},$$

where $e := (e_1, \ldots, e_s)$ and $\underline{g}^e$ stands for $g_1^{e_1} \cdots g_s^{e_s}$.

Remark 2.7.1. Note that T_S^n is the quadratic module generated by all products $\underline{g}^e$, $e \in \{0,1\}^s$.

The extensions of Theorems 2.1.10 and 2.1.12 to curves in $\mathbb{R}^d$ are the following (see [Sch03, Theorem 5.17] and [Sch05, Corollary 4.4]).

Theorem 2.7.2 (Scheiderer). *Suppose I is a prime ideal of $\mathbb{R}[\underline{x}]$ with dimension $\dim\left(\frac{\mathbb{R}[\underline{x}]}{I}\right) = 1$. Let $S := \{g_1, \ldots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{x}]$. Suppose that $K_S \cap \mathcal{Z}(I)$ is compact and that each $p \in K_S \cap \mathcal{Z}(I)$ is a non-singular zero of I. For a point $p \in K_S \cap \mathcal{Z}(I)$, let v_p denote the natural valuation on the completion of $\frac{\mathbb{R}[\underline{x}]}{I}$ at a point p. Then every hermitian polynomial $f \in H_1\left(\frac{\mathbb{C}[\underline{x}]}{I_{\mathbb{C}}}\right)$ satisfying $f(x_0) \geq 0$ for every $x_0 \in K_S \cap \mathcal{Z}(I)$ belongs to $T_S^1 + I_{\mathbb{C}}$ if and only if*

(1) For each boundary point p of $K_S \cap \mathcal{Z}(I)$ which is not an isolated point of $K_S \cap \mathcal{Z}(I)$ there exists $i \in \{1, \ldots, s\}$ such that $v_p(g_i) = 1$.

(2) For each isolated point p of $K_S \cap \mathcal{Z}(I)$ there exist $i, j \in \{1, \dots, s\}$ such that $v_p(g_i) = v_p(g_j) = 0$ and $g_i g_j \leq 0$ in some neighbourhood of p in $\mathcal{Z}(I)$.

Theorem 2.7.3 (Scheiderer)**.** *Suppose I is a prime ideal of $\mathbb{R}[\underline{x}]$ with dimension $\dim\left(\dfrac{\mathbb{R}[\underline{x}]}{I}\right) = 1$. Suppose $S \subset \mathbb{R}[\underline{x}]$ is a finite set such that $K_S \cap \mathcal{Z}(I)$ is compact and that each $p \in K_S \cap \mathcal{Z}(I)$ is a non-singular zero of I. Then $M_S^1 + I_{\mathbb{C}} = T_S^1 + I_{\mathbb{C}}$.*

Remark 2.7.4. Scheiderer in fact works with polynomials from $\mathbb{R}[\underline{x}]/I$ and his squares are usual ones, i.e., g^2 where $g \in \mathbb{R}[\underline{x}]/I$, while we work with $\mathbb{C}[\underline{x}]/I_{\mathbb{C}}$ and hermitian squares, i.e., g^*g where $g \in \mathbb{C}[\underline{x}]/I_{\mathbb{C}}$. However, since a hermitian polynomial $f \in H_1(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ is of the form $f_r + I_{\mathbb{C}}$ where $f_r \in \mathbb{R}[\underline{x}]$, Theorems 2.7.2 and 2.7.3 follow easily from [Sch03, Theorem 5.17] and [Sch05, Corollary 4.4].

Theorem 2.2.1 extends to curves in $\mathbb{R}^d$ as follows.

Theorem 2.7.5. *Suppose I is a prime ideal of $\mathbb{R}[\underline{x}]$ with dimension $\dim\left(\dfrac{\mathbb{R}[\underline{x}]}{I}\right) = 1$. Let S be a finite subset of $\mathbb{R}[\underline{x}]$. Suppose the set $K_S \cap \mathcal{Z}(I)$ is compact. Then the n-th quadratic module M_S^n contains every hermitian complex matrix polynomial $F \in H_n(\mathbb{C}[\underline{x}]/I_{\mathbb{C}})$ satisfying $F(x_0) \succeq 0$ for every $x_0 \in K_S \cap \mathcal{Z}(I)$ if and only if S satisfies the conditions (1) and (2) of Theorem 2.7.2.*

The proof of Theorem 2.7.5 is analogous to the proof of Theorem 2.2.1 and will not be given in details here. We only state the main ingredients:

(1) Theorems 2.7.2 and 2.7.3 as the $n = 1$ case.

(2) The analog of "hF"-proposition, i.e., recall Proposition 2.2.2 above. Also the proof is analogous to the proof of Proposition 2.2.2.

(3) Eliminating h in the analog of "hF"-proposition, which is also established analogously as in the case of Theorem 2.2.1.

Remark 2.7.6. After establishing Theorem 2.7.5, both Theorems 2.2.1 and 2.5.4 can be obtained as its special cases. To get Theorem 2.2.1 we take $d = 1$ and $I = 0$ in Theorem 2.7.5, while to get Theorem 2.2.1 we take $d = 2$ and $I = \langle x^2 + y^2 - 1\rangle$ in Theorem 2.7.5 and notice that $M_n(\mathbb{C}[x, y])$ is $*$-isomorphic to $M_n(\mathbb{C}[z, \frac{1}{z}])$ with a $*$-isomorphism $x \mapsto \frac{1}{2}\big(z + \frac{1}{z}\big)$ and $y \mapsto \frac{i}{2}\big(z - \frac{1}{z}\big)$.

Chapter 3

Positivstellensätze on matrix convex sets

In this chapter we study algebraic certificates of positivity for noncommutative (nc) operator polynomials on matrix convex sets, such as the solution set D_L, called a free Hilbert spectrahedron, of the linear operator inequality (LOI) $L(X) = A_0 \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0$, where A_j are self-adjoint linear operators on a separable Hilbert space, X_j matrices and I is an identity matrix. If A_j are matrices, then $L(X) \succeq 0$ is called a linear matrix inequality (LMI) and D_L a free spectrahedron. For monic LMIs, i.e., $A_0 = I$, and nc matrix polynomials the certificates of positivity were established by Helton, Klep and McCullough in a series of articles with the use of the theory of complete positivity from operator algebras and classical separation arguments from real algebraic geometry. Since the full strength of the theory of complete positivity is not restricted to finite dimensions, but works well also in the infinite-dimensional setting, it is possible to extend their results to operator polynomials. First we extend the characterization of the inclusion $D_{L_1} \subseteq D_{L_2}$ from monic *LMIs* to monic *LOIs* L_1 and L_2. As a corollary one immediately obtains the description of a polar dual of a free Hilbert spectrahedron D_L and its projection, called a free Hilbert spectrahedrop. Further on, using this characterization in a separation argument, we obtain a certificate for multivariate nc matrix polynomial F positive semidefinite on a free Hilbert spectrahedron defined by a monic LOI. Replacing the separation argument by the operator Fejér-Riesz theorem enables us to extend this certificate, in the univariate case, to operator polynomial F. Finally, focusing on the algebraic description of the equality $D_{L_1} = D_{L_2}$, we remove the assumption of boundedness from the description in the LMIs case by an extended analysis. However, the description does not extend to LOIs case by counterexamples.

This chapter is based on [HKM12, HKM13b, HKM16b, Zal17, DDSS+].

3.1 Notations and known results

3.1.1 Notations

Free sets - matrix level

Fix a positive integer $g \in \mathbb{N}$. We use $\mathbb{S}_n$ to denote real symmetric $n \times n$ matrices and $\mathbb{S}^g$ for the sequence $(\mathbb{S}_n^g)_n$. A *subset* Γ of $\mathbb{S}^g$ is a sequence $\Gamma = (\Gamma(n))_n$, where $\Gamma(n) \subseteq \mathbb{S}_n^g$ for each n. The subset Γ is *closed with respect to direct sums* if $A = (A_1, \dots, A_g) \in \Gamma(n)$ and $B = (B_1, \dots, B_g) \in \Gamma(m)$ implies

$$A \oplus B = \left(\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}, \dots, \begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix} \right) \in \Gamma(n+m).$$

It is closed with respect to *(simultaneous) unitary conjugation* if for each n, each $A \in \Gamma(n)$ and each $n \times n$ unitary matrix U,

$$U^*AU = (U^*A_1U, \dots, U^*A_gU) \in \Gamma(n).$$

The set Γ is a *free set* if it is closed with respect to direct sums and simultaneous unitary conjugation. If in addition it is closed with respect to *(simultaneous) isometric conjugation*, i.e., if for each $m \leq n$, each $A = (A_1, \dots, A_g) \in \Gamma(n)$, and each isometry $V : \mathbb{R}^m \to \mathbb{R}^n$,

$$V^*AV = (V^*A_1V, \dots, V^*A_gV) \in \Gamma(m),$$

then Γ is *matrix convex* [HKM16a].

Free sets - operator level

Fix a separable Hilbert space $\mathscr{K}$. Let $\mathrm{Lat}(\mathscr{K})$ denote the *lattice of closed subspaces* of $\mathscr{K}$. For a $K \in \mathrm{Lat}(\mathscr{K})$, we use $\mathbb{S}_K$ to denote the set of all self-adjoint operators on K. Let $\mathbb{S}_{\mathscr{K}}$ stand for the set $(\mathbb{S}_K)_K$. A collection $\Gamma = (\Gamma(K))_K$ where $\Gamma(K) \subseteq \mathbb{S}_K^g$ for each K a closed subspace of $\mathscr{K}$, is a *free operator set* [HKM16a] if it is closed under direct sums and with respect simultaneous conjugation by unitary operators. If in addition it is closed with respect to simultaneous conjugation by isometries $V : H \to K$, where $H, K \in \mathrm{Lat}(\mathscr{K})$, then Γ is *operator convex*.

Linear pencils and LOI sets

Let $\mathscr{H}$ be separable real Hilbert space and $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$. We denote by $B(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$ and by $\mathbb{S}_{\mathscr{H}}$ the set of all self-adjoint operators from $B(\mathscr{H})$. For self-adjoint operators $A_0, A_1, \dots, A_g \in \mathbb{S}_{\mathscr{H}}$, the expression

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j$$

is a *linear (operator) pencil.* If $\mathscr{H}$ is finite-dimensional, then $L(x)$ is a *linear matrix pencil.* If $A_0 = I_{\mathscr{H}}$, then L is *monic.* If $A_0 = 0$, then L is *homogeneous.* To every tuple $A = (A_1, \ldots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ we associate a homogeneous linear pencil Λ_A and a monic linear pencil L_A by

$$\Lambda_A(x) := \sum_{j=1}^{g} A_j x_j \quad \text{and} \quad L_A(x) := I_{\mathscr{H}} + \Lambda_A(x).$$

Let $C^*(\mathcal{S}_A)$ be the unital C^*-algebra generated by A, i.e., the smallest unital C^*-algebra in $B(\mathscr{H})$ which contains the operators $A_1, \ldots, A_g$.

The *matrix Hilbert convex hull* $\text{co}^{\text{mat}}{}_{\mathscr{K}}\{A\}$ of A is the matrix convex set

$$\text{co}^{\text{mat}}\{A\} = (\text{co}^{\text{mat}}\{A\}(n))_n,$$

where

$$\text{co}^{\text{mat}}\{A\}(n) := \bigcup_{(\mathscr{G},\pi,V)\in\Pi_n} (V^*\pi(A_1)V, \ldots, V^*\pi(A_g)V) \subseteq \mathbb{S}^g_n,$$

and Π_n is the set of all triples $(\mathscr{G}, \pi, V)$ of a separable real Hilbert space $\mathscr{G}$, an isometry $V : \mathbb{R}^n \to \mathscr{G}$ and a unital $*$-homomorphism $\pi : C^*(\mathcal{S}_A) \to B(\mathscr{G})$.

The *operator Hilbert convex hull* $\text{co}^{\text{oper}}_{\mathscr{K}}\{A\}$ of A in a separable real Hilbert space $\mathscr{K}$ is the operator convex set

$$\text{co}^{\text{oper}}_{\mathscr{K}}\{A\} = (\text{co}_{\mathscr{K}}\{A\}(K))_{K\in\text{Lat}(\mathscr{K})}$$

where

$$\text{co}^{\text{oper}}_{\mathscr{K}}\{A\}(K) := \bigcup_{(\mathscr{G},\pi,V)\in\Pi_K} (V^*\pi(A_1)V, \ldots, V^*\pi(A_g)V) \in \mathbb{S}^g_K,$$

and Π_K is the set of all triples $(\mathscr{G}, \pi, V)$ of a separable real Hilbert space $\mathscr{G}$, an isometry $V : K \to \mathscr{G}$ and a unital $*$-homomorphism $\pi : C^*(\mathcal{S}_A) \to B(\mathscr{G})$.

We say a bounded linear operator A on a Hilbert space $\mathscr{H}$ is *positive semidefinite* and write $A \succeq 0$ if A is self-adjoint and $\langle Ah, h\rangle_{\mathscr{H}} \geq 0$ for every $h \in \mathscr{H}$ where $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ stands for the inner product on $\mathscr{H}$. Given another real Hilbert space $\mathscr{K}$, setting

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle_{\mathscr{H}\otimes\mathscr{K}} := \langle h_1, h_2\rangle_{\mathscr{H}} \langle k_1, k_2\rangle_{\mathscr{K}}$$

and extending by linearity, we obtain an inner product on the vector space $\mathscr{H} \otimes \mathscr{K}$. The completion of $\mathscr{H} \otimes \mathscr{K}$ with respect to this inner product is a Hilbert space, which we still denote by $\mathscr{H} \otimes \mathscr{K}$. For operators $A \in B(\mathscr{H})$ and $B \in B(\mathscr{K})$ we set

$$(A \otimes B)(h \otimes k) := (Ah) \otimes (Bk),$$

and extend by linearity to an operator $A \otimes B \in B(\mathscr{H} \otimes \mathscr{K})$.

Given a tuple of self-adjoint operators $X = (X_1, \ldots, X_g) \in \mathbb{S}^g_K$ on a closed subspace K of a Hilbert space $\mathscr{K}$, the *evaluation* $L(X)$ is defined by

$$L(X) = A_0 \otimes I_K + \sum_{j=1}^{g} A_j \otimes X_j,$$

where I_K stands for the identity operator on K.

We call the set

$$D_L(1) = \{x \in \mathbb{R}^g \colon L(x) \succeq 0\}$$

a *Hilbert spectrahedron* or a *LOI domain*, the set

$$D_L = (D_L(n))_n \quad \text{where} \quad D_L(n) = \{X \in \mathbb{S}_n^g \colon L(X) \succeq 0\},$$

a *free Hilbert spectrahedron* or a *free LOI set*, the set

$$\partial D_L = (\partial D_L(n))_n \quad \text{where} \quad \partial D_L(n) = \{X \in \mathbb{S}_n^g \colon L(X) \succeq 0, L(X) \not\succ 0\},$$

the *boundary of a free Hilbert spectrahedron* and the set

$$D_L^{\mathscr{K}} = (D_L(K))_{K \in \mathrm{Lat}(\mathscr{K})} \quad \text{where} \quad D_L(K) = \{X \in \mathbb{S}_K^g \colon L(X) \succeq 0\}.$$

an *operator free Hilbert spectrahedron* or an *operator free LOI set* where $\mathscr{K}$ is a separable real Hilbert space. Note that $D_L(1) \subseteq \mathbb{R}^g$ is a closed convex set and by the classical Hahn-Banach theorem every convex closed subset of $\mathbb{R}^g$ is of this form. If L is a linear matrix pencil, then we omit the word Hilbert from the definitions.

Free Hilbert spectrahedrops

Let $\mathscr{H}$, $\mathscr{K}$ be separable real Hilbert spaces. Let D, Ω_j, $\Gamma_k \in \mathbb{S}_{\mathscr{H}}$ be self-adjoint operators and

$$L(x,y) = D + \sum_{j=1}^{g} \Omega_j x_j + \sum_{k=1}^{h} \Gamma_k y_k \in \mathbb{S}_{\mathscr{H}}\langle x, y\rangle$$

a linear pencil in the variables $(x_1, \ldots, x_g; y_1, \ldots, y_h)$. We call the set

$$\mathrm{proj}_x D_L(1) := \{x \in \mathbb{R}^g \colon \exists\, y \in \mathbb{R}^h \text{ such that } L(x,y) \succeq 0\}$$

a *Hilbert spectrahedral shadow* [BPT13], the set

$$\mathrm{proj}_x D_L = (\mathrm{proj}_x D_L(n))_n,$$

where

$$\mathrm{proj}_x D_L(n) := \{X \in \mathbb{S}_n^g \colon \exists\, Y \in \mathbb{S}_n^h \text{ such that } L(X,Y) \succeq 0\},$$

a *free Hilbert spectrahedrop*, and the set

$$\mathrm{proj}_x D_L^{\mathscr{K}} = (\mathrm{proj}_x D_L(K))_{K \in \mathrm{Lat}(\mathscr{K})},$$

where

$$\mathrm{proj}_x D_L(K) = \{X \in \mathbb{S}_K \colon \exists\, Y \in \mathbb{S}_K \text{ such that } L(X,Y) \succeq 0\},$$

an *operator free Hilbert spectrahedrop.* If L is a linear matrix pencil, then we omit the word Hilbert from the definitions.

Polar duals

Let $\mathscr{K}$ be a real separable Hilbert space. The *free polar dual* (resp. the *free Hilbert polar dual*) $\mathcal{K}^\circ = (\mathcal{K}^\circ(n))_n$ of a free set $\mathcal{K} \subseteq \mathbb{S}^g$ (resp. a free operator set $\mathcal{K} \subseteq \mathbb{S}_{\mathscr{K}}^g$) is

$$\mathcal{K}^\circ(n) = \left\{ A \in \mathbb{S}_n^g \colon L_A(X) = I_n \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0 \text{ for all } X \in \mathcal{K} \right\}.$$

The *operator free polar dual* (resp. the *operator free Hilbert polar dual*) $\mathcal{K}^{\mathscr{K},\circ} = (\mathcal{K}^\circ(K))_{K\in\mathrm{Lat}(\mathscr{K})}$ of a free set $\mathcal{K} \subseteq \mathbb{S}^g$ (resp. a free operator set $\mathcal{K} \subseteq \mathbb{S}^g_{\mathscr{K}}$) is

$$\mathcal{K}^\circ(K) = \left\{ A \in \mathbb{S}^g_K \colon L_A(X) = I_K \otimes I + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0 \text{ for all } X \in \mathcal{K} \right\}.$$

Words and nc polynomials

We write $\langle x\rangle$ for the monoid freely generated by $x = (x_1, \dots, x_g)$, i.e., $\langle x\rangle$ consists of *words* in the g noncommuting letters $x_1, \dots, x_g$. Let $\mathbb{R}\langle x\rangle$ denote the associative $\mathbb{R}$-algebra freely generated by x, i.e., the elements of $\mathbb{R}\langle x\rangle$ are polynomials in the noncommuting variables x with coefficients in $\mathbb{R}$. The elements are called *noncommutative (nc) polynomials*. Endow $\mathbb{R}\langle x\rangle$ with the natural *involution* * which fixes $\mathbb{R} \cup \{x\}$ pointwise, reverses the order of words, and acts linearly on polynomials. Polynomials invariant under this involution are *symmetric*. The length of the longest word in a noncommutative polynomial $f \in \mathbb{R}\langle x\rangle$ is denoted by $\deg(f)$. The set of all words of degree at most k is $\langle x\rangle_k$ and $\mathbb{R}\langle x\rangle_k$ is the vector space of all nc polynomials of degree at most k.

Fix separable Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$. *Operator nc polynomials* are the elements of $B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x\rangle$. We write

$$P = \sum_{w\in\langle x\rangle} A_w \otimes w \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x\rangle$$

for an element $P \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x\rangle$, where the sum is finite. The involution * extends to operator nc polynomials by

$$P^* = \sum_{w\in\langle x\rangle} A_w^* \otimes w^* \in B(\mathscr{H}_2, \mathscr{H}_1) \otimes \mathbb{R}\langle x\rangle.$$

If $\mathscr{H}_1 = \mathscr{H}_2$ and $P = P^*$, then we say P is *symmetric*.

Polynomial evaluations

If $P \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x\rangle$ is an operator nc polynomial and $X \in B(\mathscr{K})^g$, where $\mathscr{K}$ is a separable Hilbert space, then

$$P(X) \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes B(\mathscr{K})$$

is defined in the natural way by replacing x_i by X_i and sending the empty word to the identity operator on $\mathscr{K}$. Note that if $P \in \mathbb{R}^{\ell_1\times\ell_2}\langle x\rangle$ is a matrix nc polynomial, where $\ell_1, \ell_2 \in \mathbb{N}$ are natural numbers, then $P(X) : \mathscr{K}^{\ell_2} \to \mathscr{K}^{\ell_1}$ is an operator mapping form $\mathscr{K}^{\ell_2}$ to $\mathscr{K}^{\ell_1}$ and has a matrix representation $(p_{ij}(X))_{ij}$, where $P = (p_{ij}(x))_{ij}$.

Free Hilbert semialgebraic sets

A symmetric operator nc polynomial P determines the *free Hilbert semialgebraic set* by

$$D_P = (D_P(n))_n \quad \text{where} \quad D_P(n) = \{X \in \mathbb{S}^g_n \colon P(X) \succeq 0\},$$

and the *operator free Hilbert semialgebraic set* by

$$D_P^{\mathscr{K}} = (D_P(K))_{K\in\mathrm{Lat}(\mathscr{K})} \quad \text{where} \quad D_P(K) = \{X \in \mathbb{S}_K^g \colon P(X) \succeq 0\}.$$

Clearly, the sets D_P and $D_P^{\mathscr{K}}$ are a free set and a free operator set, respectively. If P is a symmetric matrix nc polynomial, then we omit the word Hilbert in the definitions of D_P and $D_P^{\mathscr{K}}$.

3.1.2 Known results

Inclusion of free spectrahedra

The question of the inclusion of free spectrahedra for matrix pencils was considered by Helton, Klep and McCullough in [HKM12] and [HKM13b]. They proved the following algebraic characterization of the inclusion $D_{L_1} \subseteq D_{L_2}$ called a *Linear Positivstellensatz.*

Theorem 3.1.1 (Helton, Klep, McCullough). *Let $L_j \in \mathbb{S}_{d_j}\langle x\rangle$, $j = 1,2$, $d_j \in \mathbb{N}$, be monic linear matrix pencils. Then $D_{L_1} \subseteq D_{L_2}$ if and only if there is $k_0 \in \mathbb{N}$, matrices $V_k \in \mathbb{R}^{d_1\times d_2}$ for $k = 1,\dots,k_0$ and a positive semidefinite matrix $S \in \mathbb{S}_{d_2}$ such that*

$$L_2(x) = S + \sum_{k=1}^{k_0} V_k^* L_1(x) V_k.$$

Moreover, if $D_{L_1}(1)$ is bounded, S can be chosen to be 0.

In [HKM13b], Theorem 3.1.1 was proved for bounded sets $D_{L_1}(1)$ with the use of the theory of complete positivity from operator algebras, while in [HKM12] a more general theorem (see Theorem 3.1.5 below) was proved with the use of classical separation arguments from real algebraic geometry, i.e., the authors generalized Theorem 3.1.1 from the case of linear matrix pencil L_2 to the case of an arbitrary noncommutative polynomial p.

Equality of free spectrahedra

Let $A_0, A_1, \dots, A_g \in \mathbb{S}_{\mathscr{H}}$ be self-adjoint operators on a separable real Hilbert space $\mathscr{H}$ and $L(x) = A_0 + \sum_{j=1}^{g} A_j x_j$ a linear pencil. Let $H \subseteq \mathscr{H}$ be a closed subspace of $\mathscr{H}$ which is *invariant* under each A_j, i.e., $A_j H \subseteq H$. Since each A_j is self-adjoint, it also follows that $A_j H^\perp \subseteq H^\perp$, i.e., H is automatically reducing for each A_j. Hence, with respect to the decomposition $\mathscr{H} = H \oplus H^\perp$, L can be written as the direct sum,

$$L = \widetilde{L} \oplus \widetilde{L}^\perp = \begin{bmatrix} \widetilde{L} & 0 \\ 0 & \widetilde{L}^\perp \end{bmatrix}, \quad \text{where} \quad \widetilde{L} = A_0|_H + \sum_{j=1}^{g} A_j|_H x_j.$$

We say that $\widetilde{L}$ is a *subpencil* of L. If H is a proper closed subspace of $\mathscr{H}$, then $\widetilde{L}$ is a *proper subpencil* of L. If $D_L = D_{\widetilde{L}}$, then $\widetilde{L}$ is a *whole subpencil* of L. If L has no proper whole subpencil, then L is *σ-minimal.*

The question of the equality of free spectrahedra for matrix pencils was considered by Helton, Klep and McCullough in [HKM13b]. They proved the following algebraic characterization of the equality $D_{L_1} = D_{L_2}$ called a *Linear Gleichstellensatz* (see [HKM13b, Theorem 1.2]).

Theorem 3.1.2 (Helton, Klep, McCullough). *Let $L_j \in \mathbb{S}_{d_j}\langle x\rangle$, $j = 1, 2$, $d_j \in \mathbb{N}$, be monic linear matrix pencils with $D_{L_1}(1)$ is bounded. Then $D_{L_1} = D_{L_2}$ if and only if every σ-minimal whole subpencil $\widetilde{L}_1$ of L_1 is unitarily equivalent to any σ-minimal whole subpencil $\widetilde{L}_2$ of L_2, i.e., there is a unitary matrix U such that $\widetilde{L}_2 = U^*\widetilde{L}_1 U$.*

Even though $D_{L_1} = D_{L_2}$ if and only if $D_{L_1} \subseteq D_{L_2}$ and $D_{L_2} \subseteq D_{L_1}$ it is not clear how to prove Theorem 3.1.2 only by using Theorem 3.1.1. The proof is more involved. One has to see how the free spectrahedron D_L is connected by the unital C^*-algebra generated by the coefficients of L. For this the authors used Arveson's noncommutative Choquet theory [Arv69, Arv08, Arv10].

Polar duals of free spectrahedra and free spectrahedrops

Helton, Klep and McCullough described the polar dual of a free spectrahedra and a free spectrahedrop (see [HKM16b, Theorem 4.6 and Corollary 4.15]).

Theorem 3.1.3 (Helton, Klep, McCullough). *Suppose $L := I_d + \sum_{j=1}^{g} A_j x_j$, $d \in \mathbb{N}$, is a monic linear matrix pencil where $A_j \in \mathbb{S}_d$. The free polar dual of D_L if the free set given by*

$$D_L^\circ(n) = \Big\{(X_1,\dots,X_g) \in \mathbb{S}_n^g \colon \exists\, \mu \in \mathbb{N} \text{ and } V_1,\dots,V_\mu \in \mathbb{R}^{(d+1)\times n} \text{ such that } \sum_{\ell=1}^{\mu} V_\ell^* V_\ell = I_n \text{ and for all } j:\ X_j = \sum_{\ell=1}^{\mu} V_\ell^*(A_j \oplus 0_{\mathbb{R}})V_\ell\Big\}.$$

Theorem 3.1.4 (Helton, Klep, McCullough). *Suppose $L := I_d + \sum_{j=1}^{g} \Omega_j x_j + \sum_{k=1}^{h} \Gamma_k y_k$, $d \in \mathbb{N}$, is a monic linear matrix pencil where $\Omega_j, \Gamma_k \in \mathbb{S}_d$ and $\mathcal{K} := \operatorname{proj}_x D_L$ its free spectrahedrop. The free polar dual of $\mathcal{K}$ is the free set given by*

$$\begin{aligned}\mathcal{K}^\circ(n) &= \{(A_1,\dots,A_g) \in \mathbb{S}_n^g \colon (A_1,\dots,A_g,0,\dots,0) \in D_L^\circ\}\\ &= \Big\{(A_1,\dots,A_g) \in \mathbb{S}_n^g \colon \exists\, \mu \in \mathbb{N} \text{ and } V_1,\dots,V_\mu \in \mathbb{R}^{(d+1)\times n} \text{ such that }\\ &\qquad \sum_{\ell=1}^{\mu} V_\ell^* V_\ell = I_n \text{ and for all } j,k:\ A_j = \sum_{\ell=1}^{\mu} V_\ell^* \widetilde{\Omega}_j V_\ell,\ 0 = \sum_{\ell=1}^{\mu} V_\ell^* \widetilde{\Gamma}_k V_\ell\Big\},\end{aligned}$$

where $\widetilde{\Omega}_j = \Omega_j \oplus 0 \in \mathbb{S}_{d+1}$ and $\widetilde{\Gamma}_k = \Gamma_k \oplus 0 \in \mathbb{S}_{d+1}$.

Moreover, if $\mathcal{K}$ is bounded, then its free polar dual of is the free set given by

$$\begin{aligned}\mathcal{K}^\circ(n) &= \{(A_1,\dots,A_g) \in \mathbb{S}_n^g \colon (A_1,\dots,A_g,0,\dots,0) \in D_L^\circ\}\\ &= \Big\{(A_1,\dots,A_g) \in \mathbb{S}_n^g \colon \exists\, \mu \in \mathbb{N} \text{ and } V_1,\dots,V_\mu \in \mathbb{R}^{d\times n} \text{ such that }\\ &\qquad \sum_{\ell=1}^{\mu} V_\ell^* V_\ell = I_n \text{ and for all } j,k:\ A_j = \sum_{\ell=1}^{\mu} V_\ell^* \Omega_j V_\ell,\ 0 = \sum_{\ell=1}^{\mu} V_\ell^* \Gamma_k V_\ell\Big\}.\end{aligned}$$

Matrix Positivstellensätze

Helton, Klep and McCullough obtained the following generalization of Theorem 3.1.1 (see [HKM12, Theorem 1.1]).

Theorem 3.1.5 (Helton, Klep, McCullough)**.** *Let $L \in \mathbb{S}_d\langle x\rangle$, $d \in \mathbb{N}$, be a monic linear matrix pencil. Then for every symmetric matrix noncommutative polynomial $F \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$, $\nu \in \mathbb{N}$, with $F|_{D_L} \succeq 0$, there exist finitely many matrix noncommutative polynomials $R_j \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ and $Q_k \in \mathbb{R}^{d\times\nu}\langle x\rangle$ all of degree at most $\frac{\deg(F)}{2}$ such that*

$$F = \sum_j R_j^* R_j + \sum_k Q_k^* L Q_k.$$

The proof of Theorem 3.1.5 uses a modification of Putinar-type argument. In [HKM16b] the authors extended Theorem 3.1.5 from free spectrahedra to free spectrahedrops (see [HKM16b, Theorem 5.1]).

Theorem 3.1.6 (Helton, Klep, McCullough)**.** *Let $L \in \mathbb{S}_d\langle x, y\rangle$, $d \in \mathbb{N}$, be a monic linear matrix pencil of the form $L(x,y) = D + \sum_{j=1}^{g} \Omega_j x_j + \sum_{k=1}^{h} \Gamma_k y_k$ and $\mathcal{K} = \operatorname{proj}_x D_L$. Then for every symmetric matrix noncommutative polynomial $F \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$, $\nu \in \mathbb{N}$, with $F|_{\mathcal{K}} \succeq 0$, there exist finitely many matrix noncommutative polynomials $R_j \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ and $Q_k \in \mathbb{R}^{d\times\nu}\langle x\rangle$ all of degree at most $\frac{\deg(F)}{2}$ such that*

$$F = \sum_j R_j^* R_j + \sum_\ell Q_\ell^* L Q_\ell$$

where $\sum_\ell Q_\ell^* \Gamma_k Q_\ell = 0$ for every k.

3.2 Linear Positivstellensatz and polar duals

The main result of this section, Theorem 3.2.13 below, is the extension of Theorem 3.1.1 from matrix pencils to operator pencils. The main techniques used are the same as in [HKM13b], i.e., complete positivity and the theory of operator algebras. We define the unital $*$-linear map τ between the linear spans of the coefficients of the given linear pencils. There are two crucial observations. The first is the connection between the inclusion $D_{L_1} \subseteq D_{L_2}$ and the complete positivity of τ given by Theorem 3.2.5, while the second is an an algebraic trick of extending the pencil to the direct sum with the monic scalar pencil 1, which makes the extended map $\widetilde{\tau}$ completely positive if and only if $D_{L_1} \subseteq D_{L_2}$. The proof of Theorem 3.2.13 then follows by invoking the real version of Arveson extension theorem and finally using the Stinespring representation theorem.

As consequence of Theorem 3.2.13 we describe operator free Hilbert polar duals of a free Hilbert spectrahedron (see Theorem 3.2.17 below) and a free Hilbert spectrahedrop (see Theorem 3.2.18 below), extending Theorems 3.1.3 and 3.1.4 from matrix to operator pencils.

3.2.1 Linear Positivstellensatz

Let $\mathscr{H}_1, \mathscr{H}_2, \mathscr{K}$ be separable real Hilbert spaces. Given L_1 and L_2 monic linear operator pencils

$$L_1(x) := I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}_1}\langle x \rangle \quad \text{and} \quad L_2(x) := I_{\mathscr{H}_2} + \sum_{j=1}^{g} B_j x_j \in \mathbb{S}_{\mathscr{H}_2}\langle x \rangle,$$

we are interested in the algebraic characterization of the inclusion of the free LOI sets (resp. operator free LOI sets)

$$D_{L_1} \subseteq D_{L_2} \quad (\text{resp. } D_{L_1}^{\mathscr{K}} \subseteq D_{L_2}^{\mathscr{K}}).$$

In this subsection we first prove the equivalence between both inclusions, then introduce the unital $*$-linear maps $\widetilde{\tau}$ and τ between the linear spans of the (extended) coefficients of both pencils, study the well-definedness and complete positivity of both maps and finally prove the main result; see Theorem 3.2.13. We also show by an example that the monicity of pencils is necessary; see Example 3.2.16.

Equivalence of the inclusions $D_{L_1} \subseteq D_{L_2}$ and $D_{L_1}^{\mathscr{K}} \subseteq D_{L_2}^{\mathscr{K}}$

Proposition 3.2.1. *We have the following equivalence:*

$$D_{L_1}^{\mathscr{K}} \subseteq D_{L_2}^{\mathscr{K}} \quad \Leftrightarrow \quad D_{L_1} \subseteq D_{L_2}.$$

To prove proposition we need a lemma.

Lemma 3.2.2. *Let $L(x) = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}}\langle x \rangle$ be a linear operator pencil and $X \in S_{\mathscr{K}}^g$ be a tuple self-adjoint operators on a Hilbert space $\mathscr{K}$. Then $X \in D_L^{\mathscr{K}}$ if and only if $V^*XV \in D_L(m)$ for every $m \in \mathbb{N}$ and every isometry $V \in B(\mathbb{R}^m, \mathscr{K})$.*

Proof. Let $X \in D_L^{\mathscr{K}}$. We have

$$\begin{aligned} L(V^*XV) &= A_0 \otimes I_{\mathbb{R}^m} + \sum_{j=1}^{g} A_j \otimes V^* X_j V \\ &= (I_{\mathscr{H}} \otimes V)^* (A_0 \otimes I_{\mathbb{R}^m} + \sum_{j=1}^{g} A_j \otimes X_j)(I_{\mathscr{H}} \otimes V) \\ &= (I_{\mathscr{H}} \otimes V)^* L(X)(I_{\mathscr{H}} \otimes V) \succeq 0. \end{aligned}$$

Hence $V^*XV \in D_L(m)$.

Let us now assume $V^*XV \in D_L(m)$ for every isometry $V \in B(\mathbb{R}^m, \mathscr{K})$, $m \in \mathbb{N}$. Suppose $X \notin D_L^{\mathscr{K}}$. Then there is $m \in \mathbb{N}$ and a vector $v := \sum_{k=1}^{m} h_k \otimes u_k \in \mathscr{H} \otimes \mathscr{K}$ such that $\langle L(X)v, v \rangle < 0$. Without loss of generality we may assume $u_1, \ldots, u_m$ are orthonormal. Define $X_0 = I_{\mathbb{R}^m}$. Hence

$$\begin{aligned}
\langle L(X)v, v\rangle &= \left\langle (\sum_{j=0}^{g} A_j \otimes X_j)(\sum_{k=1}^{m} h_k \otimes u_k), \sum_{k=1}^{m} h_k \otimes u_k \right\rangle_{\mathscr{H}\otimes\mathscr{K}} \\
&= \sum_{j=0}^{g}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \langle (A_j \otimes X_j)(h_k \otimes u_k), h_\ell \otimes u_\ell\rangle_{\mathscr{H}\otimes\mathscr{K}} \\
&= \sum_{j=0}^{g}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \langle A_j h_k, h_\ell\rangle_{\mathscr{H}} \langle X_j u_k, u_\ell\rangle_{\mathscr{K}} < 0.
\end{aligned}$$

Let e_k be the standard basis vectors for $\mathbb{R}^m$. Let us define a linear map $V : \mathbb{R}^m \to \mathscr{H}$ by $e_k \mapsto u_k$. Since $\{e_1, \ldots, e_m\}$ and $\{u_1, \ldots, u_m\}$ are orthonormal, V is an isometry. Therefore, $L(V^*XV) \succeq 0$. We have

$$\begin{aligned}
0 &\le \left\langle L(V^*XV)(\sum_k h_k \otimes e_k), \sum_k h_k \otimes e_k \right\rangle_{\mathscr{H}\otimes\mathbb{R}^m} \\
&= \left\langle (\sum_{j=1}^{g} A_j \otimes V^*X_jV)(\sum_k h_k \otimes e_k), \sum_k h_k \otimes e_k \right\rangle_{\mathscr{H}\otimes\mathbb{R}^m} \\
&= \sum_{j=0}^{g}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \langle (A_j \otimes V^*X_jV)(h_k \otimes e_k), h_\ell \otimes e_\ell\rangle_{\mathscr{H}\otimes\mathbb{R}^m} \\
&= \sum_{j=0}^{g}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \langle A_j h_k, h_\ell\rangle_{\mathscr{H}} \langle V^*X_jVe_k, e_\ell\rangle_{\mathbb{R}^m} \\
&= \sum_{j=0}^{g}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \langle A_j h_k, h_\ell\rangle_{\mathscr{H}} \langle X_jVe_k, Ve_\ell\rangle_{\mathscr{K}} \\
&= \sum_{j=0}^{g}\sum_{k=1}^{m}\sum_{\ell=1}^{m} \langle A_j h_k, h_\ell\rangle_{\mathscr{H}} \langle X_j u_k, u_\ell\rangle_{\mathscr{K}} = \langle L(X)v, v\rangle < 0.
\end{aligned}$$

This is a contradiction. Hence $X \in D_L^{\mathscr{K}}$. □

Now we prove Proposition 3.2.1.

Proof of Proposition 3.2.1. The non-trivial direction is $D_{L_1} \subseteq D_{L_2}$ implies $D_{L_1}^{\mathscr{K}} \subseteq D_{L_2}^{\mathscr{K}}$. Let us take $X \in D_{L_1}^{\mathscr{K}}$. By Lemma 3.2.2,

$$X \in D_{L_2}^{\mathscr{K}} \Leftrightarrow V^*XV \in D_{L_2}(m) \text{ for every isometry } V \in B(\mathbb{R}^m, \mathscr{K}),\ m \in \mathbb{N}.$$

By Lemma 3.2.2, $X \in D_{L_1}^{\mathscr{K}}$ implies

$$V^*XV \in D_{L_1}(m) \text{ for every isometry } V \in B(\mathbb{R}^m, \mathscr{K}),\ m \in \mathbb{N}.$$

But $D_{L_1} \subseteq D_{L_2}$ implies

$$V^*XV \in D_{L_2}(m) \text{ for every isometry } V \in B(\mathbb{R}^m, \mathscr{K}),\ m \in \mathbb{N}.$$

This concludes the proof. □

Connection with complete positivity

Given a tuple $A := (A_1, \dots, A_g) \in B(\mathscr{H})$ we denote by

$$\mathcal{S}_A = \operatorname{span}\{I_{\mathscr{H}_1}, A_1, \dots, A_g\} \subseteq B(\mathscr{H})$$

the *operator system* generated by A, i.e., the smallest subspace in $B(\mathscr{H})$ containing $I_{\mathscr{H}_1}$ and such that $X^* \in \mathcal{S}_A$ for every $X \in \mathcal{S}_A$.

We write $A \oplus \mathbf{0}_{\mathbb{R}}^g$ to denote a tuple

$$(A_1 \oplus \mathbf{0}_{\mathbb{R}}, \dots, A_g \oplus \mathbf{0}_{\mathbb{R}}) \in \mathbb{S}^g_{\mathscr{H}\oplus\mathbb{R}},$$

where $\mathbf{0}_{\mathbb{R}}$ is the zero operator on $\mathbb{R}$.

The *homogenization* ${}^{\mathrm{h}}L$ of a linear pencil $L = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ is defined by

$${}^{\mathrm{h}}L(x_0, \dots, x_g) = x_0 L(x_0^{-1}x_1, \dots, x_0^{-1}x_g).$$

Note that the evaluation of a homogeneous linear pencil ${}^{\mathrm{h}}L(x) = \sum_{j=0}^{g} A_j x_j$ on a tuple of symmetric matrices $X = (X_0, X_1, \dots, X_g) \in \mathbb{S}_n^{g+1}$ is defined by

$${}^{\mathrm{h}}L(X) = \sum_{j=0}^{g} A_j \otimes X_j$$

and

$$D_{{}^{\mathrm{h}}L} = (D_{{}^{\mathrm{h}}L}(n))_n \quad \text{where} \quad D_{{}^{\mathrm{h}}L}(n) = \{X \in \mathbb{S}_n^{g+1} \colon {}^{\mathrm{h}}L(X) \succeq 0\}$$

is its free Hilbert spectrahedron.

By Lemma 3.2.3 below the inclusion $D_{L_1}(1) \subseteq D_{L_2}(1)$ implies that the unital linear map

$$\widetilde{\tau} : \mathcal{S}_{A\oplus 0_{\mathbb{R}}^g} \to \mathcal{S}_B, \quad A_j \oplus \mathbf{0}_{\mathbb{R}} \mapsto B_j,$$

is well-defined, while the stronger inclusion $D_{{}^{\mathrm{h}}L_1}(1) \subseteq D_{{}^{\mathrm{h}}L_2}(1)$ implies the well-definedness of the unital linear map

$$\tau : \mathcal{S}_A \to \mathcal{S}_B, \quad A_j \mapsto B_j.$$

In particular, τ is well-defined if $D_{L_1}(1)$ is bounded.

Lemma 3.2.3. *Assume the notation as above.*

(1) If $D_{L_1}(1) \subseteq D_{L_2}(1)$, then the map $\widetilde{\tau}$ is well-defined.

(2) If $D_{{}^{\mathrm{h}}L_1}(1) \subseteq D_{{}^{\mathrm{h}}L_2}(1)$ or $D_{L_1}(1)$ is a bounded set, then the map τ is well-defined.

Proof. First we prove (1). It suffices to prove that

$$\mu_0(I_{\mathscr{H}_1} \oplus I_{\mathbb{R}}) + \sum_{j=1}^{g} \mu_j (A_j \oplus \mathbf{0}_{\mathbb{R}}) = 0 \quad \text{implies} \quad \mu_0 I_{\mathscr{H}_2} + \sum_{j=1}^{g} \mu_j B_j = 0,$$

where $\mu_0, \ldots, \mu_g \in \mathbb{R}$. First we notice that $\mu_0 = 0$. From $\sum_{j=1}^{g} \mu_j (A_j \oplus \mathbf{0}_{\mathbb{R}}) = 0$ it follows that $\sum_{j=1}^{g} t\mu_j (A_j \oplus \mathbf{0}_{\mathbb{R}}) = 0$ for every $t \in \mathbb{R}$. Hence,

$$(t\mu_1, \ldots, t\mu_g) \in D_{L_1}(1) \subseteq D_{L_2}(1)$$

for every $t \in \mathbb{R}$. Suppose to the contrary that $\sum_{j=1}^{g} \mu_j B_j \neq 0$. Since $\sum_{j=1}^{g} \mu_j B_j$ is self-adjoint, it follows that there is $h \in \mathscr{H}_2$ such that $\left\langle (\sum_{j=1}^{g} \mu_j B_j) h, h \right\rangle \neq 0$ by [Con90, 2.14. Corollary]. But then $t(\mu_1, \ldots, \mu_g) \notin D_{L_2}$ for $t \to \infty$ or $t \to -\infty$, which is a contradiction. Hence $\sum_{j=1}^{g} \mu_j B_j = 0$ and the map $\widetilde{\tau}$ is well-defined.

For the proof of (2) let us first consider the inclusion $D_{^{\mathrm{h}}L_1}(1) \subseteq D_{^{\mathrm{h}}L_2}(1)$. We have to prove that

$$\mu_0 I_n + \sum_{j=1}^{g} \mu_j A_j = 0 \quad \text{implies} \quad \mu_0 I_m + \sum_{j=1}^{g} \mu_j B_j = 0.$$

Suppose to the contrary that $\mu_0 I_m + \sum_{j=1}^{g} \mu_j B_j \neq 0$. Since $\mu_0 I_m + \sum_{j=1}^{g} \mu_j B_j$ is self-adjoint, it follows that there is $h \in \mathscr{H}_2$ such that $\left\langle (\mu_0 I_m + \sum_{j=1}^{g} \mu_j B_j) h, h \right\rangle \neq 0$ by [Con90, 2.14. Corollary]. Therefore $t(\mu_0, \mu_1, \ldots, \mu_g) \notin D_{^{\mathrm{h}}L_2}$ for $t \to \infty$ or $t \to -\infty$. But this is a contradiction with $t(\mu_0, \mu_1, \ldots, \mu_g) \in D_{^{\mathrm{h}}L_1} \subseteq D_{^{\mathrm{h}}L_2}$. Hence τ is well-defined.

Now we consider the case of a bounded set $D_{L_1}(1)$. In this case the set

$$\{I_{\mathscr{H}_1}, A_1, \ldots, A_g\}$$

is linearly independent; the proof is the same as in the matrix case (see [HKM13b, Proposition 2.6]). Thus τ is well-defined. □

The following example shows that for unbounded sets $D_{L_1}(1)$, the assumption $D_{L_1}(1) \subseteq D_{L_2}(1)$ does not suffice for the well-definedness of the map τ.

Example 3.2.4. Let $\ell_1 = 1 + x$ and $\ell_2 = 1$ be monic linear scalar polynomials. Note that $D_{\ell_1}(1) = [-1, \infty) \subset \mathbb{R} = D_{\ell_2}(1)$ but by the definition of the map $\tau : \mathbb{R} \to \mathbb{R}$ we have that $\tau(1) = 1$ and $\tau(1) = 0$, which is a contradiction.

Now we define the n-positivity, $n \in \mathbb{N}$, and the complete positivity of a linear map

$$\phi : \mathcal{S}_1 \to \mathcal{S}_2$$

mapping between operator systems $\mathcal{S}_j \subseteq B(\mathscr{H}_j)$, $j = 1, 2$, invariant under the transpose. For $n \in \mathbb{N}$, ϕ induces the map

$$\phi_n = I_n \otimes \phi : \mathbb{R}^{n \times n} \otimes \mathcal{S}_1 = \mathcal{S}_1^{n \times n} \to \mathcal{S}_2^{n \times n}, \quad M \otimes A \mapsto M \otimes \phi(A),$$

called an *ampliation* of ϕ. Equivalently,

$$\phi\left(\begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix}\right) = \begin{bmatrix} \phi(T_{11}) & \cdots & \phi(T_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(T_{n1}) & \cdots & \phi(T_{nn}) \end{bmatrix}.$$

We say that ϕ is *n-positive* if ϕ_n is a positive map. If ϕ is n-positive for every $n \in \mathbb{N}$, then ϕ is *completely positive*. If ϕ_n is an isometry for every $n \in \mathbb{N}$, then ϕ is *completely isometric*.

In the following theorem we prove that the n-positivity of τ is equivalent to the inclusion $D_{{}^{\mathrm{h}}L_1}(n) \subseteq D_{{}^{\mathrm{h}}L_2}(n)$. Therefore the n-positivity of $\widetilde{\tau}$ is equivalent to the inclusion $D_{L_1}(n) \subseteq D_{L_2}(n)$. However, for bounded $D_{L_1}(1)$, the map τ is n-positive if and only if the map $\widetilde{\tau}$ is n-positive.

Theorem 3.2.5. *Let*

$$L_1 = I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}_1}\langle x \rangle, \quad L_2 = I_{\mathscr{H}_2} + \sum_{j=1}^{g} B_j x_j \in \mathbb{S}_{\mathscr{H}_2}\langle x \rangle$$

be monic linear operator pencils such that $D_{{}^{h}L_1}(1) \subseteq D_{{}^{h}L_2}(1)$. Let $\tau : \mathcal{S}_A \to \mathcal{S}_B$ be the unital linear map $A_j \mapsto B_j$ and $n \in \mathbb{N}$. Then:

(1) τ is n-positive if and only if $D_{{}^{h}L_1}(n) \subseteq D_{{}^{h}L_2}(n)$.

(2) τ is completely positive if and only if $D_{{}^{h}L_1} \subseteq D_{{}^{h}L_2}$.

(3) If $\dim(\mathscr{H}_2) = n \in \mathbb{N}$, then τ is completely positive if and only if τ is n-positive.

Let $L_1 \oplus I_{\mathbb{R}}$ be the monic linear operator pencil

$$L_1 \oplus I_{\mathbb{R}} = I_{\mathscr{H}_1} \oplus I_{\mathbb{R}} + \sum_{j=1}^{g} (A_j \oplus \mathbf{0}_{\mathbb{R}}) x_j \in \mathbb{S}_{\mathscr{H}_1 \oplus \mathbb{R}}\langle x \rangle.$$

Let $\widetilde{\tau} : \mathcal{S}_{A \oplus \mathbf{0}_{\mathbb{R}}^g} \to \mathcal{S}_B$ be the unital linear map $A_j \oplus \mathbf{0}_{\mathbb{R}} \mapsto B_1$. Then:

(4) $D_{L_1}(n) \subseteq D_{L_2}(n)$ if and only if $D_{{}^{h}(L_1 \oplus I_{\mathbb{R}})}(n) \subseteq D_{{}^{h}L_2}(n)$.

(5) $\widetilde{\tau}$ is n-positive if and only if $D_{L_1}(n) \subseteq D_{L_2}(n)$.

(6) $\widetilde{\tau}$ is completely positive if and only if $D_{L_1} \subseteq D_{L_2}$.

Moreover, if $D_{{}^{h}L_1} = D_{{}^{h}(L_1 \oplus I_{\mathbb{R}})}$, then

(7) $D_{L_1}(n) \subseteq D_{L_2}(n)$ if and only if $D_{{}^{h}L_1}(n) \subseteq D_{{}^{h}L_2}(n)$.

(8) τ is n-positive if and only if $D_{L_1}(n) \subseteq D_{L_2}(n)$.

(9) τ is completely positive if and only if $D_{L_1} \subseteq D_{L_2}$.

Remark 3.2.6. (1) Note that τ and $\widetilde{\tau}$ are well-defined; see Lemma 3.2.3.

(2) Equivalent version of Theorem 3.2.5 can be obtained from the results of Davidson, Dor-On, Shalit and Solel (see [DDSS+, Proposition 5.9 and Corollary 5.10]). Their results are established by relating the existence of unital completely maps or completely contractive maps between the g-tuples of operators to *matrix ranges* which were introduced by Arveson in [Arv72a]. We will look closer to matrix ranges in the next section when studying the problem of the equality of free (Hilbert) spectrahedra.

In the proof of Theorem 3.2.5 we need some lemmas.

Lemma 3.2.7. *Suppose $T \in \mathcal{S}_A^{n\times n}$ is self-adjoint. Then there exist symmetric matrices $Y, X_1, \dots, X_g \in \mathbb{S}_n$ such that $T = Y \otimes I_{\mathscr{H}_1} + \sum_{j=1}^{g} X_j \otimes A_j$.*

Proof. By definition, T is of the form $\widetilde{Y} \otimes I_{\mathscr{H}_1} + \sum_{j=1}^{g} \widetilde{X}_j \otimes A_j$ for some $\widetilde{Y}, \widetilde{X}_1, \dots, X_g \in \mathbb{R}^{n\times n}$. From $T = T^*$, it follows that

$$\widetilde{Y} \otimes I_{\mathscr{H}_1} + \sum_{j=1}^{g} \widetilde{X}_j \otimes A_j = \frac{1}{2}\big((\widetilde{Y} + \widetilde{Y}^*) \otimes I_{\mathscr{H}_1} + \sum_{j=1}^{g} (\widetilde{X}_j + \widetilde{X}_j^*) \otimes A_j\big).$$

Defining $Y := \frac{1}{2}(\widetilde{Y} + \widetilde{Y}^*)$ and $X_j = \frac{1}{2}(\widetilde{X}_j + \widetilde{X}_j^*)$ for $j = 1, \dots, g$ proves the lemma. □

Lemma 3.2.8. *Let $L = A_0 + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ be a linear pencil. Then for a tuple $X := (X_0, X_1, \dots, X_g) \in \mathbb{S}_n^{g+1}$ we have*

$$L(X) = A_0 \otimes X_0 + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0 \quad \Leftrightarrow \quad X_0 \otimes A_0 + \sum_{j=1}^{g} X_j \otimes A_j \succeq 0.$$

Proof. The lemma follows by observing that after applying a permutation called the *canonical shuffle* [Pau02] to $L(X)$ we obtain $X_0 \otimes A_0 + \sum_{j=1}^{g} X_j \otimes A_j$. □

Lemma 3.2.9. *Let $L \in \mathbb{S}_{\mathscr{H}_k}\langle x\rangle$ be a monic linear operator pencil. Let $X := (X_0, \dots, X_g) \in \mathbb{S}_n^{g+1}$ be a tuple satisying $X_0 \succeq 0$. Then:*

$${}^{h}L(X) \succeq 0 \quad \Leftrightarrow \quad \forall \epsilon > 0 : \ L\Big((X_0 + \epsilon I_n)^{-\frac{1}{2}}(X_1, \dots, X_g)(X_0 + \epsilon I_n)^{-\frac{1}{2}}\Big) \succeq 0,$$

Proof. Let $X := (X_0, \dots, X_g) \in \mathbb{S}_n^{g+1}$ be a tuple satisying $X_0 \succeq 0$. Then we have

$$\begin{aligned} {}^{\mathrm{h}}L(X) \succeq 0 \quad &\Leftrightarrow \quad \forall \epsilon > 0 : \ {}^{\mathrm{h}}L(X + (\epsilon I_n, 0, \dots, 0)) \succeq 0 \Leftrightarrow \\ \forall \epsilon > 0 : (I_{\mathscr{H}_k} \otimes (X_0 + \epsilon I_n)^{-\frac{1}{2}})^* \, {}^{\mathrm{h}}L(X + (\epsilon I_n, 0, \dots, 0))&(I_{\mathscr{H}_k} \otimes (X_0 + \epsilon I_n)^{-\frac{1}{2}}) \succeq 0 \Leftrightarrow \\ \forall \epsilon > 0 : \ L\Big((X_0 + \epsilon I_n)^{-\frac{1}{2}}&(X_1, \dots, X_g)(X_0 + \epsilon I_n)^{-\frac{1}{2}}\Big) \succeq 0, \end{aligned}$$

where the first equivalence follows by closedeness of the free Hilbert spectrahedra. This establishes the lemma. □

Lemma 3.2.10. *Let $L \in \mathbb{S}_{\mathscr{H}_k}\langle x\rangle$ be a monic linear operator pencil and $X := (X_0, \dots, X_g) \in \mathbb{S}_n^{g+1}$ a tuple of self-adjoint matrices. Then:*

(1) $X \in D_{^{h}(L \oplus I_{\mathbb{R}})}$ if and only if $X_0 \succeq 0$ and $X \in D_{^{h}L}$.

(2) $D_{^{h}L} = D_{^{h}(L \oplus I_{\mathbb{R}})}$ if and only if $X \in D_{^{h}L}$ implies that $X_0 \succeq 0$.

Proof. (2) clearly follows from (1), while (1) is a simple observation. □

Now we prove Theorem 3.2.5.

Proof of Theorem 3.2.5. Let us prove (1) and (2). Since (2) follows from (1), it suffices to prove (1). Suppose $T \in \mathcal{S}_A^{n\times n}$ is self-adjoint. By Lemma 3.2.7 there exist symmetric matrices $Y, X_1, \dots, X_g \in \mathbb{S}_n$ such that

$$T = Y \otimes I_{\mathscr{H}_1} + \sum_{j=1}^{g} X_j \otimes A_j.$$

By definition of $\widetilde{\tau}$ we have that

$$\widetilde{\tau}(T) = Y \otimes I_{\mathscr{H}_2} + \sum_{j=1}^{g} X_j \otimes B_j.$$

By Lemma 3.2.8,

$$T \succeq 0 \quad \Leftrightarrow \quad I_{\mathscr{H}_1} \otimes Y + \sum_{j=1}^{g} A_j \otimes X_j \succeq 0 \quad \Leftrightarrow \quad (Y, X_1, \dots, X_g) \in D_{{}^{\mathrm{h}}L_1}(n),$$
$$\widetilde{\tau}(T) \succeq 0 \quad \Leftrightarrow \quad I_{\mathscr{H}_2} \otimes Y + \sum_{j=1}^{g} B_j \otimes X_j \succeq 0 \quad \Leftrightarrow \quad (Y, X_1, \dots, X_g) \in D_{{}^{\mathrm{h}}L_2}(n).$$

Therefore $\widetilde{\tau}$ is n-positive if and only if $D_{{}^{\mathrm{h}}L_1}(n) \subseteq D_{{}^{\mathrm{h}}L_2}(n)$.

(3) follows from [Pau02, Theorem 6.1]. To prove (4), (5), (6), it suffices to establish (4). The non-trivial implication of (4) is ($\Rightarrow$). Take $X := (X_0, X_1, \dots, X_g) \in D_{{}^{\mathrm{h}}\widetilde{L}_1}$. By Lemma 3.2.10 (1), $X_0 \succeq 0$. Thus, it follows by Lemma 3.2.9 and the inclusion $D_{L_1}(n) \subseteq D_{L_2}(n)$ that $X \in D_{{}^{\mathrm{h}}L_2}$. This establishes $D_{{}^{\mathrm{h}}(L_1 \oplus I_{\mathbb{R}})} \subseteq D_{{}^{\mathrm{h}}L_2}$.

Finally, to prove (7), (8), (9), it suffices to establish (7). The non-trivial implication of (7) is ($\Rightarrow$). Let us take $X := (X_0, X_1, \dots, X_n) \in D_{{}^{\mathrm{h}}L_1}$. By the assumption $D_{{}^{\mathrm{h}}L_1} = D_{{}^{\mathrm{h}}(L_1 \oplus I_{\mathbb{R}})}$ and Lemma 3.2.10 (2) it follows that $X_0 \succeq 0$. Now by Lemma 3.2.9 and the inclusion $D_{L_1}(n) \subseteq D_{L_2}(n)$, we conclude that $X \in D_{{}^{\mathrm{h}}L_2}$. Thus, $D_{{}^{\mathrm{h}}L_1} \subseteq D_{{}^{\mathrm{h}}L_2}$. □

An important case of the equality $D_{{}^{\mathrm{h}}L_1} = D_{{}^{\mathrm{h}}(L_1 \oplus I_{\mathbb{R}})}$ in the notation of Theorem 3.2.5 occurs if $D_{L_1}(1)$ is bounded.

Proposition 3.2.11. *Let $L \in \mathbb{S}_{\mathscr{H}}\langle x \rangle$ be a monic linear operator pencil such that $D_L(1)$ is bounded. Then $D_{{}^{\mathrm{h}}L} = D_{{}^{\mathrm{h}}(L \oplus I_{\mathbb{R}})}$.*

Proof. Take $X := (X_0, X_1, \dots, X_g) \in D_{{}^{\mathrm{h}}L}$. By Lemma 3.2.10 (2) we have to prove that $X_0 \succeq 0$. We argue by contradiction. Assume ${}^{\mathrm{h}}L(X) \succeq 0$ but $X_0 \not\succeq 0$. Then there exists $v \in \mathbb{R}^n$ with $\langle X_0 v, v \rangle < 0$. Define $V : \mathbb{R} \to \mathbb{R}^n$ by $r \mapsto rv$. The map $V^* : \mathbb{R}^n \to \mathbb{R}$ is given by $u \mapsto \langle u, v \rangle$. We have

$$\begin{aligned}(I_{\mathscr{H}} \otimes V)^{*\mathrm{h}} L_1(X)(I_{\mathscr{H}} \otimes V) &= I_{\mathscr{H}} \otimes V^* X_0 V + \sum_{j=1}^{g} A_j \otimes V^* X_j V \\ &= I_{\mathscr{H}} \otimes \langle X_0 v, v \rangle + \sum_{j=1}^{g} A_j \otimes \langle X_j v, v \rangle \succeq 0.\end{aligned}$$

Since $I_{\mathscr{H}} \otimes \langle X_0 v, v\rangle \prec 0$, it follows that $\sum_{j=1}^{g} A_j \otimes \langle X_j v, v\rangle \succ 0$. Thus

$$(t\langle X_1 v, v\rangle, \ldots, t\langle X_g v, v\rangle) \in D_{L_1}(1)$$

for every $t > 0$ which contradicts the boundedness of $D_{L_1}(1)$. $\square$

For L_1 and L_2 monic linear pencils such that $D_{L_1}(1)$ is unbounded and $D_{L_1} \subseteq D_{L_2}$, it is not necessary that $D_{{}^{\mathrm{h}}L_1} \subseteq D_{{}^{\mathrm{h}}L_2}$ by Example 3.2.12 below.

Example 3.2.12. For the following monic linear matrix pencils

$$L_1(x_1, x_2) = \begin{bmatrix} 1 + 2x_1 + 2x_2 & 0 & 0 \\ 0 & 1 + 2x_1 & 0 \\ 0 & 0 & 1 + 2x_2 \end{bmatrix},$$
$$L_2(x_1, x_2) = \begin{bmatrix} 1 + x_1 + x_2 & 0 & 0 \\ 0 & 1 + x_1 & 0 \\ 0 & 0 & 1 + x_2 \end{bmatrix},$$

we have

$$D_{L_1}(n) = \left\{(X_1, X_2) \in \mathbb{S}_n \colon X_1 + X_2 \succeq -\frac{1}{2}I_n,\ X_1 \succeq -\frac{1}{2}I_n,\ X_2 \succeq -\frac{1}{2}I_n\right\},$$
$$D_{L_2}(n) = \{(X_1, X_2) \in \mathbb{S}_n \colon X_1 + X_2 \succeq -I_n,\ X_1 \succeq -I_n,\ X_2 \succeq -I_n\},$$

for every $n \in \mathbb{N}$. Hence, $D_{L_1}(n) \subseteq D_{L_2}(n)$ for every $n \in \mathbb{N}$, i.e., $D_{L_1} \subseteq D_{L_2}$. But

$$\left(-1, \frac{1}{2}, \frac{1}{2}\right) \in D_{{}^{\mathrm{h}}L_1}(1) \setminus D_{{}^{\mathrm{h}}L_2}(1),$$

and hence

$$D_{{}^{\mathrm{h}}L_1}(1) \not\subseteq D_{{}^{\mathrm{h}}L_2}(1).$$

Characterization of the inclusion $D_{L_1} \subseteq D_{L_2}$

The main result of this section, Theorem 3.2.13 below, is the characterization of the inclusion $D_{L_1} \subseteq D_{L_2}$. The proof uses the connection with complete positivity explained in the previous subsection. Theorem 3.2.13 is the extension of Theorem 3.1.1 from matrix pencils to operator pencils.

Theorem 3.2.13 (Operator linear Positivstellensatz)**.** *Let*

$$L_1 = I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}_1}\langle x\rangle, \quad L_2 = I_{\mathscr{H}_2} + \sum_{j=1}^{g} B_j x_j \in \mathbb{S}_{\mathscr{H}_2}\langle x\rangle$$

be monic linear operator pencils. The following statements are equivalent:

(1) $D_{L_1} \subseteq D_{L_2}$.

(2) There exist a separable real Hilbert space $\mathscr{K}$, an isometry $V : \mathscr{H}_2 \to \mathscr{K}$ and a unital $*$-homomorphism $\pi : C^*(\mathcal{S}_{A\oplus \mathbf{0}_{\mathbb{R}}^g}) \to B(\mathscr{K})$ such that

$$L_2(x) = V^*\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)V + V^*\pi\left(\begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}\right)V.$$

(3) There exist a separable real Hilbert space $\mathscr{K}_0$, a contraction $V_0 : \mathscr{H}_2 \to \mathscr{K}_0$, a unital $*$-homomorphism $\pi_0 : C^*(\mathcal{S}_A) \to B(\mathscr{K}_0)$ and a positive semidefinite operator $S \in B(\mathscr{H}_2)$ such that

$$L_2(x) = S + V_0^* \pi_0(L_1(x))V_0.$$

Moreover, if $D_{^{\mathrm{h}}L_1} = D_{^{\mathrm{h}}(L_1 \oplus I_{\mathbb{R}})}$, then V_0 in (3) can be chosen to be isometric and $S = 0$.

Proof. First we will prove the implication (1) ⇒ (2). By Theorem 3.2.5 (5) the map $\widetilde{\tau}$ is completely positive. By the real version of Arveson's extension theorem [CZ13, Proposition 4] (take

$$E = C^*(\mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}}), \quad E_0 = \mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}}, \quad K_n(E) = \{A \in M_n(B(\mathscr{H}_1)) \colon A \succeq 0\}),$$

there exists a completely positive extension $\widetilde{\tau} : C^*(\mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}}) \to B(\mathscr{H}_2)$ for $\widetilde{\tau} : \mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}} \to \mathcal{S}_B$. By the Stinespring theorem, there exist a separable real Hilbert space $\mathscr{K}$, a $*$-homomorphism $\pi : C^*(\mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}}) \to B(\mathscr{K})$ and an isometry $V : \mathscr{H}_2 \to \mathscr{K}$ such that $\widetilde{\tau}(C) = V^*\pi(C)V$ for all $C \in C^*(\mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}})$. Hence,

$$\begin{aligned} L_2(x) = \widetilde{\tau}\left(\begin{bmatrix} L_1(x) & 0 \\ 0 & 1 \end{bmatrix}\right) &= V^*\pi\left(\begin{bmatrix} L_1(x) & 0 \\ 0 & 1 \end{bmatrix}\right)V \\ &= V^*\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)V + V^*\pi\left(\begin{bmatrix} L_1(x) & 0 \\ 0 & 0 \end{bmatrix}\right)V. \end{aligned}$$

Now we will prove the implication (2) ⇒ (3). Observe that $\pi\left(\begin{bmatrix} I_{\mathscr{H}_1} & 0 \\ 0 & 0 \end{bmatrix}\right)$ is a hermitian idempotent, hence a projection onto $\mathscr{K}_0 := \operatorname{Ran}\left(\pi\left(\begin{bmatrix} I_{\mathscr{H}_1} & 0 \\ 0 & 0 \end{bmatrix}\right)\right)$, by [Con90, 3.3 Proposition]. We define a contraction

$$V_0 := P^{\mathscr{K}}_{\mathscr{K}_0}\pi\left(\begin{bmatrix} I_{\mathscr{H}_1} & 0 \\ 0 & 0 \end{bmatrix}\right)V : \mathscr{H}_1 \to \mathscr{K}_0,$$

where $P^{\mathscr{K}}_{\mathscr{K}_0}$ is a projection from $\mathscr{K}$ to $\mathscr{K}_0$. We define a new representation

$$\pi_0 : C^*(\mathcal{S}_A) \to B(\mathscr{K}_0), \quad A \mapsto P^{\mathscr{K}}_{\mathscr{K}_0}\pi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)\Big|_{\mathscr{K}_0}.$$

Since $\operatorname{Ran}\left(\pi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)\right) \subseteq \mathscr{K}_0$, π_0 is well-defined. Thus

$$L_2(x) = S + V_0^*\pi_0(L_1(x))V_0,$$

where $S := V^*\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)V \succeq 0$.

Finally, the implication (3) ⇒ (1) is clear. If $D_{^{\mathrm{h}}L_1} = D_{^{\mathrm{h}}(L_1\oplus I_{\mathbb{R}})}$, then we work with τ instead of $\widetilde{\tau}$ to get $S = 0$ in (3). □

Remark 3.2.14. (1) If $D_{L_1}(1)$ is unbounded, then in Theorem 3.2.13 (2), V_0 cannot always be chosen to be isometric (and hence $S = 0$). See Example 3.2.12 above: if $L_2 = V_0^* \pi(L_1) V_0$ for an isometry V_0, then $D_{{}^{\mathrm{h}}L_1} \subseteq D_{{}^{\mathrm{h}}L_2}$ which is not true. If L_1 and L_2 are monic linear matrix pencils and we restrict ourselves to $*$-homomorphisms π mapping into finite dimensional spaces, then V_0 can be chosen to be isometric if and only if $\operatorname{span}\{A_1, \ldots, A_g\}$ does not contain a positive definite matrix by [HKM12, Remark 4.4].

(2) If $\mathscr{H}_1$ is finite-dimensional, then every unital $*$-homomorphism $\pi : B(\mathscr{H}_1) \to B(\mathscr{K})$ is unitarily equivalent to the direct sum of the identity $*$-homomorphism. Hence if $\mathscr{H}_2$ is infinite-dimensional, then we can replace in Theorem 3.2.13 above

$$\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right), \quad \pi\left(\begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}\right) \quad \text{and} \quad \pi_0(L_1)$$

by

$$\bigoplus_{i=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bigoplus_{i=1}^{\infty} \begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \bigoplus_{i=1}^{\infty} L_1 \quad \text{respectively.}$$

If $\mathscr{H}_2$ is finite-dimensional, then those sums are finite as in [HKM13b, Corollary 3.7] and [HKM12, Corollary 4.1].

(3) The assumption of monicity of pencils can be replaced by the assumption of nonempty D_{L_1} and the existence of an invertible positive definite element in the linear span of coefficients of L_1. In the statement of Theorem 3.2.13, V then becomes a bounded operator, which is not necesarrily a contraction.

If $\mathscr{H}_2$ is finite-dimensional of dimension n, then the inclusion $D_{L_1}(n) \subseteq D_{L_2}(n)$ is sufficient for the conclusion of Theorem 3.2.13 to hold.

Theorem 3.2.15 (Operator-to-matrix linear Postivstellensatz)**.** *Suppose that* $L_1 := I_{\mathscr{H}_1} + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}_1}\langle x \rangle$ *is a monic linear operator pencil and* $L_2 \in \mathbb{S}_n\langle x \rangle$, $n \in \mathbb{N}$, *is a monic linear matrix pencil. The following statements are equivalent:*

(1) $D_{L_1}(n) \subseteq D_{L_2}(n)$.

(2) There exist a separable real Hilbert space $\mathscr{K}$, *an isometry* $V : \mathbb{R}^n \to \mathscr{K}$, *and a unital* $*$*-homomorphism* $\pi : C^*(\mathcal{S}_{A \oplus 0_{\mathbb{R}}^g}) \to B(\mathscr{K})$ *such that*

$$L_2(x) = V^* \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) V + V^* \pi\left(\begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}\right) V.$$

(3) There exist a separable real Hilbert space $\mathscr{K}_0$, *a contraction* $V_0 : \mathbb{R}^n \to \mathscr{K}$, *a unital* $*$*-homomorphism* $\pi_0 : C^*(\mathcal{S}_A) \to B(\mathscr{K})$ *and a positive semidefinite matrix* $S \in \mathbb{S}_n$ *such that*

$$L_2(x) = S + V_0^* \pi_0(L_1) V_0.$$

Moreover, if $D_{{}^{\mathrm{h}}L_1} = D_{{}^{\mathrm{h}}(L_1 \oplus I_{\mathbb{R}})}$, *then* V_0 *in (3) can be chosen to be isometric and* $S = 0$.

Proof. By Theorem 3.2.5 (3), $D_{L_1}(n) \subseteq D_{L_2}(n)$ implies $D_{L_1} \subseteq D_{L_2}$. Now everything follows by Theorem 3.2.13. $\square$

Counterexample for non-monic pencils

We present an example which shows that the assumption of monicity of pencils in Theorem 3.2.13 is necessary. The example is a generalization of [Zal12, Example 2].

Example 3.2.16. Let $L(x) = \begin{bmatrix} 1 & x \\ x & 0 \end{bmatrix} \in \mathbb{S}_2\langle x\rangle$ be a linear matrix polynomial. Then:

(1) The free spectrahedron of L is $D_L = (\{0_n\})_{n\in\mathbb{N}}$ where 0_n stands for the $n \times n$ matrix with zero entries.

(2) The polynomial $\ell(x) = x$ is non-negative on $D_L(1)$.

(3) There do not exist a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism $\pi : B(\mathbb{R}^2) \to B(\mathscr{K})$, polynomials $r_j \in \mathbb{R}\langle x\rangle$ and operator polynomials $q_k \in B(\mathbb{R}, \mathscr{K})\langle x\rangle$ such that
$$x = \sum_j^{\text{finite}} r_j^2 + \sum_k^{\text{finite}} q_k^* \pi(L) q_k.$$

Proof. (1) and (2) are clear. Let us prove (3). For $K = \mathbb{R}^2$, the identity $*$-homomorphism $\pi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$, i.e., $\pi(X) = X$ for all $X \in M_2(\mathbb{R})$, and polynomials $r_j \in \mathbb{R}\langle x\rangle$, $q_k \in \mathbb{R}^{2\times 1}\langle x\rangle$ the proof is already done in [Zal12, Example 2]. Let us now prove a general case. If $\mathscr{K}$, π, r_j, q_k existed, we would have
$$\begin{aligned} x &= \sum_j r_j^* r_j + \sum_k q_k^* \pi(L) q_k \\ &= \sum_j r_j^* r_j + \sum_k q_k^* \pi(E_{11}) q_k + \sum_k q_k^* \pi(E_{12} + E_{21}) q_k x, \end{aligned} \tag{3.2.1}$$
where all sums are finite. Let us write
$$r_j(x) = \sum_{m=0}^{N_j} r_{j,m} x \in \mathbb{R}\langle x\rangle, \quad q_k(x) = \sum_{m=0}^{M_k} q_{k,m} x \in B(\mathbb{R}, \mathscr{K})\langle x\rangle,$$
where $N_j \in \mathbb{N}_0$ is such that $r_{j,N_j} \neq 0$ and $M_k \in \mathbb{N}_0$ is such that $q_{k,M_k} \neq 0$. Comparing the coefficients at 1 on both sides of (3.2.1) we get
$$0 = \sum_j r_{j,0}^2 + \sum_k q_{k,0}^* \pi(E_{11}) q_{k,0}.$$
Since
$$\pi(E_{11}) = \pi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \pi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2\right) = \pi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^2$$
and
$$\pi(E_{11}) = \pi(E_{11}^*) = \pi(E_{11})^*,$$
$\pi(E_{11})$ is a hermitian idempotent, hence a projection. Therefore
$$\sum_k q_{k,0}^* \pi(E_{11}) q_{k,0} = \sum_k q_{k,0}^* (\pi(E_{11}))^* \pi(E_{11}) q_{k,0} \geq 0.$$

Thus,

$$r_{j,0}^2 = \sum_k q_{k,0}^*(\pi(E_{11}))^*\pi(E_{11})q_{k,0} = 0.$$

It follows that

$$r_{j,0} = 0 \quad \text{and} \quad 0 = \pi(E_{11})q_{k,0} = q_{k,0}^*\pi(E_{11})^* \in B(\mathscr{K}, \mathbb{R}).$$

Indeed,

$$0 = \langle((\pi(E_{11})q_{k,0})^*\pi(E_{11})q_{k,0})1, 1\rangle = \langle\pi(E_{11})q_{k,0}1, \pi(E_{11})q_{k,0}1\rangle = \|\pi(E_{11})q_{k,0}1\|.$$

It follows that

$$\operatorname{Ran}(q_{k,0}) \in \ker(\pi(E_{11})(\mathscr{K})).$$

Hence,

$$\begin{aligned} q_{k,0}^*\pi(E_{12}+E_{21})q_{k,0} &= q_{k,0}^*\pi(E_{11}E_{12}+E_{21}E_{11})q_{k,0} \\ &= q_{k,0}^*\pi(E_{11}E_{12})q_{k,0} + q_{k,0}^*\pi(E_{21}E_{11})q_{k,0} \\ &= \underbrace{q_{k,0}^*\pi(E_{11})}_{=q_{k,0}^*\pi(E_{11})^*=0}\pi(E_{12})q_{k,0} + q_{k,0}^*\pi(E_{21})\underbrace{\pi(E_{11})q_{k,0}}_{0} = 0. \end{aligned}$$

The coefficient at x on the right hand side of (3.2.1) is

$$\begin{aligned} \sum_j r_{j,1}^*\underbrace{r_{j,0}}_{0} + \sum_j \underbrace{r_{j,0}^*}_{0} r_{j,1} + \sum_k q_{k,1}^*\underbrace{\pi(E_{11})q_{k,0}}_{0} + \sum_k \underbrace{q_{k,0}^*\pi(E_{11})}_{0} q_{k,1} + \\ + \sum_k \underbrace{q_{k,0}^*\pi(E_{12}+E_{21})q_{k,0}}_{0} = 0. \end{aligned}$$

The coefficient at x on the left hand side of (3.2.1) is 1 which is a contradiction. This finishes the proof. □

3.2.2 Polar duals

In this subsection we describe operator free Hilbert polar duals of an operator free Hilbert spectrahedron (see Theorem 3.2.17 below) and an operator free Hilbert spectrahedrop (see Theorem 3.2.18 below), extending Theorems 3.1.3 and 3.1.4 from matrix to operator pencils. Both results are consequence of Theorem 3.2.13.

Theorem 3.2.17. *Let $\mathscr{H}, \mathscr{K}$ be separable real Hilbert spaces and $A := (A_1, \ldots, A_g)$ a tuple of self-adjoint operators $A_i \in \mathbb{S}_{\mathscr{H}}$. Then:*

(1) $(D_{L_A})^{\mathscr{K},\circ} = \operatorname{co}_{\mathscr{K}}^{\text{oper}}\{A \oplus \mathbf{0}_{\mathbb{R}}^g\}$.

(2) If $D_{{}^hL_A} = D_{{}^hL_B}$*, then* $\operatorname{co}_{\mathscr{K}}^{\text{oper}}\{A\} = \operatorname{co}_{\mathscr{K}}^{\text{oper}}\{B\}$.

(3) $\operatorname{co}_{\mathscr{K}}^{\text{oper}}\{A\}^\circ = D_{L_A}$.

In particular, it is true that:

(4) $(D_{L_A})^\circ = \operatorname{co}^{\text{mat}}\{A \oplus \mathbf{0}_{\mathbb{R}}^g\}$.

(5) If $D_{{}^{h}L_A} = D_{{}^{h}L_B}$*, then* $\mathrm{co}^{\mathrm{mat}}\{A\} = \mathrm{co}^{\mathrm{mat}}\{B\}$.

(6) $\mathrm{co}^{\mathrm{mat}}\{A\}^{\circ} = D_{L_A}$.

Proof. First we will prove the inclusion $\mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A \oplus \mathbf{0}^g_{\mathbb{R}}\} \subseteq (D_{L_A})^{\mathscr{K},\circ}$. Let us take $X := V^*\pi(A \oplus \mathbf{0}^g_{\mathbb{R}})V \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}(K)$, where K is a closed subspace of $\mathscr{K}$, $\mathscr{G}$ a separable real Hilbert space, $V : K \to \mathscr{G}$ a contraction and $\pi : C^*(\mathcal{S}_{A\oplus\mathbf{0}^g_{\mathbb{R}}}) \to B(\mathscr{G})$ a unital $*$-homomorphism. We have to prove that $L_X|_{D_{L_A}} \succeq 0$. For every $Y \in D_{L_A}$ we have

$$L_X(Y) = L_{V^*\pi(A\oplus\mathbf{0}^g_{\mathbb{R}})V}(Y) = I_K \otimes I + \sum_{j=1}^{g} V^*\pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V \otimes Y_j$$
$$= (V \otimes I)^*(I_{\mathscr{G}} \otimes I + \sum_{j=1}^{g} \pi(A_j \oplus \mathbf{0}_{\mathbb{R}}) \otimes Y_j)(V \otimes I).$$

Using that

$$I_{\mathscr{G}} \otimes I + \sum_{j=1}^{g} \pi(A_j \oplus \mathbf{0}_{\mathbb{R}}) \otimes Y_j = (\pi \otimes I)(I_{\mathscr{K}} \otimes I + \sum_{j=1}^{g}(A_j \oplus \mathbf{0}_{\mathbb{R}}) \otimes Y_j)$$
$$= (\pi \otimes I)(L_{A\oplus\mathbf{0}_{\mathbb{R}}}(Y)) \succeq 0,$$

where the last inequality follows by $\pi \otimes I$ being a $*$-homomorphism, it follows that $X \in (D_{L_A})^{\mathscr{K},\circ}$.

Let us now prove the inclusion $(D_{L_A})^{\mathscr{K},\circ} \subseteq \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\oplus\mathbf{0}^g_{\mathbb{R}}\}$. Suppose that $X \in \mathbb{S}^g_K$ belongs to $(D_{L_A})^{\mathscr{K},\circ}(K)$ where K is a closed subspace of $\mathscr{K}$. We have to prove that $X \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A \oplus \mathbf{0}^g_{\mathbb{R}}\}(K)$. By assumption $L_X|_{D_{L_A}} \succeq 0$. Using Theorem 3.2.13 (2) there exist a separable real Hilbert space $\mathscr{G}$, an isometry $V : K \to \mathscr{G}$, a unital $*$-homomorphism $\pi : C^*(\mathcal{S}_{A\oplus\mathbf{0}^g_{\mathbb{R}}}) \to B(\mathscr{G})$ and a positive semidefinite operator $S \in \mathbb{S}_K$ such that $L_X = S + V^*\pi(L_{A\oplus\mathbf{0}^g_{\mathbb{R}}})V$. In particular,

$$X = V^*\pi(A \oplus \mathbf{0}^g_{\mathbb{R}})V \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A \oplus \mathbf{0}^g_{\mathbb{R}}\}.$$

This concludes the proof of (1).

If $D_{{}^{h}L_A} = D_{{}^{h}L_B}$, then both unital maps

$$\tau : \mathcal{S}_A \to \mathcal{S}_B, \quad A_j \mapsto B_j,$$

and its inverse

$$\tau^{-1} : \mathcal{S}_B \to \mathcal{S}_A, \quad B_j \mapsto A_j,$$

are well-defined and completely positive by Theorem 3.2.5 (2). Now for every tuple

$$X = (V^*\pi(B_1)V, \dots, V^*\pi(B_g)V) \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{B\},$$

the map

$$V^*\pi(\tau(\cdot))V : \mathcal{S}_A \to B(\mathscr{K}), \quad X = V^*\pi(\tau(\cdot))V$$

is unital completely positive and hence $X \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}$ by the use of the Arveson's extension theorem and the Stinespring's dilation theorem. Analogously we conclude that every $X \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}$ belongs to $\mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{B\}$. This proves (2).

The inclusion $\mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}^\circ \subseteq D_{L_A}$ in (3) is clear, since in particular $A \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}$ and so $L_Y(A) \succeq 0$ for every $Y \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}^\circ$. By the canonical shuffle we get $L_A(Y) \succeq 0$ which means that $Y \in D_{L_A}$. It remain to prove the inclusion $D_{L_A} \subseteq \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}^\circ$. Let us take $X = (X_1, \dots, X_g) \in D_{L_A}(n)$ and

$$Y = (V^*\pi(A_1)V, \dots, V^*\pi(A_g)V) \in (\mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\})(K).$$

We have

$$\begin{aligned} L_X(Y) = I_n \otimes I_K &= \sum_{j=1}^{g} (X_j \otimes V^*\pi(A_j)V) \\ &= (I_n \otimes V)^*\Big((I_n \otimes \pi)(I_n \otimes I_K + \sum_{j=1}^{g} X_j \otimes A_j)\Big)(I_n \otimes V). \\ &= (I_n \otimes V)^*\Big((I_n \otimes \pi)(L_X(A))\Big)(I_n \otimes V) \succeq 0, \end{aligned}$$

where we used that $L_A(X) \succeq 0$ and by the canonical shuffle also $L_X(A) \succeq 0$. Since $X \in D_{L_A}$ and $Y \in \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}$ were arbitrary, this proves that $D_{L_A} \subseteq \mathrm{co}^{\mathrm{oper}}_{\mathscr{K}}\{A\}^\circ$.

(4), (5) and the inclusion ($\supseteq$) of (6) are special cases of (1), (2) and the inclusion ($\supseteq$) of (3), while for the inclusion ($\subseteq$) of (6) we notice that $V^*AV \in \mathrm{co}^{\mathrm{mat}}\{A\}$ for every isometry $V \in B(\mathbb{R}^m, \mathscr{H})$ where $m \in \mathbb{N}$, and hence by Lemma 3.2.2 for every $Y \in \mathrm{co}^{\mathrm{mat}}\{A\}^\circ$ we have $L_A(Y) \succeq 0$, i.e., $Y \in D_{L_A}$. □

Theorem 3.2.18. *Suppose* $L = I_{\mathscr{H}} + \sum_{j=1}^{g} \Omega_j x_j + \sum_{k=1}^{h} \Gamma_k y_k \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ *is a monic linear operator pencil and* $\mathcal{K} = \mathrm{proj}_x D_L^{\mathscr{H}}$ *its operator free Hilbert spectrahedrop. The operator free Hilbert polar dual* $\mathcal{K}^{\mathscr{K},\circ}$ *is the operator free set given by*

$$\begin{aligned} \mathcal{K}^{\mathscr{K},\circ} &= \Big\{(A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{K}} : (A_1, \dots, A_g, 0, \dots, 0) \in D_L^{\mathscr{K},\circ}\Big\} \\ &= \{(A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{K}} : \exists \text{ a separable real Hilbert space } \mathscr{G}, \text{ an isometry} \\ &\quad V : \mathscr{K} \to \mathscr{G} \text{ and } *\text{-homomorphism } \pi : C^*(\mathcal{S}_{A\oplus \mathbf{0}^g_{\mathbb{R}}}) \to B(\mathscr{G}) \text{ such that} \\ &\quad \text{for all } j,k: \ A_j = V^*\pi\Big(\begin{bmatrix} \Omega_j & 0 \\ 0 & 0 \end{bmatrix}\Big)V, \ 0 = V^*\pi\Big(\begin{bmatrix} \Gamma_k & 0 \\ 0 & 0 \end{bmatrix}\Big)V\Big\}. \end{aligned}$$

Moreover, if $D_{{}^hL} = D_{{}^h(L\oplus 1_{\mathbb{R}})}$, *then*

$$\begin{aligned} \mathcal{K}^{\mathscr{K},\circ} &= \Big\{(A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{K}} : (A_1, \dots, A_g, 0) \in D_L^{\mathscr{K},\circ}\Big\} \\ &= \{(A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{K}} : \exists \text{ a separable real Hilbert space } \mathscr{G}, \text{ an isometry} \\ &\quad V : \mathscr{K} \to \mathscr{G} \text{ and } *\text{-homomorphism } \pi : C^*(\mathcal{S}_A) \to B(\mathscr{G}) \text{ such that} \\ &\quad \text{for all } j,k: \ A_j = V^*\pi(\Omega_j)V, \ 0 = V^*\pi(\Gamma_k)V\}. \end{aligned}$$

Proof. By definition,

$$\begin{aligned} \mathcal{K}^{\mathscr{K},\circ} &= \{A \in \mathbb{S}^g_{\mathscr{K}} : L_A(X) \succeq 0 \ \forall X \in \mathcal{K}\} \\ &= \big\{A \in \mathbb{S}^g_{\mathscr{K}} : L_{(A,0)}(X,Y) \succeq 0 \ \forall (X,Y) \in D_L^{\mathscr{K}}\big\} \\ &= \Big\{A \in \mathbb{S}^g_{\mathscr{K}} : (A,0) \in D_L^{\mathscr{K},\circ}\Big\}. \end{aligned}$$

Now we use Theorem 3.2.13 and obtain the statements of the theorem. □

An important case of the equality $D_{^{\mathrm{h}}L} = D_{^{\mathrm{h}}(L\oplus I_{\mathbb{R}})}$ in the notation of Theorem 3.2.18 occurs if $(\mathrm{proj}_x D_L^{\mathscr{K}})(1)$ is bounded.

Proposition 3.2.19. *Assume the notation as in Theorem 3.2.18. If the set $\mathcal{K}(1)$ is bounded, then $D_{^{\mathrm{h}}L} = D_{^{\mathrm{h}}(L\oplus I_{\mathbb{R}})}$.*

Proof. Take $Z := (X_0, X_1, \dots, X_g, Y_1, \dots, Y_h) \in D_{^{\mathrm{h}}L}$. By Lemma 3.2.10 (2) we have to prove that $X_0 \succeq 0$. We argue by contradiction. Assume that $X_0 \not\succeq 0$. As in the proof of Proposition 3.2.11 we may assume that $n = 1$, i.e., $Z \in \mathbb{R}^{g+h+1}$ and $X_0 < 0$. For every $t > 0$ we also have $^{\mathrm{h}}L(tZ) \succeq 0$. Since $tX_0 < 0$, it follows that

$$^{\mathrm{h}}L(1, tX_1, \dots, tX_g, tY_1, \dots tY_h) \succ {}^{\mathrm{h}}L(tZ) \succeq 0.$$

Therefore

$$(tX_1, \dots, tX_g) \in \mathcal{K}(1) \quad \text{for every } t > 0.$$

If $(X_1, \dots, X_g) \neq 0^g$, this contradicts the boundedness of $\mathcal{K}(1)$. Else $(X_1, \dots, X_g) = 0^g$. But then

$$\sum_{k=1}^{h} Y_k \cdot \Gamma_k \succ -X_0 \cdot I_{\mathscr{H}} = |X_0| \cdot I_{\mathscr{H}},$$

and hence for every $(X_1, \dots, X_g) \in \mathbb{R}^g$ there exists $t > 0$ such that

$$I_{\mathscr{H}} + \sum_{j=1}^{g} X_j \cdot \Omega_j + \sum_{k=1}^{h} tY_k \cdot \Gamma_k \succeq 0.$$

This again contradicts the boundedness of $\mathcal{K}(1)$. □

3.3 Linear Gleichstellensatz

In this section we consider the equality of free Hilbert spectrahedra. Our main result, Theorem 3.3.1 below, extends Theorem 3.1.2 from monic linear matrix pencils with bounded free spectrahedra to monic linear operator pencils with coefficients which are compact operators and arbitrary free Hilbert spectrahedra (not necessarily bounded ones). Moreover, we show in Subsections 3.3.3 and 3.3.4 that Theorem 3.1.2 does not extend from linear matrix pencils to arbitrary linear operator pencils. More precisely, in Subsection 3.3.3 we present a linear operator pencil that does not have a whole subpencil which is σ-minimal, while in Subsection 3.3.4 we give two σ-minimal linear operator pencils with the same free Hilbert spectrahedron but are not unitarily equivalent. In Subsection 3.3.2 we present the characterizations and existence of minimal tuples of compact operators and σ-minimal monic pencils with compact operator coefficients.

3.3.1 Compact operator coefficients

The main result of this subsection is the following.

Theorem 3.3.1 (Linear Gleichstellensatz)**.** *Let $\mathscr{H}$ and $\mathscr{K}$ be real separable Hilbert spaces. Suppose $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ and $B := (B_1, \dots, B_g) \in \mathbb{S}^g_{\mathscr{K}}$ are tuples of compact self-adjoint operators such that monic linear operator pencils L_A and L_B are σ-minimal. Then $D_{L_A} = D_{L_B}$ if and only if A and B are unitarily equivalent.*

Theorem 3.3.1 was first proved for finite dimensional $\mathscr{H}$, $\mathscr{K}$ and bounded $D_{L_A} = D_{L_B}$ in [HKM13b]. By extending the approach from [HKM13b] in two different directions, Theorem 3.3.1 was simultaneously proved for finite dimensional $\mathscr{H}$, $\mathscr{K}$ and *arbitrary* $D_{L_A} = D_{L_B}$ in [Zal17], and for separable infinite dimensional Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ and free Hilbert spectrahedra satisfying $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_B}$ in [DDSS+]. The only remaining case of Theorem 3.3.1, i.e., $\mathscr{H}$ is infinite dimensional and $D_{{}^{\mathrm{h}}L_A} \neq D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}}$ or $\mathscr{K}$ is infinite dimensional and $D_{{}^{\mathrm{h}}L_B} \neq D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}}$ cannot occur by Lemma 3.3.17 below. Therefore the results from [HKM13b], [Zal17], [DDSS+] together cover Theorem 3.3.1. In this subsection we present a unified approach which proves Theorem 3.3.1.

The proof of Theorem 3.1.2 in [HKM13b] consists of the following three major results:

(1) Characterization of monic σ-minimal matrix pencils with bounded free spectrahedra via the properties of the unital C^*-algebra generated by the coefficients of the pencils (see [HKM13b, Proposition 3.17] where the assumption *truly linear pencil* should be replaced by *monic linear pencil*). The proof uses Arveson's noncommutative Choquet theory [Arv69, Arv08, Arv10].

(2) Classification of real finite dimensional C^*-algebras (see [HKM13b, Proposition 3.14]).

(3) Classification of real $*$-isomorphisms between real finite dimensional C^*-algebras (see [HKM13b, Proposition 3.15]).

To extend Theorem 3.1.2 from monic linear matrix pencils with *bounded* free spectrahedra to monic linear matrix pencils with *arbitrary* free spectrahedra, one needs to extend the characterization in (1) above to such pencils first. In fact one needs to extend only the implication ($\Rightarrow$) of [HKM13b, Proposition 3.17] which can be done by small adaptations in the proof (see [Zal17, Proposition 3.8]). Then Theorem 3.3.1 for matrix pencils follows by an extended analysis (see [Zal17, §3.1]). In [DDSS+], Theorem 3.3.1 was extended from monic linear matrix pencils with bounded free spectrahedra to monic linear operator pencils with compact operators as the coefficients and bounded free Hilbert spectrahedra, using the same approach as in [HKM13b]. They defined the notion of a *minimal* tuple of operators, characterized minimal tuples of compact operators via the properties of the corresponding unital C^*-algebra as in (1) above (see [DDSS+, Proposition 6.3]) and replaced the use of (2) and (3) above by the classifications of C^*-algebras of compact operators [Arv79, Theorem 1.4.5] and $*$-isomorphisms between elementary C^*-algebras [Arv79, Corollary 3], respectively.

In this subsection we first show, that for tuples of compact operators, minimality of a tuple $A \in \mathbb{S}^g_{\mathscr{H}}$ coincides with σ-minimality of a monic pencil ${}^{\mathrm{h}}L_A \in \mathbb{S}^g_{\mathscr{H}}\langle x\rangle$ which in case D_{L_A} is bounded further coincides with σ-minimality of a monic pencil

$L_A \in \mathbb{S}^g_{\mathscr{H}}\langle x\rangle$. Then we prove Theorem 3.3.1 for operator pencils with compact operators as the coefficients by an analogous analysis as in [Zal17, §3.1] for matrix pencils.

Matrix ranges and minimality

Let $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators on a separable real Hilbert space $\mathscr{H}$. Recall that $C^*(\mathcal{S}_A)$ stands for the unital C^*-algebra generated by A, i.e., the smallest unital C^*-algebra in $B(\mathscr{H})$ which contains the operators $A_1, \dots, A_g$. We will use *ucp* for unital completely positive and *uci* for unital completely isometric.

The *matrix range* [DDSS+, §2.2] of a tuple A is the free set

$$\mathcal{W}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(A),$$

where

$$\mathcal{W}_n(A) = \{(\tau(A_1), \dots, \tau(A_g)) \colon \tau : C^*(\mathcal{S}_A) \to M_n(\mathbb{R}) \text{ is a ucp map}\}.$$

Proposition 3.3.2. $\mathcal{W}(A) = \mathrm{co}^{\mathrm{mat}}\{A\}$.

Proof. The equality is an easy consequence of the Stinespring's dilation theorem. □

If $\mathbf{0}_n^g \in \mathcal{W}_n(A)$ for every $n \in \mathbb{N}$ where $\mathbf{0}_n$ denotes $n \times n$ matrix with zero entries, then we write $\mathbf{0} \in \mathcal{W}(A)$.

Proposition 3.3.3. $\mathbf{0} \in \mathcal{W}(A)$ *if and only if* $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}}$.

Proof. By definition, $\mathbf{0} \in \mathcal{W}(A)$ if and only if for every $n \in \mathbb{N}$ there is a ucp map $\tau_n : C^*(\mathcal{S}_A) \to M_n(\mathbb{R})$ mapping each A_j to $\mathbf{0}_n$. Since every ucp map from $\mathcal{S}_A$ to $M_n(\mathbb{R})$ extends to some ucp map from $C^*(\mathcal{S}_A)$ to $M_n(\mathbb{R})$, we have that $\mathbf{0} \in \mathcal{W}(A)$ if and only if for every $n \in \mathbb{N}$ the map $\tau_n : \mathcal{S}_A \to M_n(\mathbb{R})$ defined by $I_{\mathscr{H}} \mapsto I_n$, $A \mapsto \mathbf{0}_n^g$, is a well-defined ucp map. By Theorem 3.2.5 (2) this is equivalent to the inclusion $D_{{}^{\mathrm{h}}L_A} \subseteq D_{{}^{\mathrm{h}}L_{\mathbf{0}_n^g}}$. Cleary, this inclusion is equivalent to the fact that $X = (X_0, \dots, X_g) \in D_{{}^{\mathrm{h}}L_A}$ implies that $X_0 \succeq 0$. Finally, by Lemma 3.2.10 (2) this is equivalent to $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}}$. □

Let $H \subseteq \mathscr{H}$ be a closed subspace of $\mathscr{H}$ which is *invariant* under each A_j, i.e., $A_j H \subseteq H$. Since each A_j is self-adjoint, it also follows that $A_j H^\perp \subseteq H^\perp$, i.e., H is automatically reducing for each A_j. Hence, with respect to the decomposition $\mathscr{H} = H \oplus H^\perp$, A can be written as the direct sum

$$A = A|_H \oplus A|_{H^\perp}.$$

We say that $A|_H$ is a *subtuple* of A. If H is a proper closed subspace of $\mathscr{H}$, then $A|_H$ is a *proper subtuple* of A. If $\mathcal{W}(A) = \mathcal{W}(A|_H)$, then $A|_H$ is a *whole subtuple* of A. If L has no proper whole subtuple, then A is *minimal.*

Equalities of free Hilbert spectrahedra, matrix ranges and uci maps

The following proposition connects the equality of free Hilbert spectrahedra, the equality of matrix ranges and the existence of a unital complete isometry between given tuples. The equivalence (2) $\Leftrightarrow$ (3) is from [DDSS+].

Proposition 3.3.4. *Let $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ and $B := (B_1, \dots, B_g) \in \mathbb{S}^g_{\mathscr{K}}$ be tuples of self-adjoint operators. Then the following statements are equivalent:*

(1) $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_B}$.

(2) $\mathcal{W}(A) = \mathcal{W}(B)$.

(3) The unital linear map $\tau : S_A \to S_B$ sending A_j to B_j is well-defined and completely isometric.

Proof. Since a uci map is the same as a ucp map with a ucp inverse, the equivalence (1) $\Leftrightarrow$ (3) follows by Theorem 3.2.5 (2). To prove the implication (2) $\Leftarrow$ (3) notice that if $\phi(B) = C \in \mathcal{W}(B)$ for some ucp map ϕ, then $(\phi \circ \tau)(A) = C$ and hence $C \in \mathcal{W}(A)$. Thus, $\mathcal{W}(B) \subseteq \mathcal{W}(A)$. Replacing the roles of A and B and τ with τ^{-1} we obtain the other inclusion, i.e., $\mathcal{W}(A) \subseteq \mathcal{W}(B)$. Finally, we prove the implication (2) $\Rightarrow$ (3). Notice that for every finite dimensional projection P we have that $PBP \in \mathcal{W}(B)$. By (2) there exists a ucp map $\tau_P : S_A \to S_{PBP}$. Let $(P_n)_{n\in\mathbb{N}}$ be an increasing sequence of projections where P_n maps onto n dimensional subspace of $\mathscr{K}$. Then the sequence $(\tau_{P_n})_n$ of ucp maps weakly converges to a ucp map $\tau : S_A \to S_B$ sending A_j to B_j. □

Remark 3.3.5. The equality $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_B}$ clearly implies $D_{L_A} = D_{L_B}$. If $D_{L_A}(1)$ is bounded, then $D_{L_A} = D_{L_B}$ implies $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_B}$. Hence, for a bounded set $D_{L_A}(1)$ we can replace $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_B}$ by $D_{L_A} = D_{L_B}$.

Minimality of A and σ-minimality of L_A, ${}^{\mathrm{h}}L_A$ and ${}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}$

We defined two notions of minimality, i.e., minimality of a tuple of self-adjoint operators and σ-minimalitiy of a linear operator pencil. The connection between minimality of $A \in \mathbb{S}^g_{\mathscr{H}}$ and σ-minimality of ${}^{\mathrm{h}}L_A \in \mathbb{S}^g_{\mathscr{H}}\langle x\rangle$ is the following.

Proposition 3.3.6. *Let $A \in \mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators. A tuple A is minimal if and only if a linear operator pencil ${}^{\mathrm{h}}L_A$ is σ-minimal.*

Proof. The equivalence in the proposition easily follows from the equivalence (1) $\Leftrightarrow$ (2) of Proposition 3.3.4. □

The following proposition, which will be crucial in the proof of Theorem 3.3.1 in case $\mathbf{0} \notin \mathcal{W}(A)$ or $\mathbf{0} \notin \mathcal{W}(B)$, states what can be said about σ-minimalities of ${}^{\mathrm{h}}L_A$ and ${}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}$ if L_A is σ-minimal.

Proposition 3.3.7. *Let $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators. Suppose L_A is a σ-minimal pencil. Then:*

(1) The pencil ${}^{\mathrm{h}}L_A$ is σ-minimal.

(2) If $D_{{}^{\mathrm{h}}L_A} \neq D_{{}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}}$, then the pencil ${}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}$ is σ-minimal.

To prove Proposition 3.3.7 we will use the following two lemmas.

Lemma 3.3.8. *Let $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators. Suppose L_A is a σ-minimal pencil. Then $\bigcap_{i=1}^{g} \ker A_i = \{0\}$.*

Proof. Let us write $J = \bigcap_{i=1}^g \ker A_i$. We have $A = A|_{J^\perp} \oplus \mathbf{0}^g_J$ where $\mathbf{0}_J$ is the zero operator on J. Note that $D_{L_A} = D_{L_{A|_{J^\perp}}}$. Since L_A is σ-minimal, we have $J^\perp = \mathscr{H}$. Hence $J = \{0\}$ which concludes the proof of the lemma. □

Lemma 3.3.9. *Let $\mathscr{H}$ be a Hilbert space, $A \in \mathbb{S}_{\mathscr{H}}$ an operator, $H \leq \mathscr{H}$ a reducing subspace of A and P_H, $P_{H^\perp}$ the projections to H and $H^\perp$, respectively. Then $h \in \ker A$ if and only if $P_H h \in \ker(A)$ and $P_{H^\perp} h \in \ker(A)$.*

Proof. Notice that $Ah = A(P_H h + P_{H^\perp} h) = AP_H h + AP_{H^\perp} h$. Since H is a reducing subspace of A, we have $AP_H h \in H$ and $AP_{H^\perp} h \in H^\perp$. Hence $Ah = 0$ if and only if $AP_H h = 0$ and $AP_{H^\perp} h = 0$ which proves the lemma. □

Proof of Proposition 3.3.7. To prove (1) first note that

$$(X_1, \dots, X_g) \in D_{L_A}(n) \quad \Leftrightarrow \quad (I_n, X_1, \dots, X_g) \in D_{{}^{\mathrm{h}}L_A}(n).$$

Hence if there is a proper closed subspace $H \subset \mathscr{H}$ which is reducing for A and such that $D_{{}^{\mathrm{h}}L_A} = D_{{}^{\mathrm{h}}L_{A|_H}}$, then in particular $D_{L_A} = D_{L_{A|_H}}$. This is a contradiction with the σ-minimality of L_A.

Now we will prove (2). We assume that $D_{{}^{\mathrm{h}}L_A} \neq D_{{}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}}$. We will prove that If ${}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}$ σ-minimal by contradiction. Let us assume then there is a proper closed subspace $H_1 \subset \mathscr{H} \oplus \mathbb{R}$ which is reducing for $A \oplus \mathbf{0}^g_{\mathbb{R}}$ and such that

$$D_{{}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_{(A\oplus\mathbf{0}^g_{\mathbb{R}})|_{H_1}}}.$$

We have $A \oplus \mathbf{0}^g_{\mathbb{R}} = (A \oplus \mathbf{0}^g_{\mathbb{R}})|_{H_1} \oplus (A \oplus \mathbf{0}^g_{\mathbb{R}})|_{H_1^\perp}$. Since L_A is σ-minimal, it follows that $\bigcap_{i=1}^{g} \ker A_i = \{0\}$ by Lemma 3.3.8. Hence, $\bigcap_{i=1}^{g} \ker(A_i \oplus \mathbf{0}_{\mathbb{R}}) = \operatorname{span}\{0_{\mathscr{H}} \oplus 1_{\mathbb{R}}\}$, where $0_{\mathscr{H}}$ denotes the zero vector in $\mathscr{H}$. By Lemma 3.3.9, we have that $h \in \bigcap_{i=1}^{g} \ker(A_i \oplus \mathbf{0}_{\mathbb{R}})$ if and only if

$$P_{H_1} h \in \bigcap_{i=1}^{g} \ker(A_i \oplus \mathbf{0}_{\mathbb{R}}) \quad \text{and} \quad P_{H_1^\perp} h \in \bigcap_{i=1}^{g} \ker(A_i \oplus \mathbf{0}_{\mathbb{R}}).$$

Since $\dim\left(\bigcap_{i=1}^{g} \ker(A_i \oplus \mathbf{0}_{\mathbb{R}})\right) = 1$, it follows that $0_{\mathscr{H}} \oplus 1 \in H_1$ or $0_{\mathscr{H}} \oplus 1 \in H_1^\perp$. If $0_{\mathscr{H}} \oplus 1 \in H_1^\perp$, then $H_1 \subseteq \mathscr{H}$. But then $L_{(A\oplus\mathbf{0}^g_{\mathbb{R}})|_{H_1}} = L_{A|_{H_1}}$ and hence

$$D_{{}^{\mathrm{h}}L_A} \supsetneq D_{{}^{\mathrm{h}}L_{A\oplus\mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_{(A\oplus\mathbf{0}^g_{\mathbb{R}})|_{H_1}}} = D_{{}^{\mathrm{h}}L_{A_{H_1}}} \supsetneq D_{{}^{\mathrm{h}}L_A}$$

which is a contradiction. Hence $0_{\mathscr{H}} \oplus 1 \in H_1$ and therefore

$$\begin{aligned}(A \oplus \mathbf{0}_{\mathbb{R}}^g)|_{H_1} &= (A \oplus \mathbf{0}_{\mathbb{R}}^g)|_{H_1 \cap \operatorname{span}\{0_{\mathscr{H}} \oplus 1\}^\perp} \oplus \mathbf{0}^g_{\operatorname{span}\{0_{\mathscr{H}} \oplus 1\}} \\ &= A|_{H_1 \cap \operatorname{span}\{0_{\mathscr{H}} \oplus 1\}^\perp} \oplus \mathbf{0}^g_{\operatorname{span}\{0_{\mathscr{H}} \oplus 1\}}.\end{aligned}$$

Therefore,

$$D_{{}^{\mathrm{h}}L_{A \oplus \mathbf{0}_{\mathbb{R}}^g}} = D_{{}^{\mathrm{h}}L_{(A \oplus \mathbf{0}_{\mathbb{R}}^g)|_{H_1}}} = D_{{}^{\mathrm{h}}L_{A|_{H_1 \cap \operatorname{span}\{0_{\mathscr{H}} \oplus 1\}^\perp}}} \bigcap D_{{}^{\mathrm{h}}L_{\mathbf{0}^g_{\operatorname{span}\{0_{\mathscr{H}} \oplus 1\}}}}$$

In particular, $D_{L_A} = D_{L_{A \oplus \mathbf{0}_{\mathbb{R}}^g}} = D_{L_{A|_{H_1 \cap \operatorname{span}\{0_{\mathscr{H}} \oplus 1\}^\perp}}}$. Since L_A is σ-minimal, it follows that $H_1 \cap \operatorname{span}\{0_{\mathscr{H}} \oplus 1\}^\perp = \mathscr{H}$. But then $H_1 = \mathscr{H} \oplus \mathbb{R}$ which is a contradiction. Hence ${}^{\mathrm{h}}L_{A \oplus \mathbf{0}_{\mathbb{R}}^g}$ is σ-minimal. □

Corollary 3.3.10. *Let $A \in \mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators. Suppose L_A is a σ-minimal pencil. Then:*

(1) A tuple A is minimal.

(2) If $\mathbf{0} \notin \mathcal{W}(A)$, then a tuple $A \oplus \mathbf{0}_{\mathbb{R}}^g$ is minimal.

Proof. Corollary follows by Propositions 3.3.3, 3.3.6 and 3.3.7. □

By Corollary 3.3.10, if L_A is σ-minimal, then A is minimal. By example below, the converse is not true. Therefore σ-minimality of L_A is a stronger requirement than minimality of A.

Example 3.3.11. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_2$. The matrix A is minimal, but the pencil L_A is not σ-minimal.

Proof. Clearly we have that

$$D_{L_A} = \bigcup_{n \in \mathbb{N}} \{X \in \mathbb{S}_n \colon X \succeq -I_n\},$$

and

$$D_{{}^{\mathrm{h}}L_A} = \bigcup_{n \in \mathbb{N}} \big\{(X_0, X_1) \in \mathbb{S}_n^2 \colon X_1 \succeq -X_0 \text{ and } X_0 \succeq 0\big\}.$$

Hence $1 + x$ is a σ-minimal whole subpencil of L_A and L_A is not σ-minimal. Since A has two one-dimensional eigenspaces, the only proper subpencil of ${}^{\mathrm{h}}L_A$ are $\ell_1 := x_0 + x_1$ with

$$D_{\ell_1} = \bigcup_{n \in \mathbb{N}} \big\{(X_0, X_1) \in \mathbb{S}_n^2 \colon X_1 \succeq -X_0\big\},$$

and $\ell_2 := x_0$ with

$$D_{\ell_2} = \bigcup_{n \in \mathbb{N}} \big\{(X_0, X_1) \in \mathbb{S}_n^2 \colon X_0 \succeq 0\big\}.$$

Since $D_{{}^{\mathrm{h}}L_A} \subsetneq D_{\ell_i}$ for $i = 1, 2$, ${}^{\mathrm{h}}L_A$ is σ-minimal. By Proposition 3.3.6, A is minimal.

Minimal tuples of compact operators with the same matrix range

Tuples $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ and $B := (B_1, \dots, B_g) \in \mathbb{S}^g_{\mathscr{K}}$ are *unitarily equivalent* if there exists a unitary operator $U : \mathscr{H} \to \mathscr{K}$ such that $A_j = U^* B_j U$ for each j. We write $A = U^* BU$. By the following theorem minimal tuples of compact operators are unitarily equivalent.

Theorem 3.3.12. *Let $\mathscr{K}$, $\mathscr{H}$ be real separable Hilbert spaces and $A \in \mathbb{S}^g_{\mathscr{H}}$, $B \in \mathbb{S}^g_{\mathscr{K}}$ minimal tuples of self-adjoint compact operators. Then $\mathcal{W}(A) = \mathcal{W}(B)$ if and only if A and B are unitarily equivalent.*

In this generality Theorem 3.3.12 appears in [DDSS+, Theorem 6.5]. The idea for the proof is the same as for the proof of [HKM13b, Theorem 1.2] which is for finite dimensional $\mathscr{H}$, $\mathscr{K}$. Namely, one characterizes minimal tuples via the properties of the (unital) C^*-algebra spanned by the coefficients of the tuple (see Theorem 3.3.14 below) and then uses classical results about $*$-homomorphism between two such C^*-algebras.

Now we are ready to prove Theorem 3.3.1. The proof does not follow directly by Corollary 3.3.10 (1) and Theorem 3.3.12, since for σ-minimal pencils L_A and L_B with $D_{L_A} = D_{L_B}$ we only know that A and B are minimal tuples but we do not know (yet) that $\mathcal{W}(A) = \mathcal{W}(B)$ to be able to use Theorem 3.3.12 directly. Therefore the proof requires some case analysis in which we essentially use Corollary 3.3.10 (2). Hence Corollary 3.3.10 (2) can be seen as the crucial observation in the proof of Theorem 3.3.1 for unbounded free spectrahedra.

Proof of Theorem 3.3.1

By Corollary 3.3.10 (1), A and B are minimal tuples. If $\mathcal{W}(A) = \mathcal{W}(B)$, then A and B are unitarily equivalent by Theorem 3.3.12 and we are done. Assume that $\mathcal{W}(A) \neq \mathcal{W}(B)$. We will prove that this leads to a contradiction. By the equivalence (1) $\Leftrightarrow$ (2) of Proposition 3.3.4 we have that $D_{{}^{\mathrm{h}}L_A} \neq D_{{}^{\mathrm{h}}L_B}$. Note that $D_{L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{L_A} = D_{L_B} = D_{L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}}$ and by Theorem 3.2.5 (7) it follows that $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}}$. Therefore at least one of $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} \subsetneq D_{{}^{\mathrm{h}}L_A}$ and $D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}} \subsetneq D_{{}^{\mathrm{h}}L_A}$ is true. We separate three cases:

Case 1: $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} \subsetneq D_{{}^{\mathrm{h}}L_A}$ and $D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_B}$.

By Proposition 3.3.7 (2), ${}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}$ is a σ-minimal pencil or equivalently by Proposition 3.3.6, $A \oplus \mathbf{0}^g_{\mathbb{R}}$ is a minimal tuple. Since $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_B}$, we have that $\mathcal{W}(A \oplus \mathbf{0}^g_{\mathbb{R}}) = \mathcal{W}(B)$ by the equivalence (1) $\Leftrightarrow$ (2) of Proposition 3.3.4. By Theorem 3.3.12, $A \oplus \mathbf{0}^g_{\mathbb{R}}$ and B are unitarily equivalent. Since $\operatorname{span}\{0_{\mathscr{H}} \oplus 1\} = \bigcap_{i=1}^{g} \ker(A_i \oplus \mathbf{0}^g_{\mathbb{R}})$, it follows that $\bigcap_{i=1}^g \ker(B_i) \neq \{0\}$ which contradicts to σ-minimality of L_B by Lemma 3.3.8.

Case 2: $D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}} \subsetneq D_{{}^{\mathrm{h}}L_B}$ and $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_A}$.

This case is analogous to the case 1 where we change the roles of A and B.

Case 3: $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} \subsetneq D_{{}^{\mathrm{h}}L_A}$ and $D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}} \subsetneq D_{{}^{\mathrm{h}}L_B}$.

By Proposition 3.3.7 (2), ${}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}$ and ${}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}$ are σ-minimal pencils or equivalently by Proposition 3.3.6, $A \oplus \mathbf{0}^g_{\mathbb{R}}$ and $B \oplus \mathbf{0}^g_{\mathbb{R}}$ are minimal tuples. Since $D_{{}^{\mathrm{h}}L_{A\oplus \mathbf{0}^g_{\mathbb{R}}}} = D_{{}^{\mathrm{h}}L_{B\oplus \mathbf{0}^g_{\mathbb{R}}}}$, we have that $\mathcal{W}(A \oplus \mathbf{0}^g_{\mathbb{R}}) = \mathcal{W}(B \oplus \mathbf{0}^g_{\mathbb{R}})$ by the equivalence (1) $\Leftrightarrow$ (2) of Proposition 3.3.4. By Theorem 3.3.12, $A \oplus \mathbf{0}^g_{\mathbb{R}}$ and $B \oplus \mathbf{0}^g_{\mathbb{R}}$ are unitarily equivalent. Thus there is a unitary operator $\widetilde{U} : \mathscr{H} \oplus \mathbb{R} \to \mathscr{K} \oplus \mathbb{R}$ such that

$$A \oplus \mathbf{0}^g_{\mathbb{R}} = \widetilde{U}^*(B \oplus \mathbf{0}^g_{\mathbb{R}})\widetilde{U}, \quad \text{and whence} \quad B \oplus \mathbf{0}^g_{\mathbb{R}} = \widetilde{U}(A \oplus \mathbf{0}^g_{\mathbb{R}})\widetilde{U}^*$$

Write $\widetilde{U}$ in the form

$$\widetilde{U} = \begin{bmatrix} U_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix},$$

where $U_{11} : \mathscr{H} \to \mathscr{K}$, $u_{12} : \mathbb{R} \to \mathscr{K}$, $u_{21} : \mathscr{H} \to \mathbb{R}$ and $u_{22} : \mathbb{R} \to \mathbb{R}$ are bounded linear operators. Therefore $\widetilde{U}^*$ is of the form

$$\widetilde{U}^* = \begin{bmatrix} U_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{bmatrix},$$

where $U_{11}^* : \mathscr{K} \to \mathscr{K}$, $u_{12}^* : \mathscr{K} \to \mathbb{R}$, $u_{21}^* : \mathbb{R} \to \mathscr{H}$ and $u_{22}^* : \mathbb{R} \to \mathbb{R}$ are the adjoints of U_{11}, u_{12}, u_{21} and u_{22}, respectively.

For every $j = 1, \dots, g$ we have

$$0 = (A_j \oplus \mathbf{0}_{\mathbb{R}})(0_{\mathscr{H}} \oplus 1) = U^*(B_j(u_{12}1) \oplus 0).$$

Since U^* is unitary, it follows that $B_j(u_{12}1) = 0$. Therefore $u_{12}1 \in \bigcap_{j=1}^{g} \ker B_j$. Since L_B is σ-minimal, we must have $\bigcap_{j=1}^{g} \ker B_j = \{0\}$ by Lemma 3.3.8. Hence $u_{12}1 = 0$ which means that $u_{12} = 0$. Analogously by changing the roles of A and B we argue that $u_{21}^* = 0$ and hence $u_{21} = 0$. But then $\widetilde{U} = U_{11} \oplus u_{22}$ and

$$\begin{aligned} I_{\mathscr{H}} \oplus I_{\mathbb{R}} &= \widetilde{U}^*\widetilde{U} = (U_{11}^*U_{11}) \oplus (u_{22}^*u_{22}), \\ I_{\mathscr{K}} \oplus I_{\mathbb{R}} &= \widetilde{U}\widetilde{U}^* = (U_{11}U_{11}^*) \oplus (u_{22}u_{22}^*). \end{aligned}$$

In particular, $U_{11}^*U_{11} = I_{\mathscr{H}}$ and $U_{11}U_{11}^* = I_{\mathscr{K}}$. Since we also have $B = U_{11}AU_{11}^*$ and $A = U_{11}^*BU_{11}$, it follows that A and B are unitarily equivalent which implies that $\mathcal{W}(A) = \mathcal{W}(B)$. But this contradicts the assumption $\mathcal{W}(A) \neq \mathcal{W}(B)$ at the beginning of the proof of Theorem 3.3.1. Hence even Case 3 does not occur. This concludes the proof of Theorem 3.3.1. □

Corollary 3.3.13. *Suppose $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ and $B := (B_1, \dots, B_g) \in \mathbb{S}^g_{\mathscr{K}}$ are tuples of compact self-adjoint operators such that monic linear operator pencils L_A and L_B are σ-minimal. Then the following statements are equivalent:*

(1) $D_{L_A} = D_{L_B}$.

(2) $D_{{}^{h}L_A} = D_{{}^{h}L_B}$.

(3) $\mathcal{W}(A) = \mathcal{W}(B)$.

Proof. By Corollary 3.3.10 (2), A and B are minimal tuples. Now the equivalence (1) $\Leftrightarrow$ (3) follows by Theorems 3.3.1 and 3.3.12, while the equivalence (2) $\Leftrightarrow$ (3) by the equivalence (1) $\Leftrightarrow$ (2) of Proposition 3.3.4. $\square$

3.3.2 Minimality

The main results of this subsection are the characterizations and existence of minimal tuples of compact operators (see Proposition 3.3.14 and Theorem 3.3.15) and σ-minimal monic pencils with compact operator coefficients (see Corollaries 3.3.20 and 3.3.16).

Characterization and existence of minimal subtuples of compact operators

Let $A := (A_1, \ldots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators on a real separable Hilbert space $\mathscr{H}$ and $\mathcal{S}_A$ the operator system generated by A. *The Šilov ideal* for the operator system $\mathcal{S}_A$ is defined as the biggest two sided ideal of $C^*(\mathcal{S}_A)$ such that the natural map

$$C^*(\mathcal{S}_A) \to C^*(\mathcal{S}_A)/K, \quad a \mapsto a + K$$

is completely isometric on $\mathcal{S}_A$. (Note that the existence and uniqueness of the Šilov ideal is nontrivial.)

Let $C^*(A)$ be the smallest C^*-algebra which contains $A_1, \ldots, A_g$. We say a $*$-homomorphism $\pi : C^*(A) \to B(\mathscr{K})$ is *nondegenerate*, if its *kernel*

$$\ker(\pi(C^*(A))) = \{k \in \mathscr{K} : \pi(X)k = 0 \text{ for all } X \in C^*(A)\}$$

is trivial. If $\mathscr{K}_0$ is a subspace of $\mathscr{K}$, invariant under all $\pi(X)$, $X \in C^*(A)$, then $\pi_0(X) := \pi(X)|_{\mathscr{K}_0}$ is a *subrepresentation* of $C^*(A)$. Let $\omega : C^*(A) \to B(\mathscr{G})$ be another $*$-homomorphism. We say π and ω are *equivalent* if there is a unitary operator $U : \mathscr{G} \to \mathscr{K}$ such that $\omega(X) = U^*\pi(X)U$ for all $X \in C^*(A)$. If $\mathscr{K} = \mathscr{G}$ and $\{\pi(X)\mathscr{K} : X \in C^*(A)\}$ is orthogonal to $\{\omega(X)\mathscr{K} : X \in C^*(A)\}$, then π and ω are *orthogonal*. We say a $*$-homomorphism π is *multiplicity free* if π does not have two non-zero orthogonal equivalent subrepresentations.

Proposition 3.3.14 below characterizes σ-minimal pencils of the form ${}^{\mathrm{h}}L_A$ or equivalently minimal tuples of compact operators. It was first proved for matrix pencils of the form L_A with bounded free spectrahedra [HKM13b, Proposition 3.17]. The implication ($\Rightarrow$) of [HKM13b, Proposition 3.17] easily extends to matrix pencils L_A with arbitrary spectrahedra [Zal17, Proposition 3.8]. On replacing matrix pencil L_A with ${}^{\mathrm{h}}L_A$ one in fact obtains the equivalence. The version for tuples of compact operators below is [DDSS+, Proposition 6.3].

Proposition 3.3.14. *Let $\mathscr{H}$ be a real separable Hilbert space and $A := (A_1, \ldots, A_g)$ a tuple of compact self-adjoint operators $A_j \in \mathbb{S}_{\mathscr{H}}$. The following statements are equivalent:*

(1) *A pencil* ${}^{\mathrm{h}}L_A$ *is* σ*-minimal.*

(2) *A tuple* A *is minimal.*

(3) *The Šilov ideal of* $\mathcal{S}_A$ *inside* $C^*(\mathcal{S}_A)$ *is trivial and the identity representation of* $C^*(A)$ *is multiplicity free.*

Minimal whole subtuple of a tuple of compact self-adjoint operators exists by the following.

Theorem 3.3.15 (Davidson, Dor-On, Shalit, Solel)**.** *Let* $\mathscr{H}$ *be a real separable Hilbert space and* $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ *a tuple of compact self-adjoint operators. There exists a minimal whole subtuple of* A*.*

Existence and characterization of σ-minimal subpencils of monic pencils with compact operator coefficients

As a consequence of Theorem 3.3.15, a σ-minimal whole subpencil of a pencil L_A with compact operator coefficients also exists and can be easily obtained in case $\mathbf{0} \in \mathcal{W}(A)$ and A is minimal.

Corollary 3.3.16. *Let* $\mathscr{H}$ *be a real separable Hilbert space and* $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ *a tuple of compact self-adjoint operators. Then a pencil* L_A *has a* σ*-minimal whole subpencil. Moreover, if* $\mathbf{0} \in \mathcal{W}(A)$ *and* A *is minimal, then one of the following statements is true:*

(1) $\dim\Big(\bigcap_{i=1}^{g} \ker(A_i)\Big) = 1$ *and* $L_{A|_{\left(\bigcap_{i=1}^{g}\ker(A_i)\right)^{\perp}}}$ *is a* σ*-minimal whole subpencil of* L_A*.*

(2) $\bigcap_{i=1}^{g} \ker(A_i) = \{0\}$ *and* L_A *is a* σ*-minimal pencil.*

In the proof of Corollary 3.3.16 we will use the following lemma.

Lemma 3.3.17. *Let* $\mathscr{H}$ *be a real separable infinite dimensional Hilbert space and* $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ *a tuple of compact self-adjoint operators. Then* $\mathbf{0} \in \mathcal{W}(A)$*.*

Proof. Assume on the contrary that $\mathbf{0} \notin \mathcal{W}(A)$. It follows that there is a tuple $(X_0, \dots, X_g) \in \mathbb{S}^{g+1}_n$ such that

$$ {}^{\mathrm{h}}L_A(X_0, \dots, X_g) \succeq 0 \quad \text{and} \quad X_0 \not\succeq 0. $$

Hence there is $u \in \mathbb{R}^n$ such that $\langle X_0 u, u\rangle = -1$ and

$$ {}^{\mathrm{h}}L_A\big(-1, \langle X_1 u, u\rangle \dots, \langle X_g u, u\rangle\big) \succeq 0. $$

But then

$$ \sum_{j=1}^{g} A_j \langle X_j u, u\rangle \succeq I_{\mathscr{H}}, $$

which is a contradiction, since the operator $\sum_{j=1}^{g} A_j\langle X_j u, u\rangle$ is compact. This concludes the proof of the lemma. $\square$

Now we are ready to prove Corollary 3.3.16.

Proof of Corollary 3.3.16. If $\mathscr{H}$ is finite dimensional, then the statement of the proposition is clear. If $\mathscr{H}$ is infinite dimensional, we have $\mathbf{0} \in \mathcal{W}(A)$ by Lemma 3.3.17. To prove the second part of Corollary 3.3.16 simultaneously, we do not assume that $\mathscr{H}$ is infinite dimensional. By Theorem 3.3.15 there is a minimal whole subtuple $\widetilde{A}$ of A. Therefore $\mathcal{W}(\widetilde{A}) = \mathcal{W}(A)$ or equivalently by Proposition 3.3.4, $D_{{}^{\mathrm{h}}L_{\widetilde{A}}} = D_{{}^{\mathrm{h}}L_A}$. In particular, $D_{L_{\widetilde{A}}} = D_{L_A}$. Therefore we may assume that A is already minimal. We separate two cases.

Case 1: L_A is not σ-minimal.

Since L_A is not σ-minimal, there exists a proper reducing subspace $H \leq \mathscr{H}$ of A such that $D_{L_{A|_H}} = D_{L_A}$. We also have $D_{{}^{\mathrm{h}}L_{A|_H}} \neq D_{{}^{\mathrm{h}}L_A}$ since otherwise it follows by Proposition 3.3.4 that $\mathcal{W}(A|_H) = \mathcal{W}(A)$ which contradicts to minimality of A.

Claim. $\bigcap_{i=1}^{g} \ker((A_i)|_H) = \{0\}$.

Proof of Claim. We denote $J := \bigcap_{i=1}^{g} \ker((A_i)|_H)$. Then $A|_H$ is unitarily equivalent to $A|_{H\cap J^\perp} \oplus \mathbf{0}_J$ where $\mathbf{0}_J$ denotes the zero operator on J. If $J \neq 0$, then $D_{{}^{\mathrm{h}}L_{A|_H}} = D_{{}^{\mathrm{h}}L_A}$, which is a contradiction. This proves Claim.

Now notice that $D_{L_{A|_H \oplus \mathbf{0}_\mathbb{R}}} = D_{L_{A|_H}} = D_{L_A}$ and since $\mathbf{0} \in \mathcal{W}(A|_H \oplus \mathbf{0}_\mathbb{R})$ and $\mathbf{0} \in \mathcal{W}(A)$, it follows by Propositions 3.3.2, 3.3.3 and 3.2.17 that

$$\mathcal{W}(A|_H \oplus \mathbf{0}_\mathbb{R}) = D^\circ_{L_{A|_H \oplus \mathbf{0}_\mathbb{R}}} = D^\circ_{L_A} = \mathcal{W}(A).$$

By Theorem 3.3.15 there is a subspace $H_1 \subseteq H \oplus \mathbb{R}$ such that $(A|_H \oplus \mathbf{0}_\mathbb{R})|_{H_1} = (A \oplus \mathbf{0}_\mathbb{R})|_{H_1}$ is a minimal whole subtuple of $A|_H \oplus \mathbf{0}_\mathbb{R}$. In particular,

$$\mathcal{W}((A \oplus \mathbf{0}_\mathbb{R})|_{H_1}) = \mathcal{W}(A),$$

and

$$A|_H \oplus \mathbf{0}_\mathbb{R} = (A \oplus \mathbf{0}_\mathbb{R})|_{H_1} \oplus (A \oplus \mathbf{0}_\mathbb{R})|_{H_1^\perp}.$$

By Claim 2 we conclude that

$$\bigcap_{i=1}^{g} \ker((A_i)|_H \oplus \mathbf{0}_\mathbb{R}) = \operatorname{span}\{0_\mathscr{H} \oplus 1\},$$

where $0_\mathscr{H} \oplus 1 \in H \oplus \mathbb{R}$. By Lemma 3.3.9, we have that $h \in \bigcap_{i=1}^{g} \ker((A_i)|_H \oplus \mathbf{0}_\mathbb{R})$ if and only if

$$P_{H_1} h \in \bigcap_{i=1}^{g} \ker((A_i)|_H \oplus \mathbf{0}_\mathbb{R}) \quad \text{and} \quad P_{H_1^\perp} h \in \bigcap_{i=1}^{g} \ker((A_i)|_H \oplus \mathbf{0}_\mathbb{R}).$$

Since $\dim\left(\bigcap_{i=1}^{g}\ker((A_i)|_H \oplus \mathbf{0}_{\mathbb{R}})\right) = 1$, it follows that $0_{\mathscr{H}} \oplus 1 \in H_1$ or $0_{\mathscr{H}} \oplus 1 \in H_1^{\perp}$. If $0_{\mathscr{H}} \oplus 1 \in H_1^{\perp}$, then $H_1 \leq H$ and hence $A|_{H_1} = (A \oplus \mathbf{0}_{\mathbb{R}})|_{H_1}$ is a minimal proper whole subtuple of A which contradicts to minimality of A. Hence we must have $0_{\mathscr{H}} \oplus 1 \in H_1$. Thus

$$(A \oplus \mathbf{0}_{\mathbb{R}})|_{H_1} = \mathbf{0}_{\mathrm{span}\{0_{\mathscr{H}} \oplus 1\}} \oplus A|_{H_1 \cap H},$$

where $\bigcap_{i=1}^{g}\ker((A_i)|_{H_1 \cap H}) = \{0\}$. By Theorem 3.3.12, A and $(A \oplus \mathbf{0}_{\mathbb{R}})|_{H_1}$ are unitarily equivalent. This implies that $\dim\left(\bigcap_{i=1}^{g}\ker(A_i)\right) = 1$ and $\mathbf{0}_{\mathrm{span}\{0_{\mathscr{H}} \oplus 1\}} \oplus A|_{H_1 \cap H}$ is unitarily equivalent to $\mathbf{0}_J \oplus A|_{H_1 \cap H}$ where $J = \bigcap_{i=1}^{g}\ker(A_i)$. Since

$$A = A|_{H_1 \cap H} \oplus A_{(H_1 \cap H)^{\perp}}$$

and $\bigcap_{i=1}^{g}\ker((A_i)|_{H_1 \cap H}) = \{0\}$, we conclude by Lemma 3.3.9 that $J \subseteq (H_1 \cap H)^{\perp}$. Therefore $\mathbf{0}_J \oplus A|_{H_1 \cap H}$ is a subtuple of A with $\mathcal{W}(\mathbf{0}_J \oplus A|_{H_1 \cap H}) = \mathcal{W}(A)$. It follows by minimiality of A that $H_1 \cap H = J^{\perp}$. Further on, $H_1 = H = J^{\perp}$ and $L_{A|_H}$ is σ-minimal. This is option (1) of Corollary 3.3.16.

Case 2: L_A is σ-minimal.

By Lemma 3.3.8 it follows that $\bigcap_{i=1}^{g}\ker(A_i) = \{0\}$ which is option (2) of Corollary 3.3.16. □

Remark 3.3.18. Note that Example 3.3.11 is an example for Corollary 3.3.16 (1), while

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{S}_2$$

is an example for Corollary 3.3.16 (2).

Let A be a minimal tuple of compact operators. If $\mathbf{0} \in \mathcal{W}(A)$, then by Corollary 3.3.16 a σ-minimal whole subpencil of L_A is its restriction to the orthoghonal complement of $\cap_{i=1}^{g}\ker(A_i)$ which is zero or one dimensional. On the other hand, if $\mathbf{0} \notin \mathcal{W}(A)$, then A must be a tuple of matrices and by the following example, there can be bigger difference in the size of matrices defining a whole minimal subtuple of a tuple A and those defining a whole σ-minimal subpencil of L_A.

Example 3.3.19. Let

$$A_1 = \mathrm{diag}\left(2, 0, \frac{\sqrt{2}}{2\sqrt{2}-1}, \sqrt{2}-1\right) \in \mathbb{S}_4,$$

$$A_2 = \mathrm{diag}\left(0, 2, \frac{2-\sqrt{2}}{2\sqrt{2}-1}, 1\right) \in \mathbb{S}_4.$$

be diagonal matrices. Then:

(1) A tuple $A := (A_1, A_2) \in \mathbb{S}_4^2$ is minimal.

(2) A σ-minimal whole subpencil of L_A is $L_{\widetilde{A}}$ where

$$\widetilde{A} = \left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right) \in \mathbb{S}_2^2.$$

Proof. Let e_i, $i = 1, 2, 3, 4$, be the standard coordinate vectors in $\mathbb{R}^4$, i.e., the only nonzero entry of e_i is the i-th one which equals to 1.

Claim. If $H \subseteq \mathbb{R}^4$ is a reducing subspace for A, then $H = \operatorname{span}\{e_{i_j} : i_j \in \{1, 2, 3, 4\}\}$.

Proof of Claim. If H is reducing for A, then in particular it is reducing for A_1 and hence also for

$$A_1 - 2I_4, \quad A_1 - \frac{\sqrt{2}}{2\sqrt{2}-1} I_4, \quad A_1 - (\sqrt{2}-1)I_4. \tag{3.3.1}$$

By Lemma 3.3.9 used for A_1 and the matrices (3.3.1), $e_i \in H$ or $e_i \in H^\perp$ for every i. This proves Claim.

Minimality of A follows by noticing that there is no proper reducing subspace H for A such that $D_{^{\mathrm{h}}L_{A|_H}}(1) = D_{^{\mathrm{h}}L_A}(1)$. This can be seen by observing that $D_{^{\mathrm{h}}L_A}(1) \cap \{-1\} \times \mathbb{R}^2$ equals to the intersection of halfspaces

$$-1 + 2x_1 \geq 0, \quad -1 + 2x_2 \geq 0,$$

$$-1 + \frac{\sqrt{2}}{2\sqrt{2}-1} x_1 + \frac{2-\sqrt{2}}{2\sqrt{2}-1} x_2 \geq 0, \quad -1 + (\sqrt{2}-1)x_1 + x_2 \geq 0.$$

The intersection has three extreme points and therefore cannot be equal to the intersection of at most three of those halfspaces. (Otherwise it would have at most two extreme points.)

It remains to prove that $L_{\widetilde{A}}$ is a whole σ-minimal subpencil of L_A. The inclusion $D_{L_A} \subseteq D_{L_{\widetilde{A}}}$ is clear. Let us prove the other inclusion, i.e., $D_{L_{\widetilde{A}}} \subseteq D_{L_A}$. Take $(X_1, X_2) \in D_{L_{\widetilde{A}}}(n)$. By definition, $X_i \succeq -\frac{1}{2} I_n$ for $i = 1, 2$ and thus

$$\frac{\sqrt{2}}{2\sqrt{2}-1} X_1 + \frac{2-\sqrt{2}}{2\sqrt{2}-1} X_2 \succeq -\frac{\sqrt{2}}{2(2\sqrt{2}-1)} I_n - \frac{2-\sqrt{2}}{2(2\sqrt{2}-1)} I_n = -\frac{1}{2\sqrt{2}-1} I_n \succeq -I_n,$$

$$(\sqrt{2}-1)X_1 + X_2 \succeq -\frac{\sqrt{2}-1}{2} I_n - \frac{1}{2} I_n = -\frac{\sqrt{2}}{2} I_n \succeq -I_n,$$

which implies that $(X_1, X_2) \in D_{L_A}(n)$. Therefore $D_{L_{\widetilde{A}}} \subseteq D_{L_A}$ and $D_{L_{\widetilde{A}}} = D_{L_A}$. Since $L_{\widetilde{A}}$ is σ-minimal, the proof is complete.

Finally we come to characterization of σ-minimal monic pencils with compact operator coefficients.

Corollary 3.3.20. *Let $\mathscr{H}$ be a real separable Hilbert space and $A := (A_1, \dots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$ a tuple of compact self-adjoint operators. Then a pencil L_A is a σ-minimal if and only if $\bigcap_{i=1}^{g} \ker(A_i) = \{0\}$, A is a minimal tuple and one of the following statements is true:*

(1) $\mathbf{0} \in \mathcal{W}(A)$.

(2) $\mathbf{0} \notin \mathcal{W}(A)$ *and* $A \oplus \mathbf{0}^g_{\mathbb{R}}$ *is a minimal tuple.*

Proof. First let us prove the implication ($\Rightarrow$). By Lemma 3.3.8, $\bigcap_{i=1}^{g} \ker(A_i) = \{0\}$. By Corollary 3.3.10 (1), A is a minimal tuple. Now we either have $\mathbf{0} \in \mathcal{W}(A)$ or $\mathbf{0} \notin \mathcal{W}(A)$ and $A \oplus \mathbf{0}^g_{\mathbb{R}}$ is a minimal tuple by Corollary 3.3.10 (2). Let us now prove the implication ($\Leftarrow$). We assume that $\bigcap_{i=1}^{g} \ker(A_i) = \{0\}$ and A is a minimal tuple. If in addition (1) is true, i.e., $\mathbf{0} \in \mathcal{W}(A)$, then L_A is σ-minimal by Corollary 3.3.16 (2). Else if in addition (2) is true, i.e., $\mathbf{0} \notin \mathcal{W}(A)$ and $A \oplus \mathbf{0}^g_{\mathbb{R}}$ is a minimal tuple, then L_A is σ-minimal by Corollary 3.3.16 (1) used for the tuple $A \oplus \mathbf{0}^g_{\mathbb{R}}$. □

3.3.3 Obstructions for minimality

Example 3.3.21 below shows that in contrast with tuples of compact operators, a general tuple $A \in \mathbb{S}^g_{\mathscr{H}}$ does not necessarily have a minimal whole subtuple and a general linear operator pencil L_A does not necessarily have a σ-minimal whole subpencil.

Example 3.3.21. Let $A = \operatorname{diag}\left(\frac{n}{n+1}\right)_{n \in \mathbb{N}} \in B(\ell^2(\mathbb{N}))$ be a diagonal linear operator on $\ell^2(\mathbb{N})$. Then:

(1) The free Hilbert spectrahedron of L_A is

$$D_{L_A} = \bigcup_{m \in \mathbb{N}} \{X \in \mathbb{S}_m \colon X \succeq -I_m\}.$$

(2) There does not exist a σ-minimal subpencil of L_A.

(3) The free Hilbert spectrahedron of ${}^{\mathrm{h}}L_A$ is

$$D_{{}^{\mathrm{h}}L_A} = \bigcup_{m \in \mathbb{N}} \left\{(X_0, X_1) \in \mathbb{S}^2_m \colon X_1 \succeq -X_0 \text{ and } X_1 \succeq -2X_0\right\}.$$

(4) There does not exist a σ-minimal subpencil of ${}^{\mathrm{h}}L_A$ or equivalently a minimal subtuple of A.

(5) The C^*-algebra generated by $\mathcal{S}_A$ is

$$C^*(\mathcal{S}_A) = \left\{\operatorname{diag}(\lambda_n)_{n \in \mathbb{N}} \in B(\ell^2(\mathbb{N})) \colon \lim_{n \to \infty} \lambda_n \text{ exists}\right\}.$$

(6) Šilov ideal K of $\mathcal{S}_A$ in $C^*(\mathcal{S}_A)$ is

$$K = \Big\{ \operatorname{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))\colon \lambda_1 = 0 \text{ and } \lim_{n\to\infty} \lambda_n = 0 \Big\}.$$

(7) The C^*-algebra $C_e^*(\mathcal{S}_A) := C^*(\mathcal{S}_A)/K$ of $\mathcal{S}_A$ is isomorphic to $\mathbb{R}^2$ under the isomorphism

$$\phi((\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)) = \big(\lambda_1, \lim_{n\to\infty} \lambda_n\big).$$

(8) The pencil L_A maps under ϕ to

$$\phi(L_A) = \begin{pmatrix} 1 + \frac{1}{2}x & 0 \\ 0 & 1 + x \end{pmatrix},$$

The pencil ${}^{\mathrm{h}}\phi(L_A)$ is σ-minimal, but $\phi(L_A)$ is not σ-minimal.

Proof: Let $X \in \mathbb{S}_m$. Then

$$L_A(X) = \operatorname{diag}\Big(I_m + \frac{n}{n+1}X\Big)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N})) \otimes M_m.$$

Note that

$$L_A(X) \succeq 0 \quad \Leftrightarrow \quad X \succeq -\frac{n+1}{n} I_m \quad \text{for every } n \in \mathbb{N} \quad \Leftrightarrow \quad X \succeq -I_m.$$

This proves (1).

Assume H is a reducing subspace for A. In particular, it is also reducing for the operators $A - \frac{n}{n+1} I_{\ell^2}$, $n \in \mathbb{N}$. Let $\{e_i\}_{i\in\mathbb{N}}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$, i.e., the only nonzero entry of e_i is the i-th one which is 1. We have $\ker(A - \frac{n}{n+1} I_{\ell^2}) = \operatorname{span}\{e_n\}$. By Lemma 3.3.9, $e_n \in H$ or $e_n \in H^\perp$. This implies that H has an orthonormal basis $\{e_{i_j}\colon i_j \in \mathbb{N}, j \in \mathbb{N}\}$. The subpencil $L_{A|_H}$ satisfies $D_{L_{A|_H}} = D_{L_A}$ if and only if the sequence $(i_j)_j$ diverges. But then $(i_j)_j$ has a diverging subsequence which means that $L_{A|_H}$ has a proper subpencil $\widetilde{L}$ satisfying $D_{\widetilde{L}} = D_{L_A}$. This proves (2).

Let $(X_0, X_1) \in \mathbb{S}_m^2$. Then

$${}^{\mathrm{h}}L_A(X_0, X_1) = \operatorname{diag}\Big(X_0 + \frac{n}{n+1}X\Big)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N})) \otimes M_m.$$

Note that

$${}^{\mathrm{h}}L_A(X_0, X_1) \succeq 0 \quad \Leftrightarrow \quad X_1 \succeq -\frac{n+1}{n} X_0 \quad \text{for every } n \in \mathbb{N}. \tag{3.3.2}$$

For $n = 1$ we get $X_1 \succeq -2X_0$. Sending n to ∞ we get $X_1 \succeq -X_0$. This two inequalities imply that all the other inequalities are true, since for every $\lambda \in [0, 1]$ we have that

$$X_1 = \lambda X_1 + (1 - \lambda)X_1 \succeq (-2\lambda - (1 - \lambda))X_0 = (-1 - \lambda)X_0.$$

This proves (3).

As was already established in the proof of (2), a reducing subspaces H for A has an orthonormal basis $\{e_{i_j}\colon i_j \in \mathbb{N}, j \in \mathbb{N}\}$. Now the subpencil ${}^{\mathrm{h}}L_{A|_H}$ satisfies $D_{{}^{\mathrm{h}}L_{A|_H}} = D_{{}^{\mathrm{h}}L_A}$ if and only if $e_1 \in H$ and the sequence $(i_j)_j$ diverges. But then $(i_j)_j$ has a diverging subsequence which means that ${}^{\mathrm{h}}L_{A|_H}$ has a proper subpencil ${}^{\mathrm{h}}\widetilde{L}$ satisfying $D_{{}^{\mathrm{h}}\widetilde{L}} = D_{{}^{\mathrm{h}}L_A}$. This proves (4).

The C^*-algebra $C^*(\mathcal{S}_A)$ is equal to the norm closure of the set $\{p(A)\colon p \in \mathbb{R}[x]\}$. For $p(x) := \sum_{i=0}^{m} a_i x^i$, the operator $p(A) = a_0 I_{\ell^2(\mathbb{N})} + \sum_{i=1}^{m} a_i A^i$ is diagonal and $\lim_{n\to\infty} p(A)_n = \sum_{i=0}^{m} a_i$, where $p(A)_n$ stands for the n-th diagonal entry of p. Hence

$$C^*(\mathcal{S}_A) \subseteq \Big\{\mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))\colon \lim_{n\to\infty} \lambda_n \text{ exists}\Big\}.$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be the standard orthonormal basis for $\ell^2(\mathbb{N})$. Let $P_k \in B(\ell^2(\mathbb{N}))$ be the projection onto the span of e_k.

Claim 1. $P_k \in C^*(\mathcal{S}_A)$.

We will prove Claim 1 by induction on k. Notice that

$$(2I_{\ell^2(\mathbb{N})} - 2A)^j = \mathrm{diag}\Big(\big(\frac{2}{n+1}\big)^j\Big)_{n\in\mathbb{N}} \in C^*(\mathcal{S}_A)$$

Therefore $\lim_{j\to\infty}(2I_{\ell^2(\mathbb{N})} - 2A)^j = P_1 \in C^*(\mathcal{S}_A)$. Now let $k \in \mathbb{N}$ and assume that $P_1, \ldots, P_{k-1} \in C^*(\mathcal{S}_A)$. Notice that

$$((k+1)I_{\ell^2(\mathbb{N})} - (k+1)A) = \mathrm{diag}\Big(\frac{k+1}{n+1}\Big)_{n\in\mathbb{N}} \in C^*(\mathcal{S}_A).$$

Since $P_1, \ldots, P_k \in C^*(\mathcal{S}_A)$, it follows that

$$B_k := \mathrm{diag}\Big(\underbrace{0, \ldots, 0}_{k-1}, 1, \frac{k+1}{k+2}, \frac{k+1}{k+3}, \ldots\Big) \in C^*(\mathcal{S}_A).$$

Therefore $\lim_{j\to\infty} B_k^j = P_k \in C^*(\mathcal{S}_A)$. This proves Claim 1.

Let $D := \mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))$ and $\lim_{n\to\infty}\lambda_n$ exists. We write $\lambda := \lim_{n\to\infty}\lambda_n$. Then $D_k := \lambda I_{\ell^2(\mathbb{N})} + \sum_{i=1}^{k}(\lambda_i - \lambda)P_i \in C^*(\mathcal{S}_A)$ and $\lim_{k\to\infty} D_k = D$. Hence $D \in C^*(\mathcal{S}_A)$ and

$$\Big\{\mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))\colon \lim_{n\to\infty}\lambda_n \text{ exists}\Big\} \subseteq C^*(\mathcal{S}_A).$$

This proves (5).

Now we will prove (6). First we will show the following claim.

Claim 2. If $D := \mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in K$, then $\lambda_1 = 0$.

Let
$$\rho : C^*(\mathcal{S}_A) \to C^*(\mathcal{S}_A)/K$$
be the natural projection. Note that $(X_0, X_1) \in D_{{}^{\mathrm{h}}L_{\rho(A)}}(m)$ if and only if
$$(I_{\ell^2(\mathbb{N})} + K) \otimes X_0 + (A + K) \otimes X_1 = B^*B$$
for some $B \in C^*(\mathcal{S}_A)/K \otimes M_m(\mathbb{R})$. In particular, if for $k \in K$ we have
$$I_{\ell^2(\mathbb{N})} \otimes X_0 + (A + k) \otimes X_1 \succeq 0,$$
then $(X_0, X_1) \in D_{{}^{\mathrm{h}}L_{\phi(A)}}(m)$. Let there exist $D = \mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in K$ such that $\lambda_1 \neq 0$. We may assume that $\lambda_1 > 0$. By (3.3.2), $(-I_m, X_1) \in D_{{}^{\mathrm{h}}L_A}(m)$ if and only if $X_1 \succeq 2I_m$. Let $t \in \mathbb{R}_{>0}$. Then
$$-I_{\ell^2(\mathbb{N})} \otimes I_m + (A + t\lambda) \otimes X_1 \succeq 0$$
if and only if
$$X_1 \succeq \Big(\frac{1}{\frac{n}{n+1} + t\lambda_n}\Big) \cdot I_m \quad \text{for every } n \in \mathbb{N}. \tag{3.3.3}$$
We have
$$\lim_{t\searrow 0} \mathrm{diag}\Big(\frac{1}{\frac{n}{n+1} + t\lambda_n}\Big) = \mathrm{diag}\Big(\frac{n+1}{n}\Big)_{n\in\mathbb{N}} \quad \text{and} \quad \frac{1}{\frac{1}{2} + t\lambda_1} < 2 \text{ for } t \in \mathbb{R}_{>0}. \tag{3.3.4}$$
Let $E_{11} \in M_m$ be the standard coordinate matrix. For $\epsilon > 0$ small enough we conclude from (3.3.3) and (3.3.4) that
$$(-I_m, 2I_m - \epsilon E_{11}) \in D_{{}^{\mathrm{h}}L_{\rho(A)}}.$$
But then $D_{{}^{\mathrm{h}}L_{\rho(A)}} \neq D_{{}^{\mathrm{h}}L_A}$, which is a contradiction with ρ being completely isometric. Therefore if $D \in K$, then $\lambda_1 = 0$ which proves Claim 2.

Claim 3. $\Big\{\mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))\colon \lambda_1 = 0 \text{ and } \lim_{n\to\infty} \lambda_n = 0\Big\} \subseteq K.$

By the proof of (3) notice that
$$\begin{aligned} {}^{\mathrm{h}}L_A(X_0, X_1) &= \mathrm{diag}\Big(X_0 + \frac{n}{n+1}X_1\Big)_{n\in\mathbb{N}} \succeq 0 \\ &\Leftrightarrow \mathrm{diag}\Big(X_0 + \frac{n}{n+1}X_1\Big)_{n\in\mathbb{N}\setminus\{2,\dots,k\}} \succeq 0 \quad \text{for } k \geq 2. \end{aligned}$$
Therefore all the projection P_k, $k \geq 2$, belong to K. Now let us take $D := \mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))$ such that $\lambda_1 = 0$ and $\lim_{n\to\infty} \lambda_n = 0$. For $k \geq 2$ we define
$$C_k := \sum_{i=2}^{k} \lambda_i P_i \in K.$$
Since K is closed and $\lim_{k\to\infty} C_k = D$, it follows that $D \in K$. This proves Claim 3.

To conclude the proof of (6) it remains to establish:

Claim 4. $K \subseteq \Big\{\mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))\colon \lambda_1 = 0 \text{ and } \lim_{n\to\infty} \lambda_n = 0\Big\}.$

Let us assume on the contrary that there is $D_1 := \mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in K$ such that $\lambda_1 = 0$ and $\lim_{n\to\infty} \lambda_n = \lambda \neq 0$. Let us take $D_2 := \mathrm{diag}(\widetilde{\lambda}_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))$ such that $\widetilde{\lambda}_1 = 0$ and $\lim_{n\to\infty} \widetilde{\lambda}_n = \widetilde{\lambda} \in \mathbb{R}$. Then

$$D_2 = \Big(D_2 - \frac{\widetilde{\lambda}}{\lambda} D_1\Big) + \frac{\widetilde{\lambda}}{\lambda} D_1 \in K + K = K,$$

where $D_2 - \frac{\widetilde{\lambda}}{\lambda} D_1 \in K$ since $\lim_{n\to\infty} \Big(D_2 - \dfrac{\widetilde{\lambda}}{\lambda} D_1\Big)_n = 0$. We conclude that

$$K = \Big\{\mathrm{diag}(\lambda_n)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N}))\colon \lambda_1 = 0 \text{ and } \lim_{n\to\infty} \lambda_n \text{ exists}\Big\},$$

and hence ${}^{\mathrm{h}}D_{L_A} = {}^{\mathrm{h}}D_{L_{\rho(A)}}$, where

$$L_{\rho(A)} = \mathrm{diag}(1 + x, 0, \ldots) + K.$$

But then

$$D_{{}^{\mathrm{h}}L_A} = \bigcup_{m\in\mathbb{N}} \{(X_0, X_1) \in \mathbb{S}_m \colon X_1 \succeq -X_0\},$$

which is a contradiction. This proves Claim 4.

(7) easily follows form (5) and (6). Minimality of ${}^{\mathrm{h}}\phi(L_A)$ is clear. The pencil $\phi(L_A)$ is not minimal, since

$$D_{\phi(L_A)} = \bigcup_{m\in\mathbb{N}} \{X \in \mathbb{S}_m \colon X \succeq -I_m\}$$

and hence $\phi(L_A)$ has a whole subpencil $\phi(L_A)|_{\mathrm{span}\{e_1\}} = 1 + x$. This proves (8) and concludes the proof of the example.

3.3.4 Counterexample

The main result of this subsection is Example 3.3.22 below which shows that σ-minimal *operator* pencils with the same free Hilbert spectrahedron are not necessarily unitarily equivalent. Hence Theorem 3.3.1 does not extend from matrix to operator pencils.

Example 3.3.22 is constructed by the use of an outer $*$-automorphism [Arc79] of the Cuntz C^*-algebra $C^*(S_1, S_2)$ [Cun77] generated by the isometries $S_1, S_2 \in B(\mathscr{H})$ on a Hilbert space $\mathscr{H}$ such that $S_1 S_1^* + S_2 S_2^* = \mathrm{Id}_{\mathscr{H}}$. Recall that a $*$-automorphism θ is *outer* if there does not exist a unitary $U \in C^*(S_1, S_2)$ such that $\theta(A) = U^* A U$ for all $A \in C^*(S_1, S_2)$.

Example 3.3.22. Let $\mathbb{N} = \{1, 2, \ldots\}$ and let e_i be a standard unit vector on a complex Hilbert space $\ell^2 := \ell^2(\mathbb{N})$, i.e., the only nonzero coordinate is the i-th one which is 1. Let S_1 and S_2 be bounded operators on ℓ^2 defined by $e_i \mapsto e_{2i-1}$ for $i \in \mathbb{N}$ and $e_i \mapsto e_{2i}$ for $i \in \mathbb{N}$ respectively. The C^*-algebra $C^*(S_1, S_2)$ was studied by Cuntz [Cun77]. He showed that there is a unique $*$-automorphism

$$\theta : C^*(S_1, S_2) \to C^*(S_1, S_2)$$

such that

$$\theta(S_1) = S_2, \quad \theta(S_2) = S_1.$$

We claim that linear operator pencils

$$L_1(x) = I_{\ell^2} + A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4,$$
$$L_2(x) = I_{\ell^2} + A_2x_1 + A_1x_2 + A_4x_3 + A_3x_4,$$

where

$$A_1 := S_1 + S_1^* \in B(\ell^2), \quad A_2 := S_2 + S_2^* \in B(\ell^2),$$
$$A_3 := i(S_1 - S_1^*) \in B(\ell^2), \quad A_4 := i(S_2 - S_2^*) \in B(\ell^2),$$

are σ-minimal pencils with $D_{L_1} = D_{L_2}$, but there is no unitary operator $U : \ell^2 \to \ell^2$ such that

$$L_2 = U^*L_1U \quad \text{or} \quad L_2 = U^*\overline{L_1}U. \tag{3.3.5}$$

Claim 1. $D_{L_1} = D_{L_2}$.

Clearly, the C^*-algebra

$$\mathcal{A} := C^*(A_1, A_2, A_3, A_4)$$

generated by A_j, $j = 1, 2, 3, 4$, equals to $C^*(S_1, S_2)$. Hence θ maps L_1 to L_2 and L_2 to L_1. From $\theta(L_1) = L_2$, it follows that $D_{L_1} \subseteq D_{L_2}$ and similarly $\theta(L_2) = L_1$ implies $D_{L_2} \subseteq D_{L_1}$. Thus $D_{L_1} = D_{L_2}$.

Claim 2. L_1 and L_2 are σ-minimal.

It is sufficient to prove that there is no common reducing subspace for the operators A_1, A_2, A_3, A_4. Let us say that H is their common reducing subspace. Then it is also reducing for the operators

$$\frac{A_1 - iA_3}{2} = S_1, \quad \text{and} \quad \frac{A_2 - iA_4}{2} = S_2.$$

By the proof of [Arc79, Theorem 1], S_1 and S_2 have no common proper reducing subspaces. Hence L_1 and L_2 are σ-minimal.

Claim 3. There does not exist a unitary operator $U : \ell^2 \to \ell^2$ satisfying (3.3.5).

If there exists a unitary operator $U : \ell^2 \to \ell^2$ satisfying (3.3.5), then in particular

$$A_4 = U^*A_3U \quad \text{or} \quad A_4 = U^*\overline{A_3}U. \tag{3.3.6}$$

We will prove that $\ker A_3 = \ker \overline{A}_3 \neq \{0\}$ and $\ker A_4 = \{0\}$ which contradicts to (3.3.6). Note that S_1^* and S_2^* are bounded operators on ℓ^2 defined by

$$e_{2i-1} \mapsto e_i,\ e_{2i} \mapsto 0 \quad \text{for } i \in \mathbb{N} \quad \text{and} \quad e_{2i-1} \mapsto 0,\ e_{2i} \mapsto e_i \quad \text{for } i \in \mathbb{N},$$

respectively. Hence,

$$A_3 e_1 = i(S_1 - S_1^*)e_1 = 0 = -i(S_1 - S_1^*)e_1 = \overline{A_3} e_1 \quad \Rightarrow \quad e_1 \in \ker A_3 = \ker(\overline{A_3}).$$

It remains to prove that $\ker A_4 = \{0\}$. Let us say

$$f := \sum_{j=1}^{\infty} \alpha_j e_j \in \ker A_4 \quad \text{where } \alpha_j \in \mathbb{C} \text{ for all } j \in \mathbb{N}.$$

We define $e_{\frac{2k-1}{2}} = 0$ for every $k \in \mathbb{N}$. We have

$$A_4 f = i \sum_{j=1}^{\infty} \alpha_j e_{2j} - i \sum_{j=1}^{\infty} \alpha_j e_{\frac{j}{2}} = 0. \tag{3.3.7}$$

If $\alpha_{j_0} \neq 0$ for some $j_0 \in \mathbb{N}$, then it follows from (3.3.7) inductively that

$$\alpha_{j_0} = \alpha_{4j_0} = \alpha_{16j_0} = \ldots = \alpha_{4^n j_0} = \ldots.$$

But then $\|f\| = \infty$ and hence $f \notin \ell^2$. Therefore $f = 0$ and $\ker A_4 = \{0\}$. □

3.4 Convex Positivstellensatz

The main result of this section, Theorem 3.4.1, is the characterization of multivariate matrix polynomials that are positive semidefinite on a free Hilbert spectrahedron which extends Theorem 3.1.5 from free spectrahedra to free Hilbert spectrahedra. Theorem 3.1.5 was proved in [HKM12] by modifying the classical Putinar-type separation argument. By essentially using Theorem 3.2.13 and a version of the Hahn-Banach theorem [HKM16b, Theorem 2.2] we are able to apply the separation argument from [HKM12] also in the case of free Hilbert spectrahedra. In Subsection 3.4.5 we extend Theorem 3.4.1 from free Hilbert spectrahedra to free Hilbert spectrahedrops (see Theorem 3.4.9).

Theorem 3.4.1 (Convex Positivstellensatz). *Let $L = I_{\mathscr{H}} + \sum_{j=1}^{g} A_j x_j \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ be a monic linear operator pencil. Then for every symmetric matrix-valued noncommutative polynomial $F \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ with $F|_{D_L} \succeq 0$, there is a separable real Hilbert space $\mathscr{K}$, a $*$-homomorphism $\pi : C^*(\mathcal{S}_A) \to B(\mathscr{K})$, finitely many matrix polynomials $R_j \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ and operator polynomials $Q_k \in B(\mathbb{R}^\nu, \mathscr{K}) \otimes \mathbb{R}\langle x\rangle$ all of degree at most $\frac{1}{2} \cdot \deg(F)$, $j_0, k_0 \in \mathbb{N}$, such that*

$$F = \sum_{j=1}^{j_0} R_j^* R_j + \sum_{k=1}^{k_0} Q_k^* \pi(L) Q_k.$$

Subsection 3.4.1-3.4.4 are devoted to the proof of Theorem 3.4.1.

3.4.1 Restatement of the theorem

To prove Theorem 3.4.1 we have to refine its statement. For this purpose we introduce some definitions.

For $P \in \mathbb{R}^{\ell\times\nu}\langle x\rangle$, an element of the form $P^*P \in \mathbb{R}^{\ell\times\nu}\langle x\rangle$ is caled a *hermitian square*. Let Σ^ν denote the cone of sums of squares of $\nu\times\nu$ matrix nc polynomials, and, given a nonnegative integer N, let $\Sigma^\nu_N \subseteq \Sigma^\nu$ denote sums of squares of polynomials of degree at most N. Thus elements of Σ^ν_N have degree at most $2N$, i.e., $\Sigma^\nu_N \subseteq \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2N}$. Since the highest order terms in a sum of squares cannot cancel, we have $\mathbb{R}^{\nu\times\nu}\langle x\rangle_{2N}\cap\Sigma^\nu = \Sigma^\nu_N$.

Fix $\nu\in\mathbb{N}$ and a separable real Hilbert space $\mathscr{H}$. Let $A\in\mathbb{S}^g_{\mathscr{H}}$ be a tuple of self-adjoint operators on $\mathscr{H}$ and $\widetilde{\Pi}_\nu$ be the set of all triples $(\mathscr{K},\pi,V)$ where $\mathscr{K}$ is a separable real Hilbert space, $V:\mathbb{R}^\nu\to\mathscr{K}$ an isometry and $\pi: C^*(\mathcal{S}_{A\oplus\mathbf{0}^g_{\mathbb{R}}})\to B(\mathscr{K})$ a $*$-homomorphism.

Let $L\in\mathbb{S}_{\mathscr{H}}\langle x\rangle$ be a monic linear operator pencil. Given $\nu_1,\nu_2,\alpha,\beta\in\mathbb{N}$, we define the $(\nu_1,\nu_2;\alpha,\beta)$ *truncated quadratic module generated by* L by

$$M^{\nu_1,\nu_2}_{\alpha,\beta}(L) := \Sigma^{\nu_1}_\alpha + \left\{\sum^{\text{finite}}_{(\mathscr{K}_k,\pi_k,V_k)\in\widetilde{\Pi}_{\nu_2}} B_k^*V_k^*\pi_k\left(\begin{bmatrix} L & 0\\ 0 & 1\end{bmatrix}\right)V_kB_k\colon B_k\in\mathbb{R}^{\nu_2\times\nu_1}\langle x\rangle_\beta\right\}.$$

In the case $D_{\mathrm{h}L} = D_{\mathrm{h}(L\oplus I_{\mathbb{R}})}$, we can replace $*$-homomorphisms from $C^*(\mathcal{S}_{A\oplus\mathbf{0}^g_{\mathbb{R}}})$ by $*$-homomorphisms from $C^*(\mathcal{S}_A)$ in the definition of the truncated quadratic module. Let Π_ν be the set of all triples $(\mathscr{K},\pi,V)$ where $\mathscr{K}$ is a separable real Hilbert space, $V:\mathbb{R}^\nu\to\mathscr{K}$ an isometry and $\pi: C^*(\mathcal{S}_A)\to B(\mathscr{K})$ a $*$-homomorphism.

Proposition 3.4.2. *If $D_{\mathrm{h}L} = D_{\mathrm{h}(L\oplus I_{\mathbb{R}})}$, then:*

$$M^{\nu_1,\nu_2}_{\alpha,\beta}(L) = \Sigma^{\nu_1}_\alpha + \left\{\sum^{\text{finite}}_{(\mathscr{K}_k,\pi_k,V_k)\in\Pi_{\nu_2}} B_k^*V_k^*\pi_k(L)V_kB_k\colon B_k\in\mathbb{R}^{\nu_2\times\nu_1}\langle x\rangle_\beta\right\}.$$

Proof. It is sufficient to prove that for every isometry $V\in B(\mathbb{R}^{\nu_2},\mathscr{K})$ and every $*$-homomorphism $\pi: C^*(\mathcal{S}_{A\oplus\mathbf{0}^g_{\mathbb{R}}})\to B(\mathscr{K})$ there exist an isometry $\widetilde{V}\in B(\mathbb{R}^{\nu_2},\widetilde{\mathscr{K}})$ and a $*$-homomorphism $\widetilde{\pi}: C^*(\mathcal{S}_A)\to B(\widetilde{\mathscr{K}})$ such that

$$V^*\pi\left(\begin{bmatrix} L & 0\\ 0 & 1\end{bmatrix}\right)V = \widetilde{V}^*\widetilde{\pi}(L)\widetilde{V}.$$

Since $V^*\pi\left(\begin{bmatrix} L & 0\\ 0 & 1\end{bmatrix}\right)V$ is a monic linear pencil positive semidefinite on D_L and $D_{\mathrm{h}L} = D_{\mathrm{h}(L\oplus I_{\mathbb{R}})}$, this is true by Theorem 3.2.13 (3). □

The following is the restatement of Theorem 3.4.1.

Theorem 3.4.3. *Let $L\in\mathbb{S}_{\mathscr{H}}\langle x\rangle$ be a monic linear operator pencil and $F\in\mathbb{R}^{\nu\times\nu}\langle x\rangle$ a symmetric matrix polynomial of degree at most $2d+1$. If $F|_{D_L}\succeq 0$, then*

$$F\in M^{\nu,\ell}_{d+1,d}(L),$$

where $\ell := \nu\cdot\sigma_\#(d)$ and $\sigma_\#(d) := \dim(\mathbb{R}\langle x\rangle_d)$.

The proof of Theorem 3.4.3 is given in Subsection 3.4.4. In the next two subsections we prove the connection between positive linear functionals and operators and show that the truncated quadratic module is closed. Both results are important ingredients for the separation argument in the proof of Theorem 3.4.3.

3.4.2 GNS construction

Proposition 3.4.4 below (see [HKM12, Proposition 2.5]), embodies the well known connection, through the Gelfand-Naimark-Segal (GNS) construction, between operators and positive linear functionals. The only difference between the statements of Proposition 3.4.4 and [HKM12, Proposition 2.5] is that the pencil L is operator-valued here but was matrix-valued in [HKM12, Proposition 2.5]. Therefore, the proof of Proposition 3.4.4 needs an additional argument. Namely, in the notation of Proposition 3.4.4 the fact that a tuple of operators X belongs to D_L if λ is nonnegative on $M_{k+1,k}^{\nu,\nu\sigma_{\#}(k)}(L)$ follows immediately by construction if L is matrix-valued but needs a proof if L is operator-valued.

Proposition 3.4.4. *If $\lambda : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2k+2} \to \mathbb{R}$ is a linear functional which is nonnegative on Σ_{k+1}^{ν} and positive on $\Sigma_k^{\nu} \setminus \{0\}$, then there exists a tuple $X = (X_1, \dots, X_g)$ of self-adjoint operators on a Hilbert space $\mathcal{X}$ of dimension at most $\nu\sigma_{\#}(k) = \nu \dim \mathbb{R}\langle x\rangle_k$ and a vector $\gamma \in \mathcal{X}^{\oplus\nu}$, such that*

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle_{\mathcal{X}^{\oplus\nu}} \tag{3.4.1}$$

for all $f \in \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2k+1}$, where $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ is the inner product on $\mathcal{X}$. Further, if $L \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ is a monic linear operator pencil and λ is nonnegative on $M_{k+1,k}^{\nu,\nu\sigma_{\#}(k)}(L)$, then $X \in D_L$.

Conversely, given $X = (X_1, \dots, X_g)$ is a tuple of self-adjoint operators on a Hilbert space $\mathcal{X}$ of dimension N, the vector $\gamma \in \mathcal{X}^{\oplus\nu}$, and k a positive integer, then the linear functional $\lambda : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2k+2} \to \mathbb{R}$, defined by

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle_{\mathcal{X}^{\oplus\nu}}$$

is nonnegative on Σ_{k+1}^{ν}. Further, if $L \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ is a monic linear operator pencil and $X \in D_L$, then λ is nonnegative also on $M_{k+1,k}^{\nu,\ell}(L)$ for every $\ell \in \mathbb{N}$.

If L is matrix-valued, then Proposition 3.4.4 becomes [HKM12, Proposition 2.5]. In the case L is operator-valued, then the main difficulty in the extension of the implication ($\Rightarrow$) of the proposition is to show that the constructed tuple X, which represents λ, belongs to D_L if λ is nonnegative on $M_{k+1,k}^{\nu,\nu\sigma_{\#}(k)}(L)$. In the matrix case this follows by a simple calculation, while in the operator case we will need a special case (see [HKM16a, Theorem 3.1] and [HM12, §6]) of the Effros and Winkler [EW97] version of the Hahn-Banach theorem, see Theorem 3.4.5 below, and Theorem 3.2.13.

Theorem 3.4.5. *If $\Gamma = (\Gamma(n))_{n\in\mathbb{N}} \subseteq \mathbb{S}^g$ is a closed matrix convex set containing 0 and $X \in \mathbb{S}_m^g$ is not in $\Gamma(m)$, then there is a monic linear pencil $\mathcal{L}$ if size m such that $\mathcal{L}(Y) \succeq 0$ for all $Y \in \Gamma$, but $\mathcal{L}(X) \not\succeq 0$.*

Proof of Proposition 3.4.4. The nontrivial direction is ($\Rightarrow$). A symmetric bilinear form defined on the vector space $K = \mathbb{R}^{1\times\nu}\langle x\rangle_{k+1}$ by

$$\langle f, h\rangle := \lambda(h^* f), \tag{3.4.2}$$

is positive semidefinite by hypothesis and hence induces a positive definite bilinear form on the quotient $\widetilde{\mathcal{X}} := K/\mathcal{N}$ where $\mathcal{N} := \{f \in K : \langle f, f\rangle = 0\}$, making it a Hilbert space. By positive definiteness of the form (3.4.2) on the subspace $\mathcal{X} = \mathbb{R}^{1\times\nu}\langle x\rangle_k$, $\mathcal{X}$ can be considered as a subspace of $\widetilde{\mathcal{X}}$ with dimension $\nu\sigma_\#(k)$. We define the operators $X_j : \mathcal{X} \to \mathcal{X}$ by

$$X_j f = P x_j f, \quad f \in \mathcal{X},\ 1 \le j \le g,$$

where P is the orthogonal projection from $\widetilde{\mathcal{X}}$ onto $\mathcal{X}$. Since the bilinear form (3.4.2) is positive definite on $\mathcal{X}$, the operators X_j are well-defined. By the calculation

$$\langle X_j f, h\rangle = \langle x_j f, h\rangle = \langle f, x_j h\rangle = \langle f, X_j h\rangle$$

for every $f, h \in \mathcal{X}$, it follows that the operators X_j are self-adjoint.

For $j = 1, \ldots, \nu$ let $\gamma_j \in \mathcal{X}$ be a vector with the j-th entry the only nonzero one which is the monomial 1. Let γ be the vector $\gamma := \bigoplus_{j=1}^{\nu} \gamma_j \in \mathcal{X}^{\oplus\nu}$. Let $e_1, \ldots, e_\nu$ be the standard coordinate vectors for $\mathbb{R}^\nu$, i.e., the only nonzero entry of e_j is the j-th entry which is 1. Take a matrix polynomial

$$f = \sum_{s,t=1}^{\nu} w_{s,t}^* v_{s,t} e_s e_t^* \in \mathbb{R}^{\nu\times\nu}\langle x\rangle \tag{3.4.3}$$

where $v_{s,t} \in \langle x\rangle_{k+1}$, $w_{s,t} \in \langle x\rangle_k$ for every s, t. For $X := (X_1, \ldots, X_g) \in \mathbb{S}^g_{\mathcal{X}}$ the following calculation holds:

$$\begin{aligned}
\langle f(X)\gamma, \gamma\rangle_{\mathcal{X}^{\oplus\nu}} &= \sum_{s,t=1}^{\nu} \big\langle w_{s,t}^*(X) v_{s,t}(X)\gamma_t, \gamma_s\big\rangle_{\mathcal{X}} = \sum_{s,t=1}^{\nu} \langle v_{s,t}(X)\gamma_t, w_{s,t}(X)\gamma_s\rangle_{\mathcal{X}} \\
&= \sum_{s,t=1}^{\nu} \langle P(v_{s,t}e_t^*), w_{s,t}e_s^*\rangle_{\mathcal{X}} = \sum_{s,t=1}^{\nu} \langle v_{s,t}e_t^*, P(w_{s,t}e_s^*)\rangle_{\mathcal{X}} \\
&= \sum_{s,t=1}^{\nu} \langle v_{s,t}e_t^*, w_{s,t}e_s^*\rangle_{\mathcal{X}} = \sum_{s,t=1}^{\nu} \lambda\big(w_{s,t}^* v_{s,t} e_s e_t^* \gamma_s\big) = \lambda(f).
\end{aligned}$$

Since every $f \in \mathbb{R}^{\nu\times\nu}\langle x\rangle$ is a linear combination of polynomials of the form (3.4.3), we established the equality (3.4.1). It remains to prove the following claim.

Claim. If λ is nonnegative on $M_{k+1,k}^{\nu,\nu\sigma_\#(k)}(L)$, then $X \in D_L$.

Proof. Suppose λ is nonnegative on $M_{k+1,k}^{\nu,\nu\sigma_\#(k)}(L)$. Write $L(x) = I_{\mathscr{H}} + \sum_{j=1}^{g} A_j x_j$ where $A := (A_1, \ldots, A_g) \in \mathbb{S}^g_{\mathscr{H}}$. Given a tuple $B := (B_1, \ldots, B_g)$ of operators we denote by $\Lambda_B(x)$ a homogeneous pencil $\sum_{j=1}^{g} B_j x_j$. Let $\ell := \nu\sigma_\#(k)$. Take an arbitrary isometry $V \in B(\mathbb{R}^\ell, \mathscr{K})$. Let e_i be the standard coordinate vectors in $\mathbb{R}^\ell$, i.e., the only nonzero entry of e_i is the i-th entry which is 1. Given

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_\ell \end{bmatrix} = \sum_{i=1}^{\ell} e_i \otimes p_i \in \mathcal{X}^{\oplus \ell},$$

note that for every triple $(\mathscr{K}, \pi, V) \in \widetilde{\Pi}_\ell$ we have

$$\begin{aligned}
&\left\langle \left(V^* \pi\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} \right) V \right)(X)p, p \right\rangle_{\mathcal{X}^{\oplus \ell}} = \left\langle \left(I_\ell + \Lambda_{V^*\pi\left(A \oplus \mathbf{0}_{\mathbb{R}}^g\right)V} \right)(X)p, p \right\rangle_{\mathcal{X}^{\oplus \ell}} \\
&= \left\langle (I_\ell \otimes I_{\mathcal{X}})p + \sum_{j=1}^{g} \left(V^* \pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V \otimes X_j \right) \left(\sum_{i=1}^{\ell} e_i \otimes p_i \right), p \right\rangle_{\mathcal{X}^{\oplus \ell}} \\
&= \left\langle (I_\ell \otimes I_{\mathcal{X}})p + \sum_{j=1}^{g} \sum_{i=1}^{\ell} ((V^* \pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V)e_i \otimes X_j p_i), p \right\rangle_{\mathcal{X}^{\oplus \ell}} \\
&= \sum_{i=1}^{\ell} \langle p_i, p_i \rangle_{\mathcal{X}} + \sum_{j=1}^{g} \sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \langle (V^* \pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V e_i, e_k \rangle_{\mathbb{R}^\ell} \cdot \langle P x_j p_i, p_k \rangle_{\mathcal{X}} \\
&= \sum_{i=1}^{\ell} \langle p_i, p_i \rangle_{\mathcal{X}} + \sum_{j=1}^{g} \sum_{i=1}^{\ell} \sum_{k=1}^{\ell} (V^* \pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V)_{ki} \cdot \langle x_j p_i, p_k \rangle_{\widetilde{\mathcal{X}}} \\
&= \sum_{i=1}^{\ell} \langle p_i, p_i \rangle_{\mathcal{X}} + \sum_{j=1}^{g} \sum_{i=1}^{\ell} \sum_{k=1}^{\ell} (V^* \pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V)_{ki} \cdot \lambda(p_k^* x_j p_i) \\
&= \sum_{i=1}^{\ell} \langle p_i, p_i \rangle_{\mathcal{X}} + \lambda\left(\sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \left(p_k^* \sum_{j=1}^{g} (V^* \pi(A_j \oplus \mathbf{0}_{\mathbb{R}})V)_{ki} x_j p_i \right) \right) \\
&= \lambda\left(\sum_{i=1}^{\ell} p_i^* p_i \right) + \lambda\left(\sum_{i=1}^{\ell} \sum_{k=1}^{\ell} \left(p_k^* \left(\Lambda_{V^*\pi\left(A \oplus \mathbf{0}_{\mathbb{R}}^g\right)V}(x) \right)_{ki} p_i \right) \right) \\
&= \lambda(p^* I_\ell p) + \lambda\left(p^* \Lambda_{V^*\pi\left(A \oplus \mathbf{0}_{\mathbb{R}}^g\right)V}(x) p \right) = \lambda\left(p^* L_{V^*\pi\left(A \oplus \mathbf{0}_{\mathbb{R}}^g\right)V}(x) p \right) \\
&= \lambda\left(p^* V^* \pi\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} \right) V p \right) \geq 0.
\end{aligned}$$

If $X \notin D_L(\ell)$, then by Theorem 3.4.5, there is a monic linear pencil $\mathcal{L}$ of size ℓ such that $\mathcal{L}(Y) \succeq 0$ for all $Y \in D_L$ and $\mathcal{L}(X) \not\succeq 0$. By Theorem 3.2.13 (1),

$$\mathcal{L} = V^* \pi\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} \right) V$$

for some triple $(\mathscr{K}, \pi, V) \in \widetilde{\Pi}_\ell$. By the calculation above, we have that $\mathcal{L}(X) \succeq 0$ which is a contradiction. Hence, $X \in D_L(\ell)$ which prove Claim and concludes the proof of Proposition 3.4.4. □

3.4.3 The truncated quadratic module is closed

Fix $\epsilon > 0$ and let

$$\mathcal{B}_\epsilon(n) := \{X \in \mathbb{S}_n^g \colon \|X\| \le \epsilon\}, \quad \text{and} \quad \mathcal{B}_\epsilon = \bigcup_{n\in\mathbb{N}} \mathcal{B}_\epsilon(n).$$

We norm $\mathbb{R}^{\nu_1\times\nu_2}\langle x\rangle_k$ by

$$\|p\|_\epsilon := \max\{\|p(X)\| \colon X \in \mathcal{B}_\epsilon\}. \tag{3.4.4}$$

(On the right-hand side of (3.4.4) the maximum is attained. This follows from the fact that the bounded nc semialgebraic set $\mathcal{B}_\epsilon$ is convex. See [HM04, Section 2.3] for details.)

Fix $\alpha, \beta, \nu_1, \nu_2 \in \mathbb{N}$ and let $\kappa = \max\{2\alpha, 2\beta + 1\}$. Let $L \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ be a monic linear operator pencil. The truncated quadratic module $M_{\alpha,\beta}^{\nu_1,\nu_2}(L)$ generated by a monic linear operator pencil L is a convex cone in $\mathbb{R}^{\nu\times\nu}\langle x\rangle_k$. There is an $\epsilon > 0$ such that for all $n \in \mathbb{N}$, if $X \in \mathbb{S}_n^g$ and $\|X\| \le \epsilon$, then $L(X) \succeq \frac{1}{2}I_{\mathscr{H}}$. In particular, $\mathcal{B}_\epsilon \subseteq D_L$. Fix $\epsilon > 0$ such that $\mathcal{B}_\epsilon \subseteq D_L$. The following proposition states, that the truncated quadratic module $M_{\alpha,\beta}^{\nu_1;\nu_2}(L)$ is closed in $\mathbb{R}^{\nu_1\times\nu_2}\langle x\rangle_\kappa$ equipped with norm $\|\cdot\|_\epsilon$.

Proposition 3.4.6. *Assume the notation above. The truncated quadratic module* $M_{\alpha,\beta}^{\nu_1;\nu_2}(L) \subseteq \mathbb{R}^{\nu_1\times\nu_1}\langle x\rangle_\kappa$ *is closed.*

Proposition 3.4.6 is an extension of [HKM12, Proposition 3.1] from matrix pencils to operator pencils. The main difficulty in the proof of the extension is to ensure that a certain convergent sequence of linear matrix pencils of the form $V_k\pi_k(L \oplus I_{\mathbb{R}})V_k$ where $\pi_k : C^*(\mathcal{S}_{A\oplus\mathbf{0}_{\mathbb{R}}^g}) \to B(\mathscr{K}_k)$ is a $*$-homomorphism, $\mathscr{K}_k$ is a separable real Hilbert space and $V_k \in B(\mathbb{R}^{\nu_2}, \mathscr{K}_k)$ is an isometry, is again of the form $V\pi(L \oplus 1_{\mathbb{R}})V$ for some $*$-homomorphism π and some isometry $V \in B(\mathbb{R}^{\nu_2}, \mathscr{K})$. For this aim we will use Theorem 3.2.13 essentially.

Proof of Proposition 3.4.6. Suppose $(P_n)_n$ is a sequence from $M_{\alpha,\beta}^{\nu_1;\nu_2}(L)$ which converges to some $P \in \mathbb{R}^{\nu_1\times\nu_1}\langle x\rangle_\kappa$. By Caratheodory's theorem on convex hulls [Bar02, Theorem I.2.3], there is $M \le \dim \mathbb{R}^{\nu_1\times\nu_1}\langle x\rangle_k + 1$ such that for each n there exist matrix-valued polynomials $R_{n,i} \in \mathbb{R}^{\nu_1\times\nu_1}\langle x\rangle_\alpha$, $T_{n,i} \in \mathbb{R}^{\nu_1\times\nu_2}\langle x\rangle_\beta$, $*$-homomorphisms $\pi_{n,i} : C^*(\mathcal{S}_{A\oplus\mathbf{0}_{\mathbb{R}}^g}) \to B(\mathscr{K}_{n,i})$ where $\mathscr{K}_{n,i}$ is a separable real Hilbert space, and isometries $V_{n,i} \in B(\mathbb{R}^{\nu_2}, \mathscr{K}_{n,i})$ such that

$$P_n = \sum_{i=1}^{M} R_{n,i}^* R_{n,i} + \sum_{i=1}^{M} T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}.$$

Claim 1. For $i = 1, \dots, M$ the sequences $(R_{n,i})_n$ and $(T_{n,i})_n$ are bounded in the norm $\|\cdot\|_\epsilon$.

Proof of Claim 1. Since the sequence $(P_n)_n$ is convegent, it is bounded in $\|\cdot\|_\epsilon$, i.e., $\|P_n\|_\epsilon \le N^2$ for every $n \in \mathbb{N}$ and some $N \in \mathbb{N}$. Now fix $i \in \{1, \dots, M\}$. For every $X \in \mathcal{B}_\epsilon$ and every $n \in \mathbb{N}$ we have

$$P_n(X) \succeq R_{n,i}^* R_{n,i}(X) \succeq 0 \quad \text{and} \quad P_n(X) \succeq T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}(X) \succeq 0.$$

Hence, for every $n \in \mathbb{N}$ we have

$$
\begin{aligned}
N^2 &\geq \|P_n\|_\epsilon \geq \left\|R_{n,i}^* R_{n,i}\right\|_\epsilon = \|R_{n,i}\|_\epsilon^2, \\
N^2 &\geq \|P_n\|_\epsilon \geq \left\|T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}\right\|_\epsilon.
\end{aligned}
$$

In particular, $(R_{n,i})_n$ is bounded. To prove Claim 1 it remains to prove that $(T_{n,i})_n$ is bounded. Let us write $L(x) = I_{\mathscr{H}} + \sum_{j=1}^g A_j x_j$. Observe that

$$
\begin{aligned}
&\|T_{n,i}\|_\epsilon^2 = \left\|T_{n,i}^* T_{n,i}\right\|_\epsilon = \frac{1}{2}\left\|T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(2\begin{bmatrix} I_{\mathscr{H}} & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}\right\|_\epsilon \\
&= \frac{1}{2}\left\|T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(2\begin{bmatrix} I_{\mathscr{H}} & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^g A_j x_j & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \sum_{j=1}^g A_j x_j & 0 \\ 0 & 0 \end{bmatrix}\right) V_{n,i} T_{n,i}\right\|_\epsilon \\
&\leq \frac{1}{2}\left\|T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} I_{\mathscr{H}} + \sum_j A_j x_j & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}\right\|_\epsilon \\
&\qquad + \frac{1}{2}\left\|T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} I_{\mathscr{H}} + \sum_j A_j (-x_j) & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}\right\|_\epsilon \\
&= \left\|T_{n,i}^* V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i} T_{n,i}\right\|_\epsilon \leq N^2.
\end{aligned}
$$

Therefore $(T_{n,i})_n$ is bounded which concludes the proof of Claim 1.

By Claim 1 and since we are in finite dimensional vector spaces, $(R_{n,i})_n$, $(T_{n,i})_n$ have convergent subsequences with limits $R_i \in \mathbb{R}^{\nu_1 \times \nu_1}\langle x \rangle_\alpha$, $T_i \in \mathbb{R}^{\nu_2 \times \nu_1}\langle x \rangle_\beta$.

Claim 2. For $i = 1, \ldots, M$ the sequences $(V_{n,i}^* \pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right) V_{n,i})_n \subseteq \mathbb{R}^{\nu_2 \times \nu_2}\langle x \rangle_1$ are bounded in the norm $\|\cdot\|_\epsilon$.

Proof of Claim 2. The following estimate holds:

$$\left\| V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right)V_{n,i}\right\|_\epsilon = \max_{X\in\mathcal{B}_\epsilon}\left\|\left(V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right)V_{n,i}\right)(X)\right\|$$

$$= \max_{X\in\mathcal{B}_\epsilon}\left\| I_{\nu_2}\otimes I + \sum_{j=1}^{g} V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}\right)V_{n,i}\otimes X_j\right\|$$

$$\leq 1 + \max_{X\in\mathcal{B}_\epsilon}\left(\sum_{j=1}^{g}\left\| V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}\right)V_{n,i}\otimes X_j\right\|\right)$$

$$= 1 + \max_{X\in\mathcal{B}_\epsilon}\left(\sum_{j=1}^{g}\left\| V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}\right)V_{n,i}\right\|\|X_j\|\right)$$

$$\leq 1 + \epsilon\sum_{j=1}^{g}\left\| V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}\right)V_{n,i}\right\|$$

$$\leq 1 + \epsilon\sum_{j=1}^{g}\|V_{n,i}^*\|\|\pi_{n,i}\|\left\|\begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}\right\|\|V_{n,i}\|$$

$$= 1 + \epsilon\sum_{j=1}^{g}\|A_j\|.$$

This proves Claim 2.

By Claim 2 and since we are in a finite dimensional vector space, the sequences $(V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right)V_{n,i})$ have convergent subsequences with limits monic linear matrix pencils of the form

$$\widehat{L}_i = I_{\nu_2} + \sum_{j=1}^{g}\widehat{A}_{j,i}x_j \in \mathbb{S}_{\nu_2}\langle x\rangle.$$

Claim 3. $D_L \subseteq D_{\widehat{L}_i}$ for $i = 1,\ldots,M$.

Proof of Claim 3. Fix $i \in \{1,\ldots,M\}$. We will prove Claim 3 by contradiction. Suppose there are $m \in \mathbb{N}$ and $X \in D_L(m)\setminus D_{\widehat{L}_i}(m)$. Then there is a vector $v \in \mathbb{R}^{\nu_2}\otimes\mathbb{R}^m$ of norm 1 such that

$$\left\langle \widehat{L}_i(X)v, v\right\rangle < 0 \tag{3.4.5}$$

Since the sequence $(V_{n,i}^*\pi_{n,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right)V_{n,i})_n$ has a convergent subsequence with a limit $\widehat{L}_i$, there exists $k_0 \in \mathbb{N}$ such that

$$\left\|\widehat{L}_i - V_{k_0,i}^*\pi_{k_0,i}\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right)V_{k_0,i}\right\|_\epsilon \leq \frac{\epsilon}{2\|X\|}\frac{\left|\left\langle \widehat{L}_i(X)v, v\right\rangle\right|}{2}. \tag{3.4.6}$$

Since $\widehat{L}_i$ and $V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}$ are monic, the following estimate holds

$$\begin{aligned}
&\Big\|\Big(\widehat{L}_i - V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}\Big)(X)\Big\| \\
&= \frac{2\|X\|}{\epsilon}\Bigg\|\Big(\widehat{L}_i - V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}\Big)\Big(\underbrace{\frac{\epsilon}{2\|X\|}X}_{\in \mathcal{B}_\epsilon}\Big)\Bigg\| \\
&\leq \frac{2\|X\|}{\epsilon}\Big\|\widehat{L}_i - V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}\Big\|_\epsilon \underbrace{\leq}_{(3.4.6)} \frac{\Big|\Big\langle \widehat{L}_i(X)v, v\Big\rangle\Big|}{2}.
\end{aligned}$$

But then, since v is of norm one, we have

$$\Big|\Big\langle\Big(\widehat{L}_i - V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}\Big)(X)v, v\Big\rangle\Big| \leq \frac{\Big|\Big\langle \widehat{L}_i(X)v, v\Big\rangle\Big|}{2},$$

and hence

$$\Big\langle\Big(V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}\Big)(X)v, v\Big\rangle \leq \frac{\Big\langle \widehat{L}_i(X)v, v\Big\rangle}{2} \underbrace{<}_{(3.4.5)} 0. \qquad (3.4.7)$$

But

$$\begin{aligned}
&\Big(V_{k_0,i}^*\pi_{k_0,i}\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_{k_0,i}\Big)(X) = \\
= \quad &(V_{k_0,i}^* \otimes I_m)\Big((\pi_{k_0,i} \otimes I_m)\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}(X)\Big)\Big)(V_{k_0,i} \otimes I_m) \\
\underbrace{\succeq}_{X \in D_L(m)} \quad &0,
\end{aligned}$$

where $\pi_{k_0,i} \otimes I_m$ is a $*$-homomorphism $A \otimes B \mapsto \pi_{k_0,i}(A) \otimes B$, leads to a contradiction with (3.4.7). This proves Claim 3.

To conclude the proof of Proposition 3.4.6 we use Theorem 3.2.13 (1). There is a triple $(\mathscr{K}_i, \pi_i, V_i)$ of a separable real Hilbert space $\mathscr{K}_i$, a $*$-homomorphism $\pi_i : C^*(\mathcal{S}_{A \oplus \mathbf{0}_{\mathbb{R}}^g}) \to B(\mathscr{K}_i)$ and an isometry V_i such that

$$\widehat{L}_i = V_i^*\pi_i\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_i.$$

Therefore $(P_n)_n$ converges to

$$\sum_{i=1}^M R_i^* R_i + \sum_{i=1}^M T_i^* V_i^* \pi_i\Big(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\Big)V_i T_i \in M_{\alpha,\beta}^{\nu_1,\nu_2}(L).$$

Thus $M_{\alpha,\beta}^{\nu_1,\nu_2}(L)$ is closed. □

3.4.4 Proof of the theorem

In this subsection we prove Theorem 3.4.3. The main idea of the proof is a classical Putinar separation-type argument [Put93] and its noncommutative version in [HM04], but there is an important difference in the separating functional λ which ensures that the Positivstellensatz holds not only for positive definite polynomials but for semidefinite ones as well and we also get degree bounds.

In the proof we need an additional lemma (see [HKM12, Lemma 3.2]) on the existence of a positive linear functional $\widehat{\lambda} : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$, i.e., $\widehat{\lambda}(F) > 0$ for all $F \in \Sigma^{\nu}_{d+1} \setminus \{0\}$, which is nonnegative on $M^{\nu,\nu\sigma_{\#}(d)}_{d+1,d}(L)$.

Lemma 3.4.7. *There exists a positive linear functional $\widehat{\lambda} : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$ which is nonnegative on $M^{\nu,\nu\sigma_{\#}(d)}_{d+1,d}(L)$.*

The proof of Lemma 3.4.7 is the same as for [HKM12, Lemma 3.2] just that $\widehat{\lambda}$ from the proof of [HKM12, Lemma 3.2] has to be nonnegative also on $M^{\nu,\nu\sigma_{\#}(d)}_{d+1,d}(L)$. This is clear from the construction of $\widehat{\lambda}$. We include the proof for the sake of completeness.

Proof of Lemma 3.4.7. Choose $\epsilon > 0$ such that $\mathcal{B}_\epsilon \subseteq D_L$. Let $X^{(1)}$, $X^{(2)}$, ... be some countable dense subset $\mathcal{B}_\epsilon$, e.g., all tuples of matrices in $\mathcal{B}_\epsilon(d+1)$ with rational entries. Let us define $\widehat{\lambda} : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$ by

$$\widehat{\lambda}(p) = \sum_{i=1}^{\infty} \frac{1}{2^i}\operatorname{tr}(X^{(i)}).$$

Obviously, $\widehat{\lambda}$ is nonnegative on $M^{\nu,\nu\sigma_{\#}(d)}_{d+1,d}(L)$. It remains to prove that $\widehat{\lambda}$ is strictly positive on nonzero hermitian squares in Σ^{ν}_{d+1}. Let $r \in \mathbb{R}^{\nu\times\nu}\langle x\rangle_{d+1}$ be arbitrary. If we have $\widehat{\lambda}(r^*r) = 0$, then by density of the set $\{X^{(1)}, X^{(2)}, \ldots\}$, r vanishes on $\mathcal{B}_\epsilon(d+1)$ and hence on $\mathbb{S}_{d+1}(\mathbb{R})^g$. But then by the nonexistence of low degree identities [Pro73], $r = 0$. ☐

Finally we are ready to prove Theorem 3.4.3.

Proof of Theorem 3.4.3. Suppose $F \notin M^{\nu,\ell}_{d+1,d}(L)$. By Proposition 3.4.6 and the Hahn-Banach separation theorem there exists a linear functional $\lambda : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$ that is nonnegative on $M^{\nu,\ell}_{d+1,d}(L)$ and negative on F. Adding a small positive multiple of $\widehat{\lambda}$ from Lemma 3.4.7 to λ, we may assume that λ is strictly positive on $\Sigma^{\nu}_{d+1} \setminus \{0\}$. By Proposition 3.4.4 used for $k = d$, there is a tuple of symmetric matrices $X \in D_L$ acting on a finite-dimensional Hilbert space $\mathcal{X}$ and a vector γ such that

$$\lambda(P) = \langle P(X)\gamma, \gamma\rangle$$

for all $P \in \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+1}$. In particular,

$$\langle F(X)\gamma, \gamma\rangle = \lambda(F) < 0,$$

so that $F(X)$ is not positive semidefinite, contradicting $D_L \subseteq D_F$ which concludes the proof of Theorem 3.4.3. ☐

3.4.5 Extension

This subsection focuses on matrix polynomials positive on a free Hilbert spectrahedrop. The main result, Theorem 3.4.9, extends Theorem 3.4.3 from free Hilbert spectrahedra to free Hilbert spectrahedrops.

Let L be a monic linear operator pencil of the form

$$L(x, y) = I_{\mathscr{H}} + \sum_{j=1}^{g} \Omega_j x_j + \sum_{k=1}^{h} \Gamma_k y_k \in \mathbb{S}_{\mathscr{H}}\langle x \rangle$$

and let $\mathcal{K} = \operatorname{proj}_x D_L$. Fix positive integers $\nu_1, \nu_2, d \in \mathbb{N}$. We define the $(\nu_1, \nu_2; d)$ *truncated quadratic module* in $\mathbb{R}^{\nu_1 \times \nu_1}\langle x \rangle_{2d+1}$ associated to L and $\mathcal{K} = \operatorname{proj}_x D_L$ by

$$M^{\nu_1,\nu_2}_{x,\alpha,\beta}(L) := \Bigg\{ \sigma + \sum_{(\mathscr{H}_\ell, \pi_\ell, V_\ell) \in \widetilde{\Pi}_{\nu_2}}^{\text{finite}} R_\ell^* V_\ell^* \pi_\ell\left(\begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix}\right) V_\ell R_\ell \colon \sigma \in \Sigma^{\nu_1}_{d+1}\langle x \rangle,$$
$$R_\ell \in \mathbb{R}^{\nu_2 \times \nu_1}\langle x \rangle_d, \sum_\ell R_\ell^* V_\ell^* \pi_\ell\left(\begin{bmatrix} \Gamma_k & 0 \\ 0 & 0 \end{bmatrix}\right) V_\ell R_\ell = 0 \text{ for all } k \Bigg\}.$$

In the case $D_{\mathfrak{h}L} = D_{\mathfrak{h}(L \oplus I_{\mathbb{R}})}$, we can replace $*$-homomorphisms of the extended pencil $L \oplus I_{\mathbb{R}}$ by $*$-homomorphisms of L in the definition of the truncated quadratic module.

Proposition 3.4.8. *If $D_{\mathfrak{h}L} = D_{\mathfrak{h}(L \oplus I_{\mathbb{R}})}$, then:*

$$M^{\nu_1,\nu_2}_{x,\alpha,\beta}(L) := \Bigg\{ \sigma + \sum_{(\mathscr{H}_\ell, \pi_\ell, V_\ell) \in \Pi_{\nu_2}}^{\text{finite}} R_\ell^* V_\ell^* \pi_\ell(L) V_\ell R_\ell \colon \sigma \in \Sigma^{\nu_1}_{d+1}\langle x \rangle,$$
$$R_\ell \in \mathbb{R}^{\nu_2 \times \nu_1}\langle x \rangle_d, \sum_\ell R_\ell^* V_\ell^* \pi_\ell(\Gamma_k) V_\ell R_\ell = 0 \text{ for all } k \Bigg\}.$$

Proof. The proof is the same as the proof of Proposition 3.4.6 using Theorem 3.2.18 instead of Theorem 3.2.13. □

The main result of this subsection is the following Positivstellensatz.

Theorem 3.4.9. *A symmetric polynomial $F \in \mathbb{R}^{\nu \times \nu}\langle x \rangle_{2d+1}$ is positive semidefinite on $\mathcal{K}$ if and only if $F \in M^{\nu, \nu\sigma_\#(k)}_{x,\alpha,\beta}(L)$ where $\sigma_\#(k) = \dim \mathbb{R}\langle x \rangle_k$.*

Remark 3.4.10. Several remarks are in order.

(1) In case there are no y-variables in L, Theorem 3.4.9 reduces to Theorem 3.4.3.

(2) If $d = 0$, i.e., F is linear, then Theorem 3.4.9 reduces to Theorem 3.2.18.

(3) If L is matrix-valued, then Theorem 3.4.9 reduces to [HKM16b, Theorem 5.1].

(4) If L is matrix-valued and variables commute, a Positivstellensatz for commutative polynomials strictly positive on spectrahedrops was established by Gouveia and Netzer in [GN11]. A major distinction is that the degrees of the R_k and σ in the commutative theorem behave very badly.

Proof of Theorem 3.4.9

The proof uses the same idea as the proof of Theorem 3.4.3, i.e., construction of a positive separating functional and then the connection with operators via the GNS construction. What has to be proved additionally is that the truncated quadratic module $M^{\nu,\nu\sigma_\#(k)}_{x,d+1,d}(L)$ is closed (see Proposition 3.4.11) and that the tuple of operators X from the GNS construction belongs to the closure of the free Hilbert spectrahedrop (see Proposition 3.4.12).

Proposition 3.4.11. *The truncated module* $M^{\nu_1,\nu_2}_{x,\alpha,\beta}(L) \subseteq \mathbb{R}^{\nu_1,\nu_1}\langle x\rangle_\kappa$ *is closed where* $\kappa = \max\{2\alpha, 2\beta+1\}$.

Proof. The proof is the same as the proof of Proposition 3.4.6 using Theorem 3.2.18 instead of Theorem 3.2.13. □

Proposition 3.4.12. *If* $\lambda : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2k+2} \to \mathbb{R}$ *is a linear functional which is nonnegative on* Σ^ν_{k+1} *and positive on* $\Sigma^\nu_k \setminus \{0\}$*, then there exists a tuple* $X = (X_1, \dots, X_g)$ *of self-adjoint operators on a Hilbert space* $\mathcal{X}$ *of dimension at most* $\nu\sigma_\#(k) = \nu \dim \mathbb{R}\langle x\rangle_k$ *and a vector* $\gamma \in \mathcal{X}^{\oplus\nu}$*, such that*

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle_{\mathcal{X}^{\oplus\nu}}$$

for all $f \in \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2k+1}$*, where* $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ *is the inner product on* $\mathcal{X}$*. Further, if* λ *is nonnegative on* $M^{\nu,\nu\sigma_\#(k)}_{x,k+1,k}(L)$*, then* X *is in the closure* $\overline{\mathcal{K}}$ *of the free spectrahedrop* $\mathcal{K} = \operatorname{proj}_x D_L$ *coming from* L.

Conversely, given $X = (X_1, \dots, X_g)$ *is a tuple of self-adjoint operators on a Hilbert space* $\mathcal{X}$ *of dimension* N*, the vector* $\gamma \in \mathcal{X}^{\oplus\nu}$*, and* k *a positive integer, then the linear functional* $\lambda : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2k+2} \to \mathbb{R}$*, defined by*

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle_{\mathcal{X}^{\oplus\nu}}$$

is nonnegative on Σ^ν_{k+1}*. Further, if* $X \in \overline{\operatorname{proj}_x D_L}$*, then* λ *is nonnegative also on* $M^{\nu,\ell\sigma_\#(k)}_{x,k+1,k}(L)$ *for every* $\ell \in \mathbb{N}$.

If L is matrix-valued, then Proposition 3.4.12 becomes [HKM16b, Proposition 5.4]. For operator-valued L there are minor changes. Namely, we need to add the explanation, why $\lambda(M^{\nu,\nu\sigma_\#(k)}_{x,k+1,k}(L)) \subseteq \mathbb{R}_{\geq 0}$ implies that $X \in \overline{\operatorname{proj}_x D_L}$.

Proof of 3.4.12. The nontrivial implication is ($\Rightarrow$). Construct $\mathcal{X}$, $\langle\cdot,\cdot\rangle_{\mathcal{X}}$, $X \in \mathbb{S}^g_{\mathcal{X}}$ and $\gamma \in \mathcal{X}^{\oplus\nu}$ exactly as in the proof of Proposition 3.4.4. The only thing which remains to be proved is the following claim.

Claim. If λ is nonnegative on $M^{\nu,\nu\sigma_\#(k)}_{x,k+1,k}(L)$, then $X \in \overline{\mathcal{K}}$.

Proof of Claim. Suppose λ is nonnegative on $M^{\nu,\nu\sigma_\#(k)}_{x,k+1,k}(L)$. We will prove that $X \in \overline{\mathcal{K}}$. Since $\overline{\mathcal{K}}$ is a closed matrix convex set and $0 \in \overline{\mathcal{K}}$, there exists, by Theorem 3.4.5, a monic linear pencil $\mathcal{L} \in \mathbb{S}_\sigma\langle x\rangle$ of size $\sigma = \dim \mathcal{X}$ such that $\mathcal{L}|_{\overline{\mathcal{K}}} \succeq 0$ and $\mathcal{L}(X) \not\succeq 0$. Thus, by Theorem 3.2.18, there exist a separable real Hilbert space $\mathscr{G}$, an isometry $V : \mathbb{R}^\sigma \to \mathscr{G}$ and $*$-homomorphism $\pi : C^*(\mathcal{S}_{A\oplus\mathbf{0}^g_{\mathbb{R}}}) \to B(\mathscr{G})$ such that

$$\mathcal{L} = V^*\pi(L \oplus I_{\mathbb{R}})V \quad \text{and} \quad 0 = V^*\pi(\Gamma_k \oplus \mathbf{0}_{\mathbb{R}})V \quad \text{for all } k.$$

Since $\mathcal{L}(X) \not\succeq 0$, there exists $u := \sum_{i=1}^{\sigma} e_i \otimes v_i \in \mathbb{R}^\sigma \otimes \mathcal{X}$ where $v_i \in \mathcal{X}$ and e_i are the standard coordinate vector in $\mathbb{R}^\sigma$, i.e., the only nonzero entry of e_i is the i-th entry which is 1, such that

$$u^*\mathcal{L}(X)u < 0. \tag{3.4.8}$$

By the construction of $\mathcal{X}$, X and γ, there exists a polynomial $p_i \in \mathbb{R}^{1\times\nu}\langle x\rangle_k$ such that $v_i = p_i(X)\gamma$. Therefore,

$$\begin{aligned} u^*\mathcal{L}(X)u &= \big(\sum_{i=1}^{\sigma} e_i \otimes v_i\big)^*\mathcal{L}(X)\big(\sum_{i=1}^{\sigma} e_i \otimes v_i\big) \\ &= \big(\sum_{i=1}^{\sigma} e_i \otimes p_i(X)\gamma\big)^*(V\otimes I_\sigma)^*\pi(L(X,Y)\oplus I_{\mathbb{R}})(V\otimes I_\sigma)\big(\sum_{i=1}^{\sigma} e_i \otimes p_i(X)\gamma\big). \end{aligned}$$

Defining $\vec{p}(x) = \sum_{i=1}^{\sigma} e_i \otimes p_i(x) \in \mathbb{R}^{\sigma\times\nu}\langle x\rangle_k$, note that $\vec{p}(X)\gamma = \sum_{i=1}^{\sigma} e_i \otimes p_i(X)\gamma$. Hence,

$$\begin{aligned} u^*\mathcal{L}(X)u &= (\vec{p}(X)\gamma)^*(V\otimes I_\sigma)^*\pi(L(X,Y)\oplus I_{\mathbb{R}})(V\otimes I_\sigma)(\vec{p}(X)\gamma) \\ &= \langle \vec{p}(X)^*(V\otimes I_\sigma)^*\pi(L(X,Y)\oplus I_{\mathbb{R}})(V\otimes I_\sigma)\vec{p}(X)\gamma, \gamma\rangle \\ &= \lambda(\vec{p}(x)^*V^*\pi(L(x,y)\oplus I_{\mathbb{R}})V\vec{p}(x)). \end{aligned}$$

Defining

$$q := \vec{p}(x)^*V^*\pi(L(X,Y)\oplus I_{\mathbb{R}})V\vec{p}(x),$$

we have that $u^*\mathcal{L}(X)u = \lambda(q)$. Since $q \in M_{x,k+1,k}^{\nu,\nu\sigma_\#(k)}(L)$, it follows that $\lambda(q) \geq 0$ which contradicts to (3.4.8). This proves Claim and concludes the proof of Proposition 3.4.12. □

In the proof of Theorem 3.4.9 we need an additional lemma on the existence of a positive linear functional $\widehat{\lambda} : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$, i.e., $\widehat{\lambda}(F) > 0$ for all $F \in \Sigma_{d+1}^{\nu} \setminus \{0\}$, which is nonnegative on $M_x^{\nu,\nu\sigma_\#(k)}(L)_d$.

Lemma 3.4.13. *There exists a positive linear functional* $\widehat{\lambda} : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$ *which is nonnegative on* $M_{x,d+1,d}^{\nu,\nu\sigma_\#(k)}(L)$.

The proof of Lemma 3.4.13 is the same as for Lemma 3.4.7 above only that in the first sentence we choose $\epsilon > 0$ such that $\mathcal{B}_\epsilon \subseteq \operatorname{proj}_x D_L$.

Now we are ready to prove Theorem 3.4.9. The proof follows the proof of [HKM16b, Theorem 5.1], only that we use Proposition 3.4.11 instead of [HKM16b, Proposition 5.3] and Proposition 3.4.12 instead of [HKM16b, Proposition 5.4].

Proof of Theorem 3.4.9. The proof is by contradiction. Let $F \in \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+1}$ be positive semidefinite on $\mathcal{K}$ such that $F \notin M_{x,d+1,d}^{\nu,\nu\sigma_\#(k)}(L)$. By the Hahn-Banach theorem and Proposition 3.4.11, there exists a functional $\lambda : \mathbb{R}^{\nu\times\nu}\langle x\rangle_{2d+2} \to \mathbb{R}$ such that $\lambda(F) < 0$ and $\lambda\Big(M_{x,d+1,d}^{\nu,\nu\sigma_\#(k)}(L)\Big) \geq 0$. By Lemma 3.4.13, we may assume that λ is strictly positive on $\Sigma^{\nu}_{d+1} \setminus \{0\}$. By Proposition 3.4.12 there are a finite dimensional Hilbert space $\mathcal{X}$, a tuple of matrices $X \in \overline{\mathcal{K}}$ and a vector $\gamma \in \mathcal{X}^{\oplus \nu_1}$ such that

$$0 > \lambda(F) = \langle F(X)\gamma, \gamma \rangle_{\mathcal{X}^{\oplus \nu_1}} \geq 0,$$

which is a contradiction and concludes the proof of Theorem 3.4.9. □

3.5 Univariate Positivstellensatz

In this section we extend Theorem 3.4.3 in the univariate case from matrix-valued polynomials to operator-valued ones, see Theorem 3.5.1 below. Namely, in the univariate case, F in Theorem 3.4.3 can be operator-valued but the conclusion still holds. The main step of the proof is the reduction to the inclusion of free Hilbert spectrahedra by the use of variants of the operator Fejér-Riesz theorem [Ros68]. By Examples 3.2.16 and 3.5.2, Theorem 3.5.1 does not extend to the non-monic case.

Theorem 3.5.1. *Let $L = I_{\mathscr{H}} + A_1 x \in \mathbb{S}_{\mathscr{H}}\langle x \rangle$ be a univariate monic linear operator pencil. Then for every symmetric operator-valued noncommutative polynomial $F \in B(\mathscr{K}) \otimes \mathbb{R}\langle x \rangle$ with $F|_{D_L(1)} \succeq 0$, there exists a separable real Hilbert space $\mathscr{G}$, a $*$-homomorphism $\pi : B(\mathscr{H}) \to B(\mathscr{G})$ and finitely many operator polynomials $R_j \in B(\mathscr{K}) \otimes \mathbb{R}\langle x \rangle$ and $Q_k \in B(\mathscr{K}, \mathscr{G}) \otimes \mathbb{R}\langle x \rangle$ all of degree at most $\frac{1}{2} \cdot \deg(F)$, $j_0, k_0 \in \mathbb{N}$, such that*

$$F = \sum_{j=1}^{j_0} R_j^* R_j + \sum_{k=1}^{k_0} Q_k^* \pi(L) Q_k.$$

Proof. Since L is monic, the set $D_L(1)$ is an interval with a non-empty interior. We separate three cases.

Case 1: $D_L(1) = [a, b]$ where $a, b \in \mathbb{R}$ and $a < b$.

By the linear change of variables we may assume that $D_L(1) = [-1, 1]$. By [CZ13, Proposition 3], there exist operator polynomials $R_j \in B(\mathscr{K})\langle x \rangle$ and $\widetilde{Q}_k \in B(\mathscr{K}, \mathscr{K}^2)\langle x \rangle$ all of degree at most $\frac{1}{2} \cdot \deg(F)$ such that

$$F(x) = \sum_{j}^{\text{finite}} R_j^* R_j + \sum_{k}^{\text{finite}} \widetilde{Q}_k^* \begin{bmatrix} (1+x)I_{\mathscr{K}} & 0 \\ 0 & (1-x)I_{\mathscr{K}} \end{bmatrix} \widetilde{Q}_k.$$

(For the degree bounds see [DS02, Theorem 2.5] and use the identity

$$x(1-x) = x^2(1-x) + (1-x)^2 x.)$$

Thus it remains to prove the statement of the theorem for $F(x) = 1 + x$ and $F(x) = 1 - x$. We use Theorem 3.2.15 and conclude the proof of Case 1.

Case 2: $D_L(1) = [a, \infty)$ or $(-\infty, a]$ where $a \in \mathbb{R}$.

By the linear change of variables we may assume that $D_L(1) = [-1, \infty)$. By [CZ13, Proposition 3], there exist operator polynomials $R_j \in B(\mathscr{K})\langle x\rangle$ and $\widetilde{Q}_k \in B(\mathscr{K})\langle x\rangle$ all of degree at most $\frac{1}{2} \cdot \deg(F)$ such that

$$F = \sum_j^{\text{finite}} R_j^* R_j + \sum_k^{\text{finite}} \widetilde{Q}_k^*(1+x) I_{\mathscr{K}} \widetilde{Q}_k.$$

(The degree bounds are easy to see by comparing the leading coefficients.) Thus it remains to prove the statement of the theorem for $F(x) = (1 + x)$. We use Theorem 3.2.15 and conclude the proof of Case 2.

Case 3: $D_L(1) = \mathbb{R}$.

By [CZ13, Proposition 3], there exist operator polynomials $R_j \in B(\mathscr{K})\langle x\rangle$ all of degree at most $\frac{1}{2} \cdot \deg(F)$ such that $F = \sum_j^{\text{finite}} R_j^* R_j$, which proves Case 3. □

If L is not monic in Theorem 3.5.1, then the conclusion is not true in general (see Example 3.2.16 above). However, by [KS13, Corollary 4.3.1], it extends to the matrix-valued pencil L with $D_L = \emptyset$. (The case $F = -1$ is the contents of [KS13, Corollary 4.3.1], while for an arbitrary F one uses the identity $\frac{1}{4}((F+1)^*(F+1) - (F-1)^*(F-1))$.) But the following counterexample shows that [KS13, Corollary 4.3.1] does not extend to the operator-valued pencil L with $D_L = \emptyset$.

Example 3.5.2. Let $L(x) = A_0 + A_1 x \in \mathbb{S}_{\ell^2(\mathbb{N})}\langle x\rangle$ be a linear operator pencil, where

$$A_0 = \operatorname{diag}\Big(-\frac{1}{n}\Big)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N})) \quad \text{and} \quad A_1 = \operatorname{diag}\Big(\frac{1}{n^2}\Big)_{n\in\mathbb{N}} \in B(\ell^2(\mathbb{N})).$$

Then:

(1) The spectrahedron $D_L(1)$ is $\emptyset$.

(2) The polynomial $\ell(y) = -1$ is non-negative on $D_L(1)$.

(3) There do not exist a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism $\pi : B(\ell^2) \to B(\mathscr{K})$, polynomials $r_j \in \mathbb{R}\langle x\rangle$ and operator polynomials $q_k \in B(\mathbb{R}, \mathscr{K})\langle x\rangle$ such that

$$-1 = \sum_j^{\text{finite}} r_j^2 + \sum_k^{\text{finite}} q_k^* \pi(L) q_k. \tag{3.5.1}$$

Proof. (1) is easy to check and (2) follows by (1). We will prove (3). Let us say that $\mathscr{K}$, π, r_j, q_k satisfying (3.5.1) exist. Observe that $A_1 = A_0^* A_0$. Therefore

$$\sum_k q_k^* \pi(L(x)) q_k = \sum_k q_k^* \pi(A_0)^* \pi(A_0) q_k \cdot x + \sum_k q_k^* \pi(A_0) q_k.$$

If $\sum_k q_k^* \pi(A_0) q_k = 0$, then

$$-1 = \sum_j r_j^2 + \sum_k q_k^* \pi(A_0)^* \pi(A_0) q_k x. \tag{3.5.2}$$

This is a contradiction since the right-hand side of (3.5.2) is nonnegative for $x \geq 0$, while the left-hand side is a constant -1. Therefore $\sum_k q_k^* \pi(A_0) q_k \neq 0$. Let us write

$$r_j(y) = \sum_{m=0}^{N_j} r_{j,m} x \in \mathbb{R}\langle x \rangle, \quad q_k(y) = \sum_{m=0}^{M_k} q_{k,m} x \in B(\mathbb{R}, \mathscr{K})\langle x \rangle,$$

where $N_j \in \mathbb{N}_0$ is such that $r_{N_j} \neq 0$ and $M_k \in \mathbb{N}_0$ is such that $\pi(A_0) q_{k,M_k} \neq 0$. We can indeed choose such M_k, since otherwise $\pi(A_0) q_{k,M_k} = q_{k,M_k}^* \pi(A_0) = 0$ and hence

$$\begin{aligned}
&q_k^* \pi(A_0) q_k = \sum_{\ell,j=0}^{M_k} q_\ell^* \pi(A_0) q_j x^{\ell+j} \\
&= \sum_{j=0}^{M_k} \underbrace{q_{M_k}^* \pi(A_0)}_{0} q_j x^{M_k+j} + \sum_{\ell=0}^{M_k-1} q_\ell^* \underbrace{\pi(A_0) q_{M_k}}_{0} x^{M_k+j} + \sum_{\ell,j=0}^{M_k-1} q_\ell^* \pi(A_0) q_j x^{\ell+j} \\
&= \sum_{\ell,j=0}^{M_k-1} q_\ell^* \pi(A_0) q_j x^{\ell+j} = (\sum_{m=0}^{M_k-1} q_{k,m} y)^* \pi(A_0) (\sum_{m=0}^{M_k-1} q_{k,m} x)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_k q_k^* \pi(A_1) q_k = \sum_{\ell,j=0}^{M_k} q_\ell^* \pi(A_0) \pi(A_0) q_j x^{\ell+j} \\
&= \sum_{j=0}^{M_k} \underbrace{q_{M_k}^* \pi(A_0)}_{0} \pi(A_0) q_j x^{M_k+j} + \sum_{\ell=0}^{M_k-1} q_\ell^* \pi(A_0) \underbrace{\pi(A_0) q_{M_k}}_{0} x^{\ell+M_k} + \\
&+ \sum_{\ell,j=0}^{M_k-1} q_\ell^* \pi(A_1) q_j x^{\ell+j} \\
&= \sum_{\ell,j=0}^{M_k-1} q_\ell^* \pi(A_1) q_j x^{\ell+j} = (\sum_{m=0}^{M_k-1} q_{k,m} x)^* \pi(A_1) (\sum_{m=0}^{M_k-1} q_{k,m} x)
\end{aligned}$$

We are repeating this calculation until $\pi(A_0) q_{k,M_k-r_k} \neq 0$ and take $q_k := \sum_{m=0}^{M_k-r_k} q_{k,m} x$.

The highest monomial according to the ordering of $\mathbb{R}\langle y \rangle$

$$dx^m \succeq cx^n \Leftrightarrow m > n \text{ or } m = n,\ d \geq c$$

in:

(1) r_j^2 is $\underbrace{r_{j,N_j}^2}_{\neq 0} x^{2N_j}$,

(2) $q_k^*\pi(A_1)q_kx$ is $\underbrace{q_{k,M_k}^*\pi(A_0)^*\pi(A_0)q_{k,M_k}}_{\neq 0} x^{2M_k+1}$,

(3) $q_k^*\pi(A_0)q_k$ is 'at most' $q_{k,M_k}^*\pi(A_0)q_{k,M_k}x^{2M_k}$ (or smaller).

Let $M := \max\{N_j, M_k \colon j, k\}$. Therefore, the highest monomial on the right-hand side of (3.5.2) is

$$\begin{cases} \displaystyle\sum_{j\colon N_j=M} \underbrace{r_{j,N_j}^2}_{>0} x^{2M}, & \text{if } M \neq M_k \text{ for every } k \\ \displaystyle\sum_{k\colon M_k=M} \underbrace{q_{k,M_k}^*\pi(A_0)^*\pi(A_0)q_{k,M_k}}_{>0} x^{2M+1}, & \text{if } M = M_k \text{ for some } k \end{cases}.$$

Since the highest monomial on left-hand side of (3.5.2) is -1, we conclude that $M = 0$ and $q_k = 0$ for every k. Thus $-1 = \sum_j r_j^2$ which is a contradiction. □

Remark 3.5.3. Theorem 3.5.1 extends to non-monic $L(x) = A_0 + A_1x \in \mathbb{S}_{\mathscr{H}}\langle x\rangle$ in the following cases:

(1) $D_L(1) \neq \emptyset$ and $\operatorname{span}\{A_0, A_1\}$ contains an invertible positive definite element.

(2) $D_L(1) = \{a\}$ and A_0, A_1 are linearly dependent.

(3) $D_L(1) = \emptyset$ and $D_{PLP}(1)$ is compact for some finite-dimensional projection $P \in B(\mathscr{H})$.

Proof. The proof of (1) is the same as the proof of Theorem 3.5.1 just that we use a non-monic version of Theorem 3.2.15 (see Remark 3.2.14 (3)).

Now we prove (2). By a linear change of variables we may assume that $D_L(1) = \{0\}$. If $A_0 \neq 0$, then we have $A_0 + A_1x = A_0(1+\lambda x)$ for some $\lambda \in \mathbb{R}$. Hence $A_0 \succeq 0$. Thus $\lim\limits_{x\to\infty} L(x) \succeq 0$ or $\lim\limits_{x\to-\infty} L(x) \succeq 0$. This is a contradiction. Hence $A_0 = 0$ and $L(x) = A_1x$. Since $D_L(1) = \{0\}$, there are $v_1, v_2 \in \mathscr{H}$ such that $\langle A_1v_1, v_1\rangle > 0$ and $\langle A_1v_2, v_2\rangle < 0$. So

$$x = \frac{\langle L(x)v_1, v_1\rangle}{\langle A_1v_1, v_1\rangle}, \quad -x = \frac{\langle L(x)v_2, v_2\rangle}{|\langle A_1v_2, v_2\rangle|}.$$

By the identity $-x^2 = \frac{1}{4}(x(x-1)^2 - x(x+1)^2)$ we have that $-x^2$ is of the form

$$-x^2 = \frac{(x-1)^2}{4\langle A_1v_1, v_1\rangle}\langle L(x)v_1, v_1\rangle + \frac{(x+1)^2}{4|\langle A_2v_1, v_1\rangle|}\langle L(x)v_2, v_2\rangle.$$

Thus, for every $\ell \in \mathbb{N}$ we have

$$\begin{aligned} -x^{2\ell} &= \frac{(x^{\ell-1}(x-1))^2}{4\langle A_1v_1, v_1\rangle}\langle L(x)v_1, v_1\rangle + \frac{(x^{\ell-1}(x+1))^2}{4|\langle A_2v_1, v_1\rangle|}\langle L(x)v_2, v_2\rangle, \\ -x^{2\ell+1} &= \frac{x^{2\ell}}{|\langle A_1v_2, v_2\rangle|}\langle L(x)v_2, v_2\rangle. \end{aligned}$$

So every $F \in B(\mathscr{K})\langle x\rangle$ satisfying $F(0) \succeq 0$ is of the from

$$\sum_{j}^{\text{finite}} R_j^* R_j + \sum_{k}^{\text{finite}} Q_k^* L Q_k,$$

where $R_j \in B(\mathscr{K})\langle x\rangle$ and $Q_k \in B(\mathscr{K}, \mathscr{H})\langle x\rangle$ are operator polynomials.

Finally we prove (3). Let $(P_n)_n$ be an increasing sequence of projections from $\mathscr{H}$ to a n dimensional subspace of $\mathscr{H}$ such that $P = P_\ell$ for $\ell = \dim \operatorname{Ran}(P)$. We have the following decreasing sequence of compact sets:

$$D_{P_\ell L P_\ell}(1) \supseteq D_{P_{\ell+1} L P_{\ell+1}}(1) \supseteq \cdots \supseteq \bigcap_{k=\ell}^{\infty} D_{P_k L P_k}(1) = D_L(1) = \emptyset.$$

Note that the equality $\bigcap_{k=\ell}^{\infty} D_{P_k L P_k}(1) = D_L(1)$ follows by the convergence of the sequence $P_k L P_k$ to L in the weak operator topology. Since $D_{P_\ell L P_\ell}(1)$ is compact and $D_{P_\ell L P_\ell}(1) \subset \bigcup_{k=\ell}^{\infty} D_{P_k L P_k}(1)^c$ is its open covering, it follows that

$$D_{P_\ell L P_\ell}(1) \subset \bigcup_{k}^{N} D_{P_k L P_k}(1)^c = D_{P_N L P_N}(1)^c$$

for some $N \in \mathbb{N}$. Hence $D_{P_N L P_N}(1)) = \emptyset$. By [KS13, Corollary 4.3.1], -1 is of the form

$$-1 = \sum_{j}^{\text{finite}} r_j^2 + \sum_{k}^{\text{finite}} Q_k^* P_N L P_N Q_k,$$

where $r_j \in \mathbb{R}\langle x\rangle$ are scalar polynomials and $Q_k \in \mathbb{R}^{N\times 1}\langle x\rangle$ are vectors of polynomials. By the equality $F = \frac{1}{4}((F+1)^*(F+1) - (F-1)^*(F-1))$, arbitrary F is of the form

$$F = \sum_{j}^{\text{finite}} R_j^* R_j + \sum_{k}^{\text{finite}} Q_k^* P_N L P_N Q_k,$$

where $R_j \in B(\mathscr{K})\langle x\rangle$ and $Q_k \in B(\mathscr{K}, \mathbb{R}^N)\langle x\rangle$ are operator polynomials. $\square$

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Razširjeni povzetek

Disertacija študira Positivstellensatze za matrične in operatorske polinome. Ime Positivestellensatz se nanaša na algebraično zagotovilo za pozitivnost danega polinoma p na dani zaprti semialgebraični množici K. Iskanje takih zagotovil spada na področje realne algebraične geometrije. Za poljubno množico K je lahko iskanje učinkovitega zagotovila zahteven problem, še posebej, če polinom p ni strogo pozitiven na K, temveč samo nenegativen. V tem delu se ukvarjamo z dvema vrstama zagotovil, nanašajoč se na množico K in nekomutativen polinom p, ki ga želimo predstaviti, pri čemer sta oba problema iz področja nekomutativne realne algebraične geometrije.

Ekvivalentni verziji matričnega Fejér-Rieszovega izreka karakterizirata pozitivno semidefinitne $n \times n$ matrične polinome na realni osi $\mathbb{R}$ in na enotski kompleksni krožnici $\mathbb{T}$. V primeru skalarnih polinomov (tj. $n = 1$) so razširitve realne verzije tega rezultata na poljubno zaprto semialgebraično množico K v $\mathbb{R}$ dobro raziskane v delih Kuhlmannove, Marshalla [KM02] in Scheidererja [Sch03]. V primeru matričnih polinomov (tj. n je poljubno naravno število) in množica K zaprt interval, sta razširitve izpeljala Dette in Studden [DS02]. Prvi problem disertacije je v primeru poljubne zaprte semialgebraične množice K v $\mathbb{R}$ rezultate razširiti iz skalarnih na matrične polinome.

Iskanje Positivstellensatzov za polinome, pozitivno semidefinitne na matrično konveksnih množicah, kot je množica rešitev linearne matrične neenakosti (LMN), spada na področje proste realne algebraične geometrije. Polinomi so ovrednoteni na tericah matrik, ovrednotenja pa so matrike ali operatorji. Za LMNje in simetrične matrične polinome so v vrsti člankov različne Positivstellensatze izpeljali Helton, Klep and McCullough, npr. [HKM12, HKM13b, HKM16b]. Ker je po [EW97] vsaka zaprta matrično konveksna množica, ki vsebuje izhodišče, natanko množica matričnih rešitev linearne operatorske neenakosti (LON), se pojavi naravno vprašanje, ali je mogoče rezultate iz LMNjev in matričnih polinomov prenesti na LONje in operatorske polinome. To je drugi problem diserticije.

Disertacija temelji na rezultatih iz [Zal15 arxiv, Zal16, Zal17].

Positivstellensatzi za matrične polinome v eni spremenljivki

V poglavju 2 študiramo matrične polinome v eni spremenljivki, ki so pozitivno semidefinitni na semialgebraičnih množicah.

Razdelek 2.1 začnemo z dobro znano karakterizacijo nenegativnih polinomov na realni osi $\mathbb{R}$. Naj bo $\mathbb{C}[x]$ množica kompleksnih polinomov v spremenljivki x. Množico $\mathbb{C}[x]$ opremimo z *involucijo* $*$, ki je na polinomu $f(x) := \sum_{m=0}^{2N} f_m x^m \in \mathbb{C}[x]$ definirana kot $f(x)^* := \sum_{m=0}^{2N} \overline{f_m} x^m$. Naslednja karakterizacija je preprosta posledica osnovnega izreka algebre.

Izrek 1 (glej Theorem 2.1.1). *Naj bo* $f(x) = \displaystyle\sum_{m=0}^{2N} f_m x^m \in \mathbb{C}[x]$ *kompleksni polinom, ki je nenegativen na realni osi* $\mathbb{R}$. *Potem obstaja tak kompleksni polinom* $g(x) = \displaystyle\sum_{m=0}^{N} g_m x^m \in \mathbb{C}[x]$, *da velja* $f(x) = g(x)^* g(x)$.

Obstaja tudi ekvivalentna verzija izreka 1, ki karakterizira Laurentove polinome, nenegativne na enotski kompleksni krožnici $\mathbb{T}$, in se imenuje *Fejér-Rieszov izrek* [Fej16]. *Kompleksni Laurentov polinom* je izraz oblike $a(z) = \sum_{m=-N_1}^{N_2} a_m z^m$, kjer je $N_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2$, in $a_m \in \mathbb{C}$, $m = -N_1, \ldots, N_2$. Množico vseh kompleksnih Laurentovih polinomov označimo z $\mathbb{C}[z, \frac{1}{z}]$ in opremimo z *involucijo* $*$, definirano s predpisom $\left(\sum_{m=-N_1}^{N_2} a_m z^m\right)^* := \sum_{m=-N_1}^{N_2} \overline{a_m} z^{-m}$.

Izrek 2 (glej Theorem 2.1.2). *Naj bo* $a(z) = \displaystyle\sum_{m=-N}^{N} a_m z^m \in \mathbb{C}\left[z, \frac{1}{z}\right]$ *kompleksni Laurentov polinom, ki je nenegativen na enotski kompleksni krožnici. Potem obstaja tak kompleksni polinom* $b(z) = \displaystyle\sum_{m=0}^{N} b_m z^m \in \mathbb{C}[z]$, *da velja* $a(z) = b(z)^* b(z)$.

Naj bo n naravno število. *Matrični polinom* velikosti n je $n \times n$ matrika z elementi iz $\mathbb{C}[x]$. Množico matričnih polinomov velikost n označimo z $M_n(\mathbb{C}[x])$ in jo opremimo z *involucijo* $*$, ki je na matričnem polinomu $F(x) := [f_{ij}(x)]_{i,j=1}^n \in M_n(\mathbb{C}[x])$ definirana kot $F(x)^* = [f_{ji}(x)^*]_{i,j=1}^n$. Pravimo, da je $F(x) \in M_n(\mathbb{C}[x])$ *hermitski*, če zadošča $F(x) = F(x)^*$. Hermitski polinom $F(x)$ je *pozitivno definiten* (oz. *pozitivno semidefiniten*) v $x_0 \in \mathbb{C}$, če je $v^* F(x_0) v > 0$ (oz. $v^* F(x_0) v \geq 0$) za vsak neničelen vektor $v \in \mathbb{C}^n \setminus \{0\}$. Leta 1966 je Popov [Pop66] izrek 1 posplošil na matrične polinome.

Izrek 3 (glej Theorem 2.1.4). *Naj bo* $F(x) = \displaystyle\sum_{m=0}^{2N} F_m x^m$, $F_m \in M_n(\mathbb{C})$, *matrični polinom velikosti* n, *ki je pozitivno semidefiniten na* $\mathbb{R}$. *Potem obstaja tak matrični polinom* $G(x) = \displaystyle\sum_{m=0}^{N} G_m x^m$, $G_m \in M_n(\mathbb{C})$, *da velja* $F(x) = G(x)^* G(x)$.

Množico realnih polinomov označimo z $\mathbb{R}[x]$. Naj bo $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ njena podmnožica. *Zaprta semialgebraična množica prirejena* S je definirana kot

$$K_S = \{x \in \mathbb{R} \colon g_j(x) \geq 0,\ j = 1, \ldots, s\} \subseteq \mathbb{R}.$$

Opazimo, da so zaprte semialgebraične množice v $\mathbb{R}$ natanko unije zaprtih intervalov. Z množico S generiramo *n-ti matrični kvadratni modul* kot

$$M_S^n := \left\{ G_0^* G_0 + \sum_{j=1}^{s} G_j^* G_j \cdot g_j \colon G_j \in M_n(\mathbb{C}[x]),\ j = 0, \ldots, s \right\} \subseteq M_n(\mathbb{C}[x]),$$

in *n-to matrično predureditev*

$$T^n_S := \left\{ \sum_{e\in\{0,1\}^s} G_e^* G_e \cdot \underline{g}^e \colon G_e \in M_n(\mathbb{C}[x]) \text{ za vsak } e \in \{0,1\}^s \right\} \subseteq M_n(\mathbb{C}[x]),$$

kjer je $e := (e_1, \ldots, e_s)$ in $\underline{g}^e$ označuje polinom $g_1^{e_1} \cdots g_s^{e_s}$. Opazimo, da je $T^n_S = M^n_{\prod S}$, kjer je $\prod S := \{\underline{g}^e \colon e \in \{0,1\}^s\}$ množica vseh možnih produktov polinomov iz S. S $\mathrm{Pos}^n_{\succeq 0}(K_S)$ označimo množico vseh hermitskih matričnih polinomov velikosti n, ki so pozitivno semidefinitni v vsaki točki iz K_S. Pravimo, da je množica M^n_S (oz. T^n_S) *nasičena*, če je $M^n_S = \mathrm{Pos}^n_{\succeq 0}(K_S)$ (oz. $T^n_S = \mathrm{Pos}^n_{\succeq 0}(K_S)$).

Za vsako semialgebraično množico $K \subseteq \mathbb{R}$ obstaja naravna izbira množice $S \subset \mathbb{R}[x]$, ki zadošča $K = K_S$. Za $a, b \in \mathbb{R}$, $a < b$, z (a, b) označimo odprt interval $\{x \in \mathbb{R} \colon a < x < b\}$. Končna podmnožica $S \subset \mathbb{R}[x]$ je *naravni opis* za K, če zadošča naslednjim pogojem:

(a) Če ima K najmanjši element a, potem je $x - a \in S$.

(b) Če ima K največji element b, potem je $b - x \in S$.

(c) Za vsaka taka $a \neq b \in K$, ki zadoščata $(a, b) \cap K = \emptyset$, je $(x - a)(x - b) \in S$.

(d) Polinomi iz točk (a), (b), (c) so edini elementi množice S.

Razširitev izreka 1 na poljubno zaprto semialgebraično množico $K \subseteq \mathbb{R}$ sta leta 2002 dokazala Kuhlmannova in Marshall [KM02, Theorem 2.2].

Izrek 4 (glej Theorem 2.1.7)**.** *Naj bo $K \subset \mathbb{R}$ neprazna zaprta semialgebraična množica in $S \subset \mathbb{R}[x]$ taka končna podmnožica realnih polinomov, da je $K_S = K$. Če je S naravni opis množice K, potem je predureditev T^1_S nasičena. Še več, če K ni kompaktna, potem je T^1_S nasičena natanko tedaj, ko S vsebuje pozitivni večkratnik vsakega polinoma iz naravnega opisa za K.*

Drugi del izreka 4 karakterizira nasičene predureditve T^1_S v primeru nekompaktne semialgebraične množice K. Obstaja pa tudi karakterizacija v kompaktnem primeru. Končna podmnožica $S \subset \mathbb{R}[x]$ je *nasičen opis* kompaktne semialgebraične množice K natanko tedaj, ko je $K = K_S$ in veljata naslednja pogoja:

(a) Za vsako levo krajišče $x_j \in K$ obstaja tak polinom $g \in S$, da je $g(x_j) = 0$ in $g'(x_j) > 0$.

(b) Za vsako desno krajišče $y_j \in K$ obstaja tak polinom $h \in S$, da je $h(y_j) = 0$ in $h'(y_j) < 0$.

Leta 2003 je Scheiderer (glej [Sch03, Theorem 5.17] in [Sch05, Corollary 4.4]) v kompaknem primeru dokazal naslednjo karakterizacijo nasičenih kvadratnih modulov.

Izrek 5 (glej Theorem 2.1.10)**.** *Naj bo $K \subset \mathbb{R}$ neprazna kompaktna semialgebraična množica in $S \subset \mathbb{R}[x]$ taka končna podmnožica realnih polinomov, da je $K_S = K$. Potem velja:*

(1) $T^1_S = M^1_S$.

(2) Kvadratni modul M_S^1 je nasičen natanko tedaj, ko je S nasičen opis za K.

Izrek 5 (1) pove, da je kvadratni modul M_S^1, generiran s tako množico $S \subset \mathbb{R}[x]$, da je $K_S \subseteq \mathbb{R}$ kompaktna, zaprt za množenje.

V primeru, ko je semialgebraična množica K interval, sta matrični analog izreka 4 leta 2002 izpeljala Dette and Studden [DS02].

Izrek 6 (glej Theorem 2.1.14)**.** *Kvadratna modula $M^n_{\{x,1-x\}}$ in $M^n_{\{x\}}$ sta nasičena za vsako naravno število $n \in \mathbb{N}$.*

Izrek 6 motivira vprašanje, ali matrični analog izreka 4 obstaja za poljubno zaprto semialgebraično množico K. To je glavni problem poglavja 2.

V razdelku 2.2 dokažemo glavni rezultat tega poglavja, ki izrek 5 razširi na matrične polinome.

Izrek 7 (Kompaktni Positivstellensatz; glej Theorem 2.2.1)**.** *Naj bo $K \subseteq \mathbb{R}$ neprazna kompaktna semialgebraična množica in $S \subset \mathbb{R}[x]$ taka končna podmnožica realnih polinomov, da je $K_S = K$. Naj bo $n \in \mathbb{N}$ naravno število. Potem je n-ti matrični kvadratni modul M_S^n nasičen natanko tedaj, ko je S nasičen opis množice K.*

Glavni elementi v dokazu izreka 7 so:

(1) primer $n = 1$ (glej izrek 5 zgoraj);

(2) *"hF-trditev"* (glej Proposition 2.2.2);

(3) odprava h v "hF-trditvi".

"hF-trditev" je dokazana v podrazdelku 2.2.1 s pomočjo predelave Schurovih komplementov in uporabe izreka 5. Odprava h v "hF-trditvi" je predstavljena v podrazdelku 2.2.2, za kar potrebujemo izrek 5.

V razdelku 2.3 dokažemo, da se izrek 4 ne razširi na poljubne neomejene zaprte semialgebraične množice, saj obstajajo protiprimeri.

Izrek 8 (glej Theorem 2.3.1)**.** *Naj bo $K \subseteq \mathbb{R}$ neomejena zaprta semialgebraična množica in $K_1, \ldots, K_r$ njene komponente za povezanost. Naj velja eden od naslednjih pogojev:*

(1) Obstajata taki števili $i, j \in \{1, \ldots, r\}$, da imata K_i in K_j neprazni notranjosti in je K_i omejena, K_j pa neomejena.

(2) Komponente so vsaj 3, pri čemer je natanko ena od njih neomejen interval, preostale pa so točke.

(3) Komponente so vsaj 4, pri čemer sta natanko dve neomejena intervala, preostale pa so točke.

Potem ne obstaja taka končna množica $S \subset \mathbb{R}[x]$, da je $K_S = K$ in je matrična predureditev T_S^2 nasičena.

V razdelku 2.4 se osredotočimo na vprašanje stopenj sumandov v nasičenih matričnih predureditvah T_S^n, generiranih z naravnim opisom S semialgebraične množice K. Naslednji izrek pove, da je v primeru disjunktne unije dveh neomejenih intervalov K predureditev T_S^n nasičena, stopnje sumandov pa so najboljše možne. Pravimo, da je matrični polinom $F(x) = \sum_{m=0}^{N} F_m x^m$, $N \in \mathbb{N} \cup \{0\}$, $F_m \in M_n(\mathbb{C})$, *stopnje* N, če je $F_N \neq 0$ in pišemo $\deg F = M$.

Izrek 9 (glej Theorem 2.4.2)**.** *Naj bosta dani realni števili $a, b \in \mathbb{R}$, $a < b$, in naj bo $K = (-\infty, a] \cup [b, \infty)$ disjunktna unija dveh neomejenih intervalov. Naj bo $n \in \mathbb{N}$ poljubno naravno število. Potem za vsak hermitski matrični polinom $F \in M_n(\mathbb{C}[x])$, ki je pozitivno semidefiniten v vsaki točki iz K, obstajata taka matrična polinoma $G, H \in M_n(\mathbb{C}[x])$, da je*

$$F = G^*G + H^*H \cdot (x-a)(x-b),$$

pri čemer je $\deg G \leq \frac{\deg F}{2}$ *in* $\deg H \leq \frac{\deg F}{2} - 1$.

V razdelku 2.4 obravnavamo tudi meje na stopnje sumandov v predureditvi za končne množice K. Naj bo $S = \{g_1, \ldots, g_s\} \subset \mathbb{R}[x]$ končna množica polinomov. Za $d \in \mathbb{N} \cup \{0\}$ definiramo *d-ti del n-te matrične predureditve T_S^n* s predpisom

$$T_{S,d}^n := \left\{ \sum_{e \in \{0,1\}^s} G_e^* G_e \cdot \underline{g}^e \colon G_e \in M_n(\mathbb{C}[x]) \text{ in } \deg(G_e^* G_e \cdot \underline{g}^e) \leq d \ \ \forall e \in \{0,1\}^s \right\}.$$

Naslednja trditev pove, v katerem delu predureditve leži matrični polinom F iz matrične predureditve T_S^n, kjer je S naravni opis končne množice K.

Trditev 10 (glej Proposition 2.4.3)**.** *Naj bo $K = \bigcup_{j=1}^m \{x_j\} \subseteq \mathbb{R}$ disjunktna unija m točk z naravnim opisom S in $n \in \mathbb{N}$ naravno število. Potem za vsak hermitski matrični polinom $F \in M_n(\mathbb{C}[x])$, ki je pozitivno semidefiniten v vsaki točki iz K, velja $F \in T_{S,k}^n$, kjer je k večje izmed števil* $\deg(F)$ *in* $m - 1$.

Če so v množici K vsaj štiri točke, potem stopnja matričnega polinoma v trditvi 10 ne zadošča (glej Example 2.4.5). Za preostale množice K je vprašanje stopenj še vedno odprto.

V razdelku 2.5 se osredotočimo na kompleksne Laurentove matrične polinome. Naj bo n naravno število. Elemente množice $M_n(\mathbb{C}[z, \frac{1}{z}])$ vseh $n \times n$ matrik nad $\mathbb{C}[z, \frac{1}{z}]$ imenujemo *kompleksni Laurentovi matrični polinomi*. Množico $M_n(\mathbb{C}[z, \frac{1}{z}])$ opremimo z *involucijo* $*$, ki je na $A(z) := [a_{ij}(z)]_{i,j=1}^n \in M_n(\mathbb{C}[z, \frac{1}{z}])$ definirana kot $A(z)^* = [a_{ji}(z)^*]_{i,j=1}^n$. Pravimo, da je $A(z) \in M_n(\mathbb{C}[z, \frac{1}{z}])$ *hermitski*, če zadošča $A(z) = A(z)^*$. Naj bo $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$ enotska kompleksna krožnica. Hermitski polinom $A(z)$ je *pozitivno definiten* (oz. *pozitivno semidefiniten*) v $z_0 \in \mathbb{T}$, če je $v^* A(z_0) v > 0$ (oz. $v^* A(z_0) v \geq 0$) za vsak neničelen vektor $v \in \mathbb{C}^n \setminus \{0\}$.

Leta 1958 je Rosenblatt [Ros58] izrek 2 posplošil na Laurentove matrične polinome, pozitivno definitne na $\mathbb{T}$, leta 1964 pa je Helson [Hel64] predpostavko pozitivne definitnosti nadomestil s šibkejšo predpostavko pozitivne semidefinitnosti.

Izrek 11 (glej Theorem 2.1.3)**.** *Naj bo* $A(z) = \sum_{m=-N}^{N} A_m z^m$, $A_m \in M_n(\mathbb{C})$, *hermitski Laurentov matrični polinom, ki je pozitivno semidefiniten na* $\mathbb{T}$. *Potem obstaja tak matrični polinom* $B(z) = \sum_{m=0}^{N} B_m z^m$, $B_m \in M_n(\mathbb{C})$, *da velja* $A(z) = B(z)^* B(z)$.

Naj bo $\mathcal{S} = \{b_1, \ldots, b_s\} \subset \mathbb{C}[z, \frac{1}{z}]$ podmnožica hermitskih polinomov. *Zaprta semialgebraična množica prirejena* $\mathcal{S}$ je definirana kot

$$\mathscr{K}_{\mathcal{S}} = \{z \in \mathbb{T} \colon b_j(z) \geq 0,\ j = 1, \ldots, s\} \subseteq \mathbb{T}.$$

Opazimo, da so zaprte semialgebrične množice v $\mathbb{T}$ natanko unije zaprtih lokov. Z množico $\mathcal{S}$ generiramo *n-ti matrični kvadratni modul* kot

$$\mathcal{M}_{\mathcal{S}}^n := \left\{ A_0^* A_0 + \sum_{j=1}^{s} A_j^* A_j \cdot b_j \colon A_j \in M_n(\mathbb{C}[z]),\ j = 0, \ldots, s \right\} \subseteq M_n\left(\mathbb{C}\left[z, \frac{1}{z}\right]\right).$$

Naj bo $\mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{\mathcal{S}})$ množica vseh hermitskih Laurentovih matričnih polinomov velikosti n, ki so pozitivno semidefinitni v vsaki točki iz $\mathscr{K}_{\mathcal{S}}$. Pravimo, da je množica $\mathcal{M}_{\mathcal{S}}^n$ *nasičena*, če je $\mathcal{M}_{\mathcal{S}}^n = \mathrm{Pos}^n_{\succeq 0}(\mathscr{K}_{\mathcal{S}})$.

Končna podmnožica $\mathcal{S} \subset \mathbb{C}\left[z, \frac{1}{z}\right]$ hermitskih polinomov je *nasičen opis* zaprte semialgebraične množice $\mathscr{K} \subseteq \mathbb{T}$ natanko tedaj, ko veljajo naslednji pogoji:

(a) $\mathscr{K} = \mathscr{K}_{\mathcal{S}}$.

(b) Za vsako robno točko $a \in \mathscr{K}$, ki ni izolirana, obstaja tak $b \in \mathcal{S}$, da je $b(a) = 0$ and $\frac{db}{dz}(a) \neq 0$.

(c) Za vsako izolirano točko $a \in \mathscr{K}$, obstaja taka $b_1, b_2 \in \mathcal{S}$, da je $b_1(a) = b_2(a) = 0$, $\frac{db_1}{dz}(a) \neq 0$, $\frac{db_2}{dz}(a) \neq 0$ in na neki okolici točke a velja $b_1 b_2 \leq 0$.

Naslednji izrek je razširitev izreka 11 na poljubno zaprto semialgebraično množico v $\mathbb{T}$.

Izrek 12 (glej Theorem 2.5.4)**.** *Naj bo* $\mathscr{K} \subseteq \mathbb{T}$ *neprazna zaprta semialgebraična množica in* $\mathcal{S} \subset \mathbb{C}[z, \frac{1}{z}]$ *taka končna podmnožica hermitskih polinomov, da je* $\mathscr{K} = \mathscr{K}_{\mathcal{S}}$. *Naj bo* $n \in \mathbb{N}$ *naravno število. Potem je n-ti matrični kvadratni modul* $\mathcal{M}_{\mathcal{S}}^n$ *nasičen natanko tedaj, ko je* $\mathcal{S}$ *nasičen opis množice* $\mathscr{K}$.

Pred dokazom izreka 12 sprva s pomočjo Möbiusovih transformacij v podrazdelku 2.5.1 izpeljemo povezave med semialgebraičnimi množicami v $\mathbb{T}$ in $\mathbb{R}$, njihovimi nasičenimi opisi, in matričnimi polinomi $M_n(\mathbb{C}[z, \frac{1}{z}])$ in $M_n(\mathbb{C}[x])$. Nato te povezave uporabimo v podrazdelku 2.5.2, da izpeljemo analog "hF-trditve" za matrične Laurentove polinome (glej Proposition 2.5.5). Na koncu s pomočjo Scheidererjevega rezultata (glej Proposition 2.5.6) odpravimo imenovalce.

V razdelku 2.6 se ponovno osredotočimo na matrične polinome iz $M_n(\mathbb{C}[x])$ in dokažemo Positivstellensatz za neomejeno zaprto semialgebraično množico v $\mathbb{R}$.

Izrek 13 (Nekompaktni Positivstellensatz; glej Theorem 2.6.1)**.** *Naj bo $K \subset \mathbb{R}$ prava neomejena zaprta semialgebraična množica in S njen naravni opis. Naj bo $n \in \mathbb{N}$ naravno število. Potem so za vsak hermitski matrični polinom $F \in M_n(\mathbb{C}[x])$ naslednje trditve ekvivalentne:*

(1) F je pozitivno semidefiniten v vsaki točki $x_0 \in K$.

(2) Za vsako točko $w \in \mathbb{C} \setminus K$ obstaja tako nenegativno celo število $k_w \in \mathbb{N} \cup \{0\}$, da velja $|x-w|^{2k_w} \cdot F \in M_S^n$.

(3) Obstaja tako nenegativno celo število $k \in \mathbb{N} \cup \{0\}$, da velja $(1+x^2)^k \cdot F \in M_S^n$.

Da dokažemo izrek 13, sprva v podrazdelku 2.6.1 izpeljemo obratne povezave povezav iz podrazdelka 2.5.1. To nam omogoči, da v dokazu izreka 13 uporabimo izrek 12. Zanimiva posledica izreka 13 je dejstvo, da obstaja imenovalec, ki je do eksponenta natančno ustrezen za vse množice K in vse matrične polinome F, ki zadoščajo predpostavkam izreka.

Na koncu poglavja v razdelku 2.7 na kratko razložimo, kako izrek 7 razširiti na krivulje v $\mathbb{R}^d$ (glej Theorem 2.7.5).

Positivstellensatzi na matrično konveksnih množicah

V poglavju 3 študiramo algebraična zagotovila za pozitivnost nekomutativnih operatorskih polinomov na matrično konveksnih množicah.

V razdelku 3.1 uvedemo potrebne definicije in predstavimo znana zagotovila pozitivnosti. Naj bo $\mathscr{H}$ separabilen realen Hilbertov prostor in $I_{\mathscr{H}}$ identični operator na njem. Z $B(\mathscr{H})$ označimo množico vseh omejenih linearnih operatorjev na $\mathscr{H}$, z $\mathbb{S}_{\mathscr{H}}$ pa množico vseh sebi adjungiranih opertorjev iz $B(\mathscr{H})$. *Linearni operatorski šop* je izraz oblike

$$L(x) = A_0 + \sum_{j=1}^{g} A_j x_j,$$

kjer so $A_0, A_1, \dots, A_g \in \mathbb{S}_{\mathscr{H}}$ sebi adjungirani operatorji. Če je $\dim \mathscr{H} = n \in \mathbb{N}$, potem $B(\mathscr{H})$ identificiramo z realnimi matrikami $M_n(\mathbb{R})$, $L(x)$ pa imenujemo *linearni matrični šop.* V primeru $A_0 = I_{\mathscr{H}}$, je L *eničen.* Če pa je $A_0 = 0$, je L *homogen.*

Pravimo, da je $A \in B(\mathscr{H})$ *pozitivno semidefiniten* in pišemo $A \succeq 0$, če je A sebi adjungiran in velja $\langle Ah, h\rangle_{\mathscr{H}} \geq 0$ za vsak $h \in \mathscr{H}$, pri čemer je $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ skalarni produkt na $\mathscr{H}$. Naj bo $\mathscr{K}$ realen Hilbertov prostor. Če definiramo

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle_{\mathscr{H} \otimes \mathscr{K}} := \langle h_1, h_2 \rangle_{\mathscr{H}} \langle k_1, k_2 \rangle_{\mathscr{K}},$$

in razširimo po linearnosti, potem dobimo skalarni produkt na vektorskem prostoru $\mathscr{H} \otimes \mathscr{K}$. Napolnitev $\mathscr{H} \otimes \mathscr{K}$ v tem skalarnem produktu je Hilbertov prostor, ki ga še vedno označimo z $\mathscr{H} \otimes \mathscr{K}$. Za operatorja $A \in B(\mathscr{H})$ in $B \in B(\mathscr{K})$ definiramo

$$(A \otimes B)(h \otimes k) := (Ah) \otimes (Bk),$$

in razširimo po linearnosti do operatorja $A \otimes B \in B(\mathscr{H} \otimes \mathscr{K})$.

Z $\mathbb{S}_n$ označimo množico simetričnih realnih matrik velikosti $n \in \mathbb{N}$. Naj bo $L(x) = A_0 + \sum_{j=1}^{g} A_j x_j$ linearni šop kot zgoraj in $X = (X_1, \ldots, X_g) \in \mathbb{S}_n^g$ terica simetričnih matrik. *Ovrednotenje* $L(X)$ je definirano kot

$$L(X) = A_0 \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j,$$

kjer je I_n identična matrika velikosti n. *Prost Hilbertov spektraeder* je zaporedje množic

$$D_L = (D_L(n))_n, \quad \text{kjer je} \quad D_L(n) = \{X \in \mathbb{S}_n^g \colon L(X) \succeq 0\}.$$

Množico $D_L(1)$ imenujemo *Hilbertov spektraeder*. Če je L linearni matrični šop, potem zaporedje D_L imenujemo *prost spektraeder*, množico $D_L(1)$ pa *spektraeder*.

Prvi pomemben problem študija linearnih šopov je karakterizacija vsebovanosti $D_{L_1} \subseteq D_{L_2}$, kjer sta L_j, $j = 1, 2$, linearna šopa. Za enične matrične šope so karakterizacijo našli Helton, Klep in McCullough v [HKM12, HKM13b].

Izrek 14 (glej Theorem 3.1.1). *Naj bosta* $L_j = I_{d_j} + \sum_{k=1}^{g} A_{j,k} x_k$, $j = 1, 2$, $d_j \in \mathbb{N}$, $A_{j,k} \in \mathbb{S}_{d_j}$, *enična linearna matrična šopa. Potem je* $D_{L_1} \subseteq D_{L_2}$ *natanko tedaj, ko obstajajo tako naravno število* $k_0 \in \mathbb{N}$, *matrike* $V_k \in \mathbb{R}^{d_1 \times d_2}$, $k = 1, \ldots, k_0$, *in pozitivno semidefinitna matrika* $S \in \mathbb{S}_{d_2}$, *da velja*

$$L_2 = S + \sum_{k=1}^{k_0} V_k^* L_1 V_k.$$

Še več, če je spektraeder $D_{L_1}(1)$ *omejen, potem obstajajo take matrike* V_k, *da je* $\sum_{k=1}^{k_0} V_k^* V_k = I_{d_2}$ *in je* S *ničelna matrika.*

Tu se pojavi naravno vprašanje, ali je možno zgornji izrek posplošiti iz matričnih šopov na operatorske. To vprašanje obravnavamo v razdelku 3.2. Odgovor nanj je pritrdilen in je vsebina naslednjega izreka.

Izrek 15 (Linearni Positivstellensatz; glej Theorem 3.2.13). *Naj bosta* $\mathscr{H}_j$, $j = 1, 2$, *separabilna realna Hilbertova prostora in* $L_j = I_{\mathscr{H}_j} + \sum_{k=1}^{g} A_{j,k} x_k$, $j = 1, 2$, $A_{j,k} \in \mathbb{S}_{\mathscr{H}_j}$, *linearna operatorska šopa. Naslednje trditve so ekvivalentne:*

(1) $D_{L_1} \subseteq D_{L_2}$.

(2) Naj bo $\widetilde{\mathcal{C}}$ *najmanjša unitalna* C^**-algebra v* $B(\mathscr{H}_1 \oplus \mathbb{R})$, *ki vsebuje operatorje*

$$\begin{bmatrix} A_k & 0 \\ 0 & 1 \end{bmatrix}, \quad k = 1, \ldots, g.$$

Obstajajo tak separabilen realen Hilbertov prostor $\mathscr{K}$, *izometrija* $V : \mathscr{H}_2 \to \mathscr{K}$ *in unitalni* $*$*-homomorfizem* $\pi : \widetilde{\mathcal{C}} \to B(\mathscr{K})$, *da velja*

$$L_2 = V^* \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) V + V^* \pi\left(\begin{bmatrix} L_1 & 0 \\ 0 & 0 \end{bmatrix}\right) V.$$

(3) *Naj bo $\mathcal{C}$ najmanjša unitalna C^*-algebra v $B(\mathscr{H}_1)$, ki vsebuje operatorje A_k, $k = 1, \ldots, g$. Obstajajo tak separabilen realen Hilbertov prostor $\mathscr{K}_0$, skrčitev $V_0 : \mathscr{H}_2 \to \mathscr{K}_0$, unitalni $*$-homomorfizem $\pi_0 : \mathcal{C} \to B(\mathscr{K}_0)$ in pozitivno semidefiniten operator $S \in B(\mathscr{H}_2)$, da velja*

$$L_2 = S + V_0^* \pi_0(L_1) V_0.$$

Dokaz izreka 15 je predstavljen v podrazdelku 3.2.1. Glavne uporabljene tehnike so podobne kot v [HKM13b], to sta popolna pozitivnost in teorija operatorskih algeber. Definiramo unitalno $*$-linearno preslikavo τ, ki slika med linearnima ogrinjačama koeficientov danih linearnih šopov in povežemo vsebovanost $D_{L_1} \subseteq D_{L_2}$ in popolno pozitivnost preslikave τ (glej Theorem 3.2.5). Potem izrek 15 sledi z uporabo realne verzije Arvesonovega izreka o razširitvi in Stinespringovega reprezentacijskega izreka. Kot posledico izreka 15 v podrazdelku 3.2.2 opišemo vse terice $(A_{2,1}, \ldots, A_{2,g})$, ki za dano terico $(A_{1,1}, \ldots, A_{1,g})$, zadoščajo trditvi (1) izreka 15.

Drugi pomemben problem študija linearnih šopov je karakterizacija enakosti $D_{L_1} = D_{L_2}$, kjer sta L_j, $j = 1, 2$, linearna šopa. Za enične matrične šope z omejenimi spektraedri so karakterizacijo našli Helton, Klep in McCullough [HKM13b, Theorem 1.2].

Izrek 16 (glej Theorem 3.1.2). *Naj bosta $L_j = I_{d_j} + \sum_{k=1}^{g} A_{j,k} x_k$, $j = 1, 2$, $d_j \in \mathbb{N}$, $A_{j,k} \in \mathbb{S}_{d_j}$, taka enična linearna matrična šopa, da sta spektraedra $D_{L_j}(1)$ omejena. Potem sta naslednji trditvi ekvivalentni:*

(1) $D_{L_1} = D_{L_2}$.

(2) *Naj bosta $H_j \subseteq \mathbb{R}^{d_j}$, $j = 1, 2$, podprostora, ki zadoščata naslednjim pogojem:*

  (a) *H_j je invarianten za vsako matriko $A_{j,k}$, $k = 1, \ldots, g$.*

  (b) *$D_{L_j|_{H_j}} = D_{L_j}$, kjer je $L_j|_{H_j}$ zožitev šopa L_j na podprostor H_j.*

  (c) *Ne obstaja pravi podprostor $H_j' \subset H_j$, ki zadošča (2a) in (2b).*

Potem obstaja taka unitarna matrika $U : H_2 \to H_1$, da za vsak $k = 1, \ldots, g$ velja

$$A_{2,k}|_{H_2} = U^* A_{1,k}|_{H_1} U.$$

Tu se pojavita dve vprašanji. Prvo vprašanje je, ali je predpostavka omejenosti spektraederov $D_{L_j}(1)$ v zgornjem izreku potrebna. Drugo vprašanje pa je, ali je možno zgornji izrek posplošiti iz matričnih šopov na operatorske. Glavni rezultat razdelka 3.3. je posplošitev iz omejenih spektraedrov na neomejene in iz matričnih šopov na operatorske, katerih koeficienti (razen identičnega operatorja) so kompaktni operatorji.

Izrek 17 (Linearni Gleichstellensatz; glej Theorem 3.3.1). *Naj bosta $\mathscr{H}_j$, $j = 1, 2$, separabilna realna Hilbertova prostora in $L_j(x) = I_{\mathscr{H}_j} + \sum_{k=1}^{g} A_{j,k} x_k$, $j = 1, 2$, linearna operatorska šopa, kjer so $A_{j,k} \in \mathbb{S}_{\mathscr{H}_j}$ sebi adjungirani kompaktni operatorji. Potem sta naslednji trditvi ekvivalentni:*

(1) $D_{L_1} = D_{L_2}$.

(2) Naj bosta $H_j \subseteq \mathscr{H}_1$, $j = 1, 2$, *zaprta podprostora, ki zadoščata naslednjim pogojem:*

(a) H_j *je invarianten za vsak operator* $A_{j,k}$, $k = 1, \ldots, g$.

(b) $D_{L_j|_{H_j}} = D_{L_j}$, *kjer je* $L_j|_{H_j}$ *zožitev šopa* L_j *na podprostor* H_j.

(c) Ne obstaja pravi zaprt podprostor $H'_j \subset H_j$, *ki zadošča (2a) in (2b).*

Potem obstaja tak unitarni operator $U : H_2 \to H_1$, *da za vsak* $k = 1, \ldots, g$ *velja*

$$A_{2,k}|_{H_2} = U^* A_{2,k}|_{H_1} U.$$

Glavna tehnika v dokazu izreka 17 je razumevanje unitalnih C^*-algeber, generiranih s koeficienti $A_{j,k}|_{H_j}$, $k = 1, \ldots, g$, in izomorfizmov med takimi C^*-algebrami. Ključno opažanje za razširitev iz omejenih na neomejene spektraedre je povezava med tericami $(A_{j,1}|_{H_j}, \ldots, A_{j,g}|_{H_j})$ in $(A_{j,1}|_{H_j} \oplus \mathbf{0}_{\mathbb{R}}, \ldots, A_{j,g}|_{H_j} \oplus \mathbf{0}_{\mathbb{R}})$, kjer je $\mathbf{0}_{\mathbb{R}}$ ničelni operator na $\mathbb{R}$ (glej Proposition 3.3.7). Obstoj in karakterizacija zaprtih podprostorov H_j, $j = 1, 2,$, ki zadoščata predpostavkam (2a)-(2c) v izreku 17, sta dokazana v podrazdelku 3.3.2 (glej Corollary 3.3.16 in Corollary 3.3.20). Nato v podrazdelkih 3.3.3 in 3.3.4 študiramo, ali se izrek 17 posploši na operatorske šope, ki nimajo kompaktnih koeficientov. V podrazdelku 3.3.3 dokažemo, da podprostora H_j, $j = 1, 2$, ki zadoščata predpostavkam (2a)-(2c) v izreku 17, ne obstajata vedno (glej Example 3.3.21). Toda tudi če taka podprostora obstajata, se zaključek izreka 17 ne posploši na šope z nekompaktnimi koeficienti (glej Example 3.3.22 v podrazdelku 3.3.4).

Tretji zanimiv problem študija linearnih šopov je vprašanje, kako posplošiti karakterizacijo vsebovanosti $D_{L_1} \subseteq D_{L_2}$, kjer sta L_j, $j = 1, 2$, linearna šopa, na primer poljubnega nekomutativnega operatorskega polinoma L_2. Z $\mathbb{R}\langle x \rangle := \mathbb{R}\langle x_1, \ldots, x_g \rangle$ označimo množico polinomov v nekomutativnih spremenljivkah x_1, $\ldots$, x_g, s koeficienti v $\mathbb{R}$. Naj bosta $\mathscr{H}_j, j = 1, 2$, Hilbertova prostora in $B(\mathscr{H}_1, \mathscr{H}_2)$ množica vseh omejenih linearnih operatorjev iz $\mathscr{H}_1$ v $\mathscr{H}_2$. Posebej, množici $\mathbb{R}\langle x \rangle$ in $B(\mathscr{H}_1, \mathscr{H}_2)$ sta $\mathbb{R}$-modula. *Nekomutativni (nk) operatorski polinom* je element $\mathbb{R}$-modula $B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle$, opremljenega z *involucijo* $*$, ki je trivialna na $\mathbb{R}$, obrača vrstni red spremenljivk in je običajni adjungirani operator na $B(\mathscr{H}_1, \mathscr{H}_2)$. Če je $\dim \mathscr{H}_j = \nu_j \in \mathbb{N}$, $j = 1, 2$, potem elemente iz $B(\mathscr{H}_1, \mathscr{H}_2)$ identificiramo z množico $M_{\nu_2 \times \nu_1}(\mathbb{R})$ realnih $\nu_2 \times \nu_1$ matrik, elementov iz $M_{\nu_2 \times \nu_1}(\mathbb{R}) \otimes \mathbb{R}\langle x \rangle$ pa pravimo *nk matrični polinomi.*

Naj bo $\mathscr{K}$ realen Hilbertov prostor, $n \in \mathbb{N}$ naravno število in I_n identična matrika velikosti n. Nc operatorski polinom $F \in B(\mathscr{H}_1, \mathscr{H}_2) \otimes \mathbb{R}\langle x \rangle$ ovrednotimo na terici $X := (X_1, \ldots, X_g) \in \mathbb{S}_n^g$ simetričnih $n \times n$ matrik na naraven način, tako da zamenjamo spremenljivke x_i z matrikami X_i, prosti člen $F_0 \otimes 1$ pa spremenimo v $F_0 \otimes I_n$. Tako $F(X)$ postane element iz $B(\mathscr{H}_1, \mathscr{H}_2) \otimes M_n(\mathbb{R})$.

Naj bo sedaj $\mathscr{H}$ realni Hilbertov prostor in $F \in B(\mathscr{H}) \otimes \mathbb{R}\langle x \rangle$ nc operatorski polinom. *Prosta Hilbertova semialgebraična množica generirana z* F je zaporedje množic

$$D_F = (D_F(n))_n, \quad \text{kjer je} \quad D_F(n) = \{X \in \mathbb{S}_n^g \colon F(X) \succeq 0\}.$$

Helton, Klep and McCullough so v primeru matričnega šopa dokazali naslednjo posplošitev izreka 14 (glej [HKM12, Theorem 1.1]).

Izrek 18 (glej Theorem 3.1.5)**.** *Naj bo $L \in \mathbb{S}_d \otimes \mathbb{R}\langle x \rangle$, $d \in \mathbb{N}$, eničen linearni matrični šop. Za vsak tak nk matrični polinom $F \in M_\nu(\mathbb{R}) \otimes \mathbb{R}\langle x \rangle$, $\nu \in \mathbb{N}$, ki zadošča $F = F^*$ in $D_L \subseteq D_F$, obstaja končno mnogo takih nk matričnih polinomov $R_j \in M_\nu(\mathbb{R}) \otimes \mathbb{R}\langle x \rangle$ in $Q_k \in M_{d\times\nu}(\mathbb{R}) \otimes \langle x \rangle$ stopnje največ $\frac{\deg(F)}{2}$, $j_0, k_0 \in \mathbb{N}$, da velja*

$$F = \sum_{j=1}^{j_0} R_j^* R_j + \sum_{k=1}^{k_0} Q_k^* L Q_k.$$

V razdelku 3.4 izrek 18 posplošimo iz matričnega na operatorski šop.

Izrek 19 (Konveksni Positivstellensatz; glej Theorem 3.4.1)**.** *Naj bo $\mathscr{H}$ separabilen realen Hilbertov prostor in $L(x) = I_{\mathscr{H}} + \sum_{k=1}^{g} A_k x_k$ eničen linearni operatorski šop, kjer je $I_{\mathscr{H}}$ identični operator na $\mathscr{H}$ in so A_k, $k = 1, \dots, g$, sebi adjungirani operatorji na $\mathscr{H}$. Naj bo $\mathcal{C} \subseteq B(\mathscr{H})$ unitalna C^*-algebra, generirana z operatorji A_k, $k = 1, \dots, g$. Za vsak nk matrični polinom $F \in M_\nu(\mathbb{R}) \otimes \mathbb{R}\langle x \rangle$, $\nu \in \mathbb{N}$, ki zadošča $F = F^*$ in $D_L \subseteq D_F$, obstajajo tak separabilen realen Hilbertov prostor $\mathscr{K}$, $*$-homomorfizem $\pi : \mathcal{C} \to B(\mathscr{K})$, končno mnogo nk matričnih $R_j \in M_\nu(\mathbb{R}) \otimes \mathbb{R}\langle x \rangle$ in nk operatorskih polinomov $Q_k \in B(\mathbb{R}^\nu, \mathscr{K}) \otimes \mathbb{R}\langle x \rangle$ stopnje največ $\frac{1}{2} \cdot \deg(F)$, $j_0, k_0 \in \mathbb{N}$, da velja*

$$F = \sum_{j=1}^{j_0} R_j^* R_j + \sum_{k=1}^{k_0} Q_k^* \pi(L) Q_k.$$

Izrek 19 dokažemo s podobnimi tehnikami kot v [HKM12], pri čemer je ključna ideja predelava klasičnega Putinarjevega separacijskega argumenta. S pomočjo izreka 15 in verzije Hahn-Banachovega izreka [HKM16b, Theorem 2.2] lahko uporabimo separacijski argument iz [HKM12] tudi v primeru prostih Hilbertovih spektraedrov.

V razdelku 3.5 razširimo izrek 19 v primeru ene spremenljivke iz nk matričnih na nk operatorski polinome (glej Theorem 3.5.1). Glavni korak v dokazu je redukcija na primer inkluzije prostih Hilbertovih spektraedrov, kar dosežemo z uporabo verzije operatorskega Fejér-Rieszovega izreka [Ros68]. Primera Example 3.2.16 in Example 3.5.2 pokažeta, da se izrek 19 ne razširi na neenične šope.