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Minimalne ploskve in Plateaujev problem

Delo diplomskega seminarja

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Superfici minime ed il problema di Plateau

Tesi finale

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Minimal Surfaces and Plateau's Problem

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Abstract

In this thesis we study minimal surfaces and the related Plateau's problem. The first section is devoted to the introduction and to a short historical overview of the theory of minimal surfaces and of the related issues.

In the second section we give the definition of minimal surface from the variational point of view. We consider surfaces given as graphs of functions over bounded domains. The minimisation of the area yields the Euler-Lagrange equations. We give the geometric description of the Euler-Lagrange equations by means of the mean curvature.

In the third section we study the Plateau's problem, i.e. the problem of finding a minimal surface with a fixed boundary curve. In this section we consider surfaces given by a general regular parameterization. We also present some examples of the well-known minimal surfaces.

In the next two sections we show that the Gauss map of a minimal surface is conformal and that, in terms of conformal coordinates, the parameterization of a minimal surface is harmonic.

The following two sections deal with the definition and examples of the Weierstrass representation, which is a correspondence between minimal surfaces and pairs of functions. Every such pair consists of a holomorphic and a meromorphic function which satisfy a special condition concerning their zeros and poles.

In the last section we introduce the notion of perimeter, which is a generalisation of the area to a large class of hypersurfaces, including many singular hypersurfaces. We study the regularity of minimal surfaces in higher dimensions and we prove that there exist singular minimal surfaces in eight (or higher) dimensional spaces.

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Minimalne ploskve in Plateaujev problem

Razširjeni povzetek

Diplomska naloga obravnava minimalne ploskve in Plateaujev problem. Prvi razdelek je namenjen uvodu in zgodovinskemu orisu. V njem so tudi navedena področja, kjer lahko najdemo praktično uporabo problema.

V drugem razdelku pridemo do definicije minimalnih ploskev z uporabo variacijskega računa. Ploskve so dane kot grafi funkcij na omejenih območjih. Minimiziramo funkcional, ki vsaki ploskvi dodeli njeno ploščino ter z upoštevanjem teorije variacijskega računa pridemo do Euler-Lagrangevih enačb. Najprej obravnavamo le dvodimenzionalne ploskve, nato pa še $(n-1)$ dimenzionalne ploskve, kjer je n dimenzija ambientnega prostora $\mathbb{R}^n$. Sledi geometrijska interpretacija Euler-Lagrangevih enačb: dokažemo, da so Euler-Lagrangeve enačbe ekvivalentne pogoju, da je povprečna ukrivljenost enaka nič v vsaki točki ploskve.

V tretjem razdelku se srečamo s Plateaujevim problemom, to je s problemom iskanja minimalnih ploskev s fiksno robno krivuljo. V tem razdelku študiramo ploskve, podane s poljubnimi parametrizacijami. Pri obravnavi problema uporabimo variacijo ploskve. Najprej konstruiramo družine ploskev, katerih elementi so perturbacije neke dane ploskve. Te družine so parametrizirane z enim parametrom. Nato opazujemo funkcional, ki vsaki ploskvi družine dodeli njeno ploščino. Posledično je tudi funkcional odvisen le od enega parametra, kar poenostavi obravnavo. Ploskev, ki podaja iskano rešitev zadošča pogoju, da je povprečna ukrivljenost v vsaki njeni točki enaka nič. Na koncu razdelka obravnavo problema poenostavimo tako, da namesto splošne variacije ploskve upoštevamo samo normalno variacijo ploskve oz. variacijo v smeri normale, ki je pravzaprav edina relevantna smer. Sledi nekaj primerov minimalnih ploskev: katenoid, helikoid, Enneperjeva ploskev, Scherkova ploskev, Hennebergova ploskev in druge.

V naslednjih dveh razdelkih dokažemo, da je Gaussova preslikava minimalne ploskve konformna in da je parametrizacija minimalne ploskve, izražena v konformnih koordinatah, harmonična.

Naslednja dva razdelka se ukvarjata s konstrukcijo in primeri Weierstrassove reprezentacije, ki je korespondenca med minimalnimi ploskvami in pari funkcij. Vsak par sestavljata holomorfna in meromorfna funkcija, katerih ničle in poli zadoščajo posebnem pogoju. Če je minimalna ploskev parametrizirana s konformno parametrizacijo $\sigma(u, v)$, potem je vektorska funkcija $\sigma_u - i\sigma_v$ holomorfna funkcija treh komponent, ki pa jih lahko izrazimo z dvema skalarnima funkcijama, in sicer z eno holomorfno in z eno meromorfno. Vsako minimalno ploskev določata torej ti dve funkciji, ki pa nista enolično določeni. Sledijo primeri Weierstrassove reprezentacije prej predstavljenih primerov minimalnih ploskev.

V zadnjem razdelku preučujemo regularnost minimalnih ploskev tudi v višjih dimenzijah ter dokažemo eksistenco singularnih minimalnih ploskev v osem ali več dimenzionalnem prostoru. Z uporabo perimetrov, minimalnih in sub-minimalnih množic ter sub-kalibracij dokažemo, da je sedemdimenzionalni Simonsov stožec minimalna ploskev. Perimeter je posplošitev ploščine. Pokažemo kako so ti pojmi povezani in kako prek njih pridemo do dokaza minimalnosti Simonsovega stožca.

Ključne besede: ploskev, ukrivljenost, Plateau, variacija, minimalna

Superfici minime ed il problema di Plateau

Sintesi Estesa

Nella presente tesi vengono introdotte le superfici minime e il problema di Plateau. Il primo capitolo è dedicato all' introduzione dei contenuti e dei relativi problemi. Viene data anche un'introduzione storica e vengono citate alcune discipline in cui il problema trova un'appicazione pratica.

Nel secondo capitolo viene data la definizione delle superfici minime dal punto di vista del calcolo delle variazioni. Vengono considerate superfici che sono date come grafici di funzioni su un dominio limitato e viene minimizzato il funzionale che a ogni superficie associa la propria area. Usando la teoria del calcolo delle variazioni in dimensione n si ottengono le equazioni di Eulero-Lagrange. Per maggiore chiarezza i calcoli vengono svolti prima solo per superfici di dimensione due e poi il problema viene generalizzato per superfici $(n-1)$-dimensionali, dove n è la dimensione dello stazion $\mathbb{R}^n$. In seguito viene mostrata l'interpretazione geometrica delle equazioni di Eulero-Lagrange: tali equazioni sono equivalenti alla condizione che la curvatura media si annulli in ogni punto della superficie.

Il terzo capitolo tratta il problema di Plateau, ovvero il problema di trovare una superficie di area minima avente come bordo una curva chiusa nello spazio. Le superfici che si considerano sono parametrizzate. Il problema viene affrontato facendo uso della variazione di una superficie. Si considera una famiglia ad un parametro di superfici, i cui elementi sono variazioni di una data superficie. Sucessivamente si considera il funzionale che a ogni superficie di tale famiglia associa la rispettiva area; anche tale funzionale dipende da un solo parametro. Infine si verifica, che la superficie che rappresenta la soluzione cercata soddisfa la condizione della curvatura media nulla. Alla fine della sezione proponiamo una trattazione del problema in cui invece di considerare la variazione generale di una superficie consideriamo la variazione normale, cioè ottenuta variando la superficie lungo la direzione normale, che è la direzione più significativa. Seguono alcuni esempi di superfici minime: la catenoide, l'elicoide, la superficie di Enneper, quella di Scherk, quella di Henneberg e altre.

Nei due capitoli seguenti si dimostra che la mappa di Gauss di una superficie minima è conforme e che in termini di coordinate conformi la parametrizzazione di una superficie minima è armonica.

Il sesto e settimo capitolo espongono la definizione e degli esempi della rappresentazione di Weierstrass, che è una corrispondenza tra superfici minime e coppie di funzioni. Ogni coppia è costituita da una funzione olomorfa e da una meromorfa, le quali soddisfano una condizione particolare riguardante gli zeri e i poli. Si dimostra che considerando una superficie minimale parametrizzata con una parametrizzazione conforme $\sigma(u, v)$, la funzione avente forma $\sigma_u - i\sigma_v$ è una funzione olomorfa di tre componenti, le quali sono esprimibili tramite due funzioni scalari, una olomorfa e l'altra meromorfa. Quindi ogni superficie minima è caratterizzata da queste due funzioni, che non sono univoche. Seguono esempi della rappresentazione di Weierstrass per le superfici minime considerate precedentemente.

Nell'ultimo capitolo viene studiata la regolarità di una superficie minima anche in dimensioni superiori e viene dimostra l'esistenza di superfici minime singolari negli spazi otto (o più) dimensionali. Più precisamente si dimostra che il cono di Simons di sette dimensioni è una superficie minima utilizzando i perimetri, che sono

una generalizzazione dell'area, gli insiemi minimali, gli insiemi sub-minimali e le sub-calibrazioni. Dimostrando dei risultati che riguardano le relazioni tra questi concetti si ottiene la minimalità del cono di Simons.

Parole chiave: superficie, curvatura, Plateau, variazione, minima

1. Introduction

A minimal surface is a surface which has the mean curvature equal to zero at every point. The theory of minimal surfaces is closely related to problems of minimal area. We consider all the surfaces in R^3 sharing the same boundary and search for the one whose area is minimal. We find out that the surfaces, which are solutions of this problem, have zero mean curvature everywhere. This is the reason why the surfaces with zero mean curvature are called minimal surfaces.

In nature, examples of minimal surfaces can be obtained by dipping a wire frame into a soap solution: when withdrawing the frame carefully, the soap solution will assume a position, such that at the regular points the mean curvature will be equal to zero. This has a physical interpretation (explanation): a soap film has energy by virtue of surface tension, and this energy is proportional to its area. Thus, minimizing the energy is equivalent to minimizing the area. In the sequel we shall prove that the surfaces which minimize the area share a common differential geometric feature, namely, their mean curvature is everywhere equal to zero.

The converse is not always true: not all minimal surfaces with a given boundary curve γ are surfaces of minimal area.

1.1. **History of the problem.** The first mathematician who started the study of surfaces of minimal area was Leonhard Euler. His pioneering work goes back to the year 1744. In his work 'Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes' he considered the problem of finding a surface with minimal area that has, as the boundary curve, two circles in $\mathbb{R}^3$ located in two parallel planes and such that the line that goes through the centers of the two circles is orthogonal to the planes. Euler showed that the solution is a part of the catenoid.

The theory of minimal surfaces has developed from such problems. It was officially born in 1762, when Lagrange published his memoir 'Essai d'une nouvelle méthode pour determiner les maxima et les minima des formules integrales indéfinies'. In this work Lagrange determined the differential equation which must be satisfied by the parameterization of the surface which minimizes the area, with an arbitrary boundary curve. This equation is called the Euler-Lagrange equation. Every surface that satisfied the Euler-Lagrange equation was called by Lagrange a 'minimal surface'; in fact, Lagrange's minimal surfaces form a larger family than the area minimizing ones.

In 1776 Meusnier realized the geometrical meaning of the Euler-Lagrange equation: if a point of a surface satisfies the Euler-Lagrange equation, then the mean curvature of the surface at that point is zero. Meusnier also discovered a new 'minimal surface', namely the helicoid. The helicoid is an example of a ruled surface. For a long time, the helicoid and the catenoid remained the only known minimal surfaces.

In 1783 Monge proved in his work 'Sur une méthode d'intégrer les équations aux différences ordinaires' that a 'minimal surface', which is the graph of a differentiable function and has as boundary curve a closed simple curve, is also of minimal area. In the years 1831 to 1835 Monge discovered also new examples of minimal surfaces including the Scherk's surface.

In the mid 19th century the Belgian physicist J. Plateau began the study of the forms taken by the soap films. He constructed a lot of models of minimal surfaces, including the catenoid and the helicoid. Plateau got a soap film for every form of the wire. His experiments empirically proved that the ‘minimal surfaces’ discovered until that time were just a small part of the existing ‘minimal surfaces’, for which the mathematical expressions still had to be discovered. The simplest minimal surface we can get from soap films is the plane, obtained by dipping a wire that forms a closed circle. Since Plateau’s experiments had a big success, the problem of finding the surface, which has a closed curve as boundary, with minimal area is called the ‘Plateau’s Problem’.

In the years from 1850 to 1880 a lot of mathematicians focused on the Plateau’s problem and looked for solutions: Schwarz, Riemann and Weierstrass solved the problem for several polygonal contours.

In 1914 Sergei Bernstein considered ‘minimal surfaces’ without boundary and studied their regularity. He proved that if a minimal surface is represented as a graph of a function f, defined for all $(x, y) \in \mathbb{R}^2$ and with continuous first and second partial derivatives, then the surface is a plane. Such a problem was in the sequel called the Bernstein’s problem. It is equivalent to saying that the only solutions to the Euler-Lagrange equations for two-dimensional minimal surfaces in $\mathbb{R}^3$, which are defined for all (x, y) and have continuous first and second partial derivatives, are linear functions.

The natural question that arose concerned the regularity of such surfaces in higher dimensions i.e. the generalisation of the Bernstein’s problem. Ennio De Giorgi solved the Bernstein’s problem for three-dimensional surfaces, Frederick Almgren for four-dimensional surfaces and the mathematician James Simons for surfaces up to dimension seven.

In order to study the regularity of higher-dimensional hypersurfaces, in the early 1950s Renato Caccioppoli introduced the concept of perimeter, i.e. a generalisation of the idea of area to a larger class of hypersurfaces. The mathematician De Giorgi studied the Bernstein’s problem with the technique of the perimeters and together with E. Bombieri and E. Giusti in 1968 worked out the proof of minimality of a cone in $\mathbb{R}^8$. They also proved, as a nontrivial consequence, that there exist minimal surfaces in $\mathbb{R}^9$ represented as a graph of a function f, defined for all $(x_1, ..., x_8) \in \mathbb{R}^8$, which have continuous first and second partial derivatives, and are not planes. This means that the generalisation of the Bernstein’s problem holds for hypersurfaces in $\mathbb{R}^n$ for $n \leq 8$ and not more.

1.2. **Practical application.** Minimal surfaces and soap films are used in architecture: in the 1960s Frei Otto invented the tensile structure, that is a construction of elements carrying only tension. His most famous creation is the Olympic Stadium in Munich: its form is made by models of soap films.

Surfaces that locally minimize the area have also been extensively used to construct models for physical phenomena, including the already mentioned soap films, black holes, compound polymers and protein folding. Recently, this mathematical field has become an area of intense mathematical and scientific study, specifically in the areas of molecular engineering, materials science, and nanotechnology because of their many anticipated applications.

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2. Definition of minimal surfaces with the calculus of variations

In this section we will get to the definition of minimal surfaces for surfaces given as a graph of a function over a bounded domain as follows: since we are searching for surfaces with minimal area we will minimize the area functional, that to every surface given as a graph of a function associates its area. More precisely, we will not find the minimum of the area functional but its stationary points. We will solve the problem as a variational problem, find the appropriate Euler-Lagrange equations and prove that they are equivalent to the condition of zero mean curvature, that is the formal definition of minimal surfaces.

Remark: with the calculus of variation we will consider our problem for surfaces, given as a graph of a function over a bounded domain, in all dimensions. With the second method in the next section (with the surface variation) we will study only surfaces in $\mathbb{R}^3$.

At first, we will find the tools for solving a general n-dimensional variational problem and get the general Euler-Lagrange equations.

2.1. Euler-Lagrange equations for an n-dimensional variational problem.

In this subsection, we will first see the definition of the variational derivative of a functional $I[f] = \int_D L\left(x^\beta; f^j\left(x^\beta\right); f^i_{x^\alpha}\left(x^\beta\right)\right) dx^1 \wedge ... \wedge dx^n$ and the definition of a stationary function.

Let D be a region of the Euclidean space $\mathbb{R}^n$ with coordinates $x^1, ..., x^n$, such that the boundary ∂D is piecewise smooth.

Let F be a linear space of smooth functions $f(x^1, ..., x^n) = (f^1, ..., f^k)$ defined on D i.e.

$$F = \{f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^k\},$$

and let

$$L(x^\beta; p^j; q^i_\alpha),$$

with $1 \leq \beta \leq n, 1 \leq j \leq k, 1 \leq i \leq k, 1 \leq \alpha \leq n$, be a smooth real-valued function of $n + k + nk$ arguments; in an explicit way we shall write

$$L(x^1, ..., x^n, p^1, ..., p^k, q^1_1, ..., q^k_1, ..., q^1_n, ..., q^k_n),$$

and we call L a Lagrangian.

We define the functional

$$I[f] := \int_D L\left(x^\beta; f^j\left(x^\beta\right); f^i_{x^\alpha}\left(x^\beta\right)\right) dx^1 \wedge ... \wedge dx^n,$$

where

$$f^i_{x^\alpha}\left(x^\beta\right) = \left(\frac{\partial}{\partial x^\alpha}\right) f^i\left(x^\beta\right).$$

More briefly we will write $I[f] = \int_D L\left(x^\beta; f^j; f^i_{x^\alpha}\right) d^n x$.

Definition 2.1. *The variational derivative of the functional I at the point f in the direction η is*

$$\frac{\delta I}{\delta f} = \lim_{\epsilon \to 0} \frac{\Delta I[f]}{\epsilon} = \lim_{\epsilon \to 0} \frac{I[f + \epsilon\eta] - I[f]}{\epsilon},$$

where $f \in F$, ϵ is a small (real) parameter and $\eta = \delta f \in F$ is a function vanishing on ∂D.

Remark: the function η is used as increment or "perturbation" of f and it determines the "direction" of the variation from f to $f + \epsilon\eta$.

Definition 2.2. *A function $f_0 \in F$ is said to be stationary (or extremal, or critical) for a functional I, if*

$$\frac{\delta I[f_0]}{\delta f} \equiv 0$$

for every perturbation $\delta f = \eta$ identically zero on the boundary of D.

Theorem 2.3. *A function $f_0 \in F$ is extremal for the functional $I[f]$ if and only if it satisfies the system of equations*

$$\frac{\partial L}{\partial f^i} - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x^\alpha}}\right)\Big|_{f=f_0} = 0 \quad (1 \leq i \leq k),$$

where $\Big|_{f=f_0}$ indicates that the functional is restricted to $f = f_0$. These equations are called the Euler-Lagrange equations for the functional I.

Proof. Let $f_0 \in F$ be extremal for the functional $I[f]$, i.e. $\frac{\delta I[f_0]}{\delta f} \equiv 0$. So

$$\frac{\delta I}{\delta f} = \lim_{\epsilon \to 0} \frac{\Delta I[f_0]}{\epsilon} \equiv 0.$$

Let us find an explicit expression for $\Delta I[f]$:

$$\begin{aligned}\Delta I[f] &= I[f + \epsilon\eta] - I[f] \\ &= \int_D L(x^\beta; f^j + \epsilon\eta^j; (f^i + \epsilon\eta^i)_{x^\alpha}) - L(x^\beta; f^j; f^i_{x^\alpha}) d^n x.\end{aligned}$$

If we consider the first term in the integrand as a function of ϵ and we expand it in a Taylor series of order 1 with respect to ϵ around the point $\epsilon = 0$, we obtain

$$\begin{aligned}\Delta I[f] &= \int_D L + \left(\sum_{j=1}^{k} \frac{\partial L}{\partial f^j}\eta^j + \sum_{\alpha=1}^{n}\sum_{i=1}^{k} \frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i_{x^\alpha}\right)\epsilon + o(\epsilon) - L\, d^n x \\ &= \epsilon \int_D \sum_{i=1}^{k}\left(\frac{\partial L}{\partial f^i}\eta^i + \sum_{\alpha=1}^{n} \frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i_{x^\alpha}\right) d^n x + \int_D o(\epsilon) d^n x.\end{aligned}$$

If we use the fact that

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i\right) = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\right)\eta^i + \frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i_{x^\alpha}$$

we obtain

$$\Delta I[f] = \epsilon \int_D \sum_{i=1}^{k}\left(\frac{\partial L}{\partial f^i}\eta^i + \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i\right) - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\right)\eta^i\right) d^n x$$

$$+\int_D o(\epsilon)d^n x.$$

Since all functions involved are assumed to be piecewise smooth (as also the boundary of D) we can use the Fubini's theorem and integrate first with respect to any variable x^α. We apply it on the second term of the main sum and get

$$\int_D \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i\right)d^n x = \int_{x^1,\dots,\widehat{x}^\alpha,\dots,x^n}\left(\int_P^Q \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i\right)dx^\alpha\right)d^{n-1}x,$$

where the hat over a symbol indicates that symbol is omitted, where P and Q depend on $x^1,...,\widehat{x}^\alpha,...,x^n$, and where $d^{n-1}x = dx^1 \wedge ... \wedge d\widehat{x}^\alpha \wedge ... \wedge dx^n$. It follows that

$$\int_D \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i\right)d^n x = \int_{x^1,\dots,\widehat{x}^\alpha,\dots,x^n}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i|_Q - \frac{\partial L}{\partial f^i_{x_\alpha}}\eta^i|_P\right)d^{n-1}x \equiv 0$$

because P and Q are on ∂D where the perturbation η vanishes. Thus

$$\Delta I[f] = \epsilon\int_D \sum_{i=1}^k \left(\frac{\partial L}{\partial f^i}\eta^i - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\right)\eta^i\right)d^n x + \int_D o(\epsilon)d^n x.$$

Since

$$\lim_{\epsilon\to 0}\frac{\int_D o(\epsilon)d^n x}{\epsilon} = 0,$$

the expression of the variational derivative of the functional will be

$$\frac{\delta I[f]}{\delta f} = \int_D \sum_{i=1}^k \left(\frac{\partial L}{\partial f^i} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\right)\right)\eta^i d^n x.$$

Since f_0 is an extremal of I, we have

$$\frac{\delta I[f_0]}{\delta f} = \int_D \sum_{i=1}^k \left(\frac{\partial L}{\partial f^i} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\right)\right)\eta^i d^n x\Big|_{f=f_0} = 0$$

for all perturbations η. We remind that on ∂D we have $\eta = 0$. It follows:

$$\frac{\partial L}{\partial f^i} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha}\left(\frac{\partial L}{\partial f^i_{x_\alpha}}\right)\Big|_{f=f_0} = 0,$$

which are the Euler-Lagrange equations for the functional I. □

2.2. **Euler-Lagrange equations for two-dimensional minimal surfaces.** Let V be a smooth surface in the Euclidean space $\mathbb{R}^3$ with Euclidean coordinates (x, y, z) given in the graphical form $z = f(x, y)$, where the domain of definition of the function f is a bounded region $D \subset \mathbb{R}^2$. A parameterization of this surface is $r : U \longrightarrow V \in \mathbb{R}^3$, $r(u, v) = (u, v, f(u, v))$ and the area is by definition

$$A(V) = \int_U \parallel r_u \times r_v \parallel du\ dv = \int_U \parallel (1, 0, f_u) \times (0, 1, f_v) \parallel du\ dv$$

$$= \int_U \parallel (f_u, f_v, 1) \parallel du\ dv = \int_U \sqrt{1 + f_u^2 + f_v^2}\ du\ dv.$$

We would like to minimize the area functional

$$A(f) = \int_U \sqrt{1 + f_u^2 + f_v^2}\ du\ dv,$$

which associates to every function $f : U \longrightarrow \mathbb{R}^3$ the area of the surface given by $z = f(x, y)$. By using the theorem of the previous section, we get the Euler-Lagrange equations of our variational problem:

$$\frac{\partial L}{\partial f} - \left(\frac{\partial}{\partial x} \left(\frac{\partial L}{\partial f_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial L}{\partial f_y} \right) \right) = 0,$$

where $L = \sqrt{1 + f_x^2 + f_y^2}$, that is

$$\frac{\partial}{\partial x} \left(\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}} \right) = 0,$$

which implies

$$\frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}} - \frac{f_x(2 f_x f_{xx} + 2 f_y f_x y)}{2\sqrt{1 + f_x^2 + f_y^2}}$$
$$+ \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}} - \frac{f_y(2 f_x f_{xy} + 2 f_y f_{yy})}{2\sqrt{1 + f_x^2 + f_y^2}} = 0$$

so

$$(1 + f_x^2) f_{yy} - 2 f_{xy} f_x f_y + (1 + f_y^2) f_{xx} = 0,$$

which are the Euler-Lagrange equations for two-dimensional minimal surfaces in $\mathbb{R}^3$.

2.3. **Euler-Lagrange equations for $(n-1)$-dimensional minimal surfaces.** Let V^{n-1} be a smooth hypersurface in Euclidean $\mathbb{R}^n$ with Euclidean coordinates $x^1, ..., x^{n-1}, x^n$ given in the graphical form $x^n = f(x^1, ..., x^{n-1})$, where the domain of definition of the function f is a bounded region $D \subseteq \mathbb{R}^{n-1}$. We want to minimize the area functional $S[f] = \int_D d\tau^{n-1}$, where $d\tau^{n-1}$ is a form representing the (n-1)-dimensional volume element of V^{n-1}. If for each point P of V^{n-1} we denote by $n(P)$ the unit normal to V^{n-1} at P and by $\alpha(P)$ the angle between $n(P)$ and $e_n = (0, ..., 0, 1)$, then we can write

$$S[f] = \int_D d\tau^{n-1} = \int_D \frac{d^{n-1}x}{\cos \alpha(P)}.$$

Since $\cos \alpha(P) = \langle e_n, n(P) \rangle$ we get

$$S[f] = \int_D \frac{d^{n-1}x}{\langle e_n, n(P) \rangle}.$$

Now let us find a more explicit form of the normal: if we consider the function

$$F(x^1, ..., x^n) := x^n - f(x^1, ..., x^{n-1}),$$

then our surface V^{n-1} is a level set of the function F, $V^{n-1} = F^{-1}(0)$. Recall that the gradient of F

$$\nabla F = \left(\frac{\partial F}{\partial x_1}, ..., \frac{\partial F}{\partial x_{n-1}}, \frac{\partial F}{\partial x_n} \right) = (-f_{x^1}, ..., -f_{x^{n-1}}, 1)$$

is orthogonal to the level sets. If we normalize such a vector, we get a normal vector to V^{n-1}:

$$n(P) = \frac{(-f_{x^1}, ..., -f_{x^{n-1}}, 1)}{\sqrt{1 + \sum_{i=1}^{n-1} (f_{x^i})^2}}.$$

So the area functional that we would like to minimize is

$$S[f] = \int_D \sqrt{1 + \sum_{i=1}^{n-1} (f_{x^i})^2} \; dx^1 \wedge ... \wedge dx^{n-1}.$$

By Theorem 2.3 we get

$$\sum_{j=1}^{n-1} \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial f_{x^j}} \left(\sqrt{1 + \sum_{i=1}^{n-1} (f_{x^i})^2} \right) \right) = 0,$$

which gives

$$\sum_{j=1}^{n-1} \frac{\partial}{\partial x^j} \left(\frac{f_{x^j}}{\sqrt{1 + \sum_{j=1}^{n-1} (f_{x^j})^2}} \right) = 0.$$

These are the Euler-Lagrange equations for the extremal surface $x^n = f(x^1, ..., x^{n-1})$ over D. Now we can give a first definition of minimal surfaces given in the graphical form, if we consider them as minimizers of the area functional:

Definition 2.4. *Minimal surfaces given in the graphical form $x^n = f(x^1, ..., x^{n-1})$ are surfaces which are extremal with respect to the area functional*

$$S(f) = \int_D \sqrt{1 + \sum_{i=1}^{n-1} (f_{x^i})^2} \; dx^1 \wedge ... \wedge dx^{n-1}.$$

2.4. **From the Euler-Lagrange equations to the formal definition of minimal surfaces.** In the present section we will find the necessary and sufficient geometric condition for a surface in $\mathbb{R}^3$ to be minimal. This condition can be used to define minimal surfaces. It is given in terms of one of the surfaces' embedding invariants in $\mathbb{R}^n$, the mean curvature.

Definition 2.5. *Let X be a surface in $\mathbb{R}^3$. The normal curvature of a curve $\gamma(t) : [a, b] \longmapsto X \subset \mathbb{R}^3$ is given by*

$$K_n(\gamma)_{(t)} = \langle \ddot{\gamma}, n \rangle_{(t)},$$

where $\ddot{\gamma} = \frac{d^2\gamma}{dt^2}$ and n is the normal of the surface X.

Proposition 2.6. *If $\gamma(t) : [a, b] \longmapsto X$ is a naturally parameterized curve (i.e. $\| \dot{\gamma} \| = 1$) and $\gamma(t) = r(u(t), v(t))$, then*

$$K_n(t) = (\dot{u}(t), \dot{v}(t)) \begin{pmatrix} L & M \\ M & N \end{pmatrix}_{(\gamma(t))} \begin{pmatrix} \dot{u}(t) \\ \dot{v}(t) \end{pmatrix},$$

where $L = \langle r_{uu}, n \rangle_{(u,v)}$, $M = \langle r_{uv}, n \rangle_{(u,v)}$, $N = \langle r_{vv}, n \rangle_{(u,v)}$ are the coefficients of the second fundamental form of the parameterization r of the surface X.

Proof. Let $\gamma(t) = r(u(t), v(t))$ so that $\dot{\gamma} = r_u \dot{u} + r_v \dot{v}$ and $\ddot{\gamma} = r_{uu}\dot{u}^2 + 2r_{uv}\dot{u}\dot{v} + r_{vv}\dot{v}^2 + r_u \ddot{u} + r_v \ddot{v}$. The normal curvature is given by

$$\begin{aligned} K_n = \langle \ddot{\gamma}, n \rangle &= \langle r_{uu}, n \rangle \dot{u}^2 + 2\langle r_{uv,n} \dot{u}\dot{v} + \langle r_{vv}, n \rangle \dot{v}^2 \\ &= (\dot{u}, \dot{v}) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}. \end{aligned}$$

□

Remark:: We see that the normal curvature can be viewed as a function $K_n : T_mX \longrightarrow \mathbb{R}$ defined on the tangent plane T_mX.

Definition 2.7. *The principal curvatures are the values of the function*

$$K_n : T_mX \longrightarrow \mathbb{R}$$
$$K_n(\xi,\eta) = (\xi,\eta)\begin{pmatrix} L & M \\ M & N \end{pmatrix}_{(m)}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi^2 L + 2\xi\eta M + \eta^2 N$$

in those unit vectors where the function reaches the minimum and the maximum. Here T_mX is the tangent plane to the surface X at the point m and (ξ,η) are the coordinates of a tangent vector to the surface X at the point m with respect to the basis $\{r_u, r_v\}_{(m)}$. The vector (ξ,η) has to be of unit length, that is

$$(\xi,\eta)\begin{pmatrix} E & F \\ F & G \end{pmatrix}_{(m)}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 1.$$

Proposition 2.8. *The principal curvatures are the zeros of the equation*

$$\det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Proof. Let K_1 be a principal curvature, that is the value of the function $K_n(\xi,\eta) = \xi^2 L + 2\xi\eta M + \eta^2 N$ at a unit vector, where the function reaches an extremum. So we first try to find the points (ξ,η) in which the function

$$K_n(\xi,\eta) = (\xi,\eta)\begin{pmatrix} L & M \\ M & N \end{pmatrix}_{(m)}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi^2 L + 2\xi\eta M + \eta^2 N$$

reaches an extremum with the requirement

$$(\xi,\eta)\begin{pmatrix} E & F \\ F & G \end{pmatrix}_{(m)}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi^2 E + 2\xi\eta F + \eta^2 G = 1.$$

Let us use the method of Lagrange multipliers, that is, let us search for the stationary points of

$$f(\xi,\eta) = (\xi^2 L + 2\xi\eta M + \eta^2 N) - \lambda(\xi^2 E + 2\xi\eta F + \eta^2 G - 1).$$

Derivation gives

$$\frac{\partial f}{\partial \xi} = 2((\xi L + \eta M) - \lambda(\xi E + \eta F)) = 0$$
$$\frac{\partial f}{\partial \eta} = 2((\xi M + \eta N) - \lambda(\xi F + \eta G)) = 0,$$

which implies

$$\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \lambda\begin{pmatrix} E & F \\ F & G \end{pmatrix}\right)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0. \qquad (1)$$

We observe that we can find a non-trivial solution (ξ,η) of this equation if and only if

$$\det\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \lambda\begin{pmatrix} E & F \\ F & G \end{pmatrix}\right) = 0.$$

That is

$$\det\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

which means that λ is a eigenvalue of the matrix

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

.

So $K_1 = K_n(\xi, \eta)$, where $(\xi, \eta) \neq 0$ is a unit vector such that

$$\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix} \right) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0,$$

where λ is an eigenvalue of A. If we multiply (1) from the left with (ξ, η), we get

$$(\xi, \eta) \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} - \lambda(\xi, \eta) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0.$$

Since (ξ, η) is a unit vector and $K_1 = K_n(\xi, \eta)$, we obtain $K_1 = \lambda_1$. □

Definition 2.9. *The mean curvature H of a surface at the point m is given by*

$$H(m) = \frac{K_1(m) + K_2(m)}{2},$$

where K_1 and K_2 are the principal curvatures of the surface at the point m.

Proposition 2.10. *The expression of the mean curvature in terms of the coefficients of the first and the second fundamental form is*

$$H_{(m)} = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

Proof. By definition $H(m) = \frac{K_1(m)+K_2(m)}{2}$. We have seen that the principal curvatures are the zeros of the equation

$$det(A - \lambda I) = 0,$$

where $A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$. So

$$det(A - \lambda I) = (\lambda - K_1)(\lambda - K_2) = \lambda^2 - (K_1 + K_2)\lambda + K_1 K_2.$$

Since $det(A - \lambda I) = \lambda^2 - tr(A)\lambda + det(A)$, where tr(A) is the trace of A, we have

$$\frac{1}{2}K_1 + K_2 = \frac{1}{2}tr(A) = \frac{1}{2}tr(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix})$$

$$= \frac{1}{2}tr(\frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix}) = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

□

The following result states that the Euler-Lagrange equations are equivalent to the condition $H = 0$.

Theorem 2.11. *The mean curvature H of a smooth hypersurface V^2 in $\mathbb{R}^3$ is identical to zero if and only if for each of its points there exists some neighbourhood such that the surface V^2 can be represented as a graph of a function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$, and f is an extremal of the area functional (that is, a solution of the Euler-Lagrange equations).*

Proof. : Let P be a non-singular point of the surface. We choose Euclidean coordinates x, y, z in a neighbourhood of P by taking the origin to be P, the x-axis and y-axis tangent to the surface at P and the z-axis orthogonal to V^2 at P. Then, in this neighbourhood the surface is given by $z = f(x, y)$ and so a parameterization of the surface is $r(u, v) = (x, y, f(x, y))$. We will calculate the coefficients of the first and second fundamental form and use the formula of the mean curvature in terms of these coefficients:

$E = \langle r_x, r_y \rangle = \langle (1, 0, f_x), (1, 0, f_x) \rangle = 1 + f_x^2$,
$F = \langle r_x, r_y \rangle = \langle (1, 0, f_x), (0, 1, f_x) \rangle = f_x f_y$,
$G = 1 + f_y^2$,

$$n = \frac{r_x \times r_y}{\| r_x \times r_y \|} = \frac{(1, 0, f_x) \times (0, 10 f_y)}{\| r_x \times r_y \|} = \frac{(f_x, f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}},$$

$$L = \langle r_{xx}, n \rangle = \langle (0, 0, f_{xx}, n) = \frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}},$$

$$M = \langle r_{yx}, n \rangle = \langle (0, 0, f_{yx}, n) = \frac{f_{yx}}{\sqrt{f_x^2 + f_y^2 + 1}},$$

$$N = \frac{f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

So the mean curvature is

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0$$

i.e.

$$\frac{(1 + f_x^2) f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}} - \frac{2 f_x f_y f_{xy}}{\sqrt{f_x^2 + f_y^2 + 1}} + \frac{(1 + f_x^2) f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}} = 0$$

i.e.

$$(1 + f_x^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0.$$

□

This theorem suggests the formal definition of the minimal surfaces:

Definition 2.12. *A minimal surface is a surface whose mean curvature is zero everywhere.*

3. Plateau's problem

In this section we will consider the Plateau's problem which consists of finding a surface with the minimal area among all the surfaces which share the common boundary curve. The main result of this section is the proof of the fact that a surface, which solves the Plateau's problem, has mean curvature zero.

Remark: we consider more general surfaces that in the previous section. We consider all regularly parameterized surfaces. The results will be more general than in the previous section where we considered only surfaces given in a graphical form.

Plateau's idea was to divide all the surfaces that have a fixed boundary in families. Intuitively, we imagine that every family is connected with a direction. Every element of a family is a surface that is moved for a certain quantity in the direction corresponding to its family. So each family is characterised by a parameter: different values of the parameter represent different surfaces. To minimize the area, we will differentiate the area functional that depends only on the parameter and we will do so for each family.

Definition 3.1. *Let U be an open subset of $\mathbb{R}^2$. Let*

$$\sigma^\tau : U \longrightarrow \mathbb{R}^3$$
$$(u, v) \longmapsto (x^\tau(u, v), y^\tau(u, v), z^\tau(u, v))$$

be a family of surface patches, where τ lies in some open interval $(-\delta, \delta)$, where $\delta > 0$ i.e.

$$\sigma^\tau : U \longrightarrow \mathbb{R}^3, \tau \in (-\delta, \delta), \delta > 0.$$

Let the family be smooth, that is, let the map $(u, v, \tau) \mapsto \sigma^\tau(u, v)$ be smooth. We define the surface variation of the family as the map

$$\varphi : U \longrightarrow \mathbb{R}^3$$

$$\varphi = \frac{d}{d\tau}\sigma^\tau|_{\tau=0}.$$

Theorem 3.2. *If a surface S has least area among all surfaces with the same boundary curve, then the mean curvature of S is zero everywhere.*

Proof. Let U be an open subset of $\mathbb{R}^2$. Let S be parameterized by the patch

$$\sigma : U \longrightarrow \mathbb{R}^3.$$

Let

$$\sigma^\tau : U \longrightarrow \mathbb{R}^3$$
$$(u, v) \longmapsto (x^\tau(u, v), y^\tau(u, v), z^\tau(u, v))$$

be a smooth family of surface patches, where τ lies in some open interval $(-\delta, \delta)$, where $\delta > 0$. Let $\sigma = \sigma^0$. Let π be a simple closed curve that is contained, along with its interior $int(\pi)$ in U. If we apply σ^τ to π, we get a simple closed curve $\gamma^\tau = \sigma^\tau \circ \pi$. Since we are searching for a minimal surface with a fixed boundary curve we fix $\gamma^\tau = \gamma$ for every τ.

We define the area function

$$A(\tau) = \int\int_{int(\pi)} dA_{\sigma^\tau},$$

which associates to a value of the parameter τ the area of the part of σ^τ inside γ. By the definition of surface area we can write

$$A(\tau) = \iint_{int(\pi)} \| \sigma_u^\tau \times \sigma_v^\tau \| \, dudv$$
$$= \int\int_{int(\pi)} \frac{\| \sigma_u^\tau \times \sigma_v^\tau \|}{\| \sigma_u^\tau \times \sigma_v^\tau \|^2} \| \sigma_u^\tau \times \sigma_v^\tau \|^2 \, dudv$$
$$= \int\int_{int(\pi)} \frac{(\sigma_u^\tau \times \sigma_v^\tau)}{\| \sigma_u^\tau \times \sigma_v^\tau \|}(\sigma_u^\tau \times \sigma_v^\tau) dudv = \int\int_{int(\pi)} N^\tau(\sigma_u^\tau \times \sigma_v^\tau) dudv,$$

where N^τ is the standard unit normal of σ^τ. If we differentiate, we obtain

$$\dot{A}(\tau) = \frac{dA(\tau)}{d\tau} = \int\int_{int(\pi)} \frac{\partial}{\partial\tau} N^\tau(\sigma_u^\tau \times \sigma_v^\tau) dudv.$$

To simplify the notation, we drop the subscript τ for the rest of this calculation and we denote $\frac{\partial}{\partial \tau}$ with a dot:

$$\dot{A}(\tau) = \int\int_{int(\pi)} \dot{N}(\sigma_u \times \sigma_v) + N(\dot{\sigma_u} \times \sigma_v) + N(\sigma_u \times \dot{\sigma_v}) dudv.$$

We observe that the first element in the integral $\dot{N}(\sigma_u \times \sigma_v)$ is zero because $\langle \dot{N}, N \rangle = 0$, since N is a unit vector. If we write the second and the third element inside the integral in terms of the coefficients of the first fundamental form, we get

$$N(\dot{\sigma_u} \times \sigma_v) = \frac{(\sigma_u \times \sigma_v)(\dot{\sigma_u} \times \sigma_v)}{\| \sigma_u \times \sigma_v \|} = \frac{(\sigma_u \dot{\sigma}_u)(\sigma_v \sigma_v) - (\sigma_u \sigma_v)(\sigma_v \dot{\sigma}_u)}{\| \sigma_u \times \sigma_v \|}$$
$$= \frac{G(\sigma_u \dot{\sigma}_u) - F(\sigma_v \dot{\sigma}_u)}{\sqrt{EG - F^2}}$$

$$N(\sigma_u \times \dot{\sigma_v}) = \frac{E(\sigma_v \dot{\sigma}_v) - F(\sigma_u \dot{\sigma}_v)}{\sqrt{EG - F^2}}.$$

So, the derivative of the area function is (if we add the superscript τ)

$$\dot{A}(\tau) = \int\int_{int(\pi)} \frac{E^\tau(\sigma_v^\tau \dot{\sigma^\tau}_v) - F^\tau(\sigma_u^\tau \dot{\sigma^\tau}_v + \sigma_u^\tau \dot{\sigma^\tau}_v) + G^\tau(\sigma_u^\tau \dot{\sigma^\tau}_u)}{\sqrt{E^\tau G^\tau - (F^\tau)^2}} dudv.$$

Now we will use the surface variation φ which we have defined as the derivative of the family of surface patches with respect to τ, evaluated at $\tau = 0$. We define $\varphi^\tau = \frac{d}{d\tau}\sigma^\tau$, so that $\varphi^0 = \varphi$. Observe that the surface variation is an infinitesimal vector in $\mathbb{R}^3$ that depends on the parameter τ of the family and of the point (u, v). We can express it as a linear combination of the basis vectors $\sigma_u^\tau,\ \sigma_v^\tau,\ N^\tau$. It follows that there exist smooth functions $\alpha^\tau,\ \beta^\tau$ and γ^τ of (u, v, τ) such that

$$\varphi^\tau = \alpha^\tau N^\tau + \beta^\tau \sigma_u^\tau + \gamma^\tau \sigma_v^\tau.$$

If we differentiate σ^τ with respect to u and v we get

$$\dot{\sigma^\tau}_u = \varphi_u^\tau = \alpha_u^\tau N^\tau + \beta_u^\tau \sigma_u^\tau + \gamma_u^\tau \sigma_v^\tau + \alpha^\tau N_u^\tau + \beta^\tau \sigma_{uu}^\tau + \gamma^\tau \sigma_{uv}^\tau.$$

To simplify the notation, let us drop again the subscript τ. We have

$$\dot{\sigma_u} = \alpha_u N + \beta_u \sigma_u + \gamma_u \sigma_v + \alpha N_u + \beta \sigma_{uu} + \gamma \sigma_{uv},$$
$$\dot{\sigma_v} = \alpha_v N + \beta_v \sigma_u + \gamma_v \sigma_v + \alpha N_v + \beta \sigma_{uv} + \gamma \sigma_{vv},$$

and further

$$\sigma_u \dot{\sigma}_u = \beta_u E + \gamma_u F + \alpha(\sigma_u N_u) + \beta(\sigma_u \sigma_{uu}) + \gamma(\sigma_u \sigma_{uv})$$
$$= \beta_u E + \gamma_u F - \alpha L + \frac{1}{2}\beta E_u + \frac{1}{2}\gamma E_v$$

$$\sigma_v \dot{\sigma}_u = \beta_u F + \gamma_u G - \alpha M + \beta(F_u - \frac{1}{2}E_v) + \frac{1}{2}\gamma G_u$$

$$\sigma_u \dot{\sigma}_v = \beta_v E + \gamma_v F - \alpha M + \frac{1}{2}\beta E_v + \gamma(F_v - \frac{1}{2}G_u)$$

$$\sigma_v \dot{\sigma}_v = \beta_v F + \gamma_v G - \alpha N + \frac{1}{2}\beta G_u + \frac{1}{2}\gamma G_v$$

$$E(\sigma_v \dot{\sigma}_v) - F(\sigma_u \dot{\sigma}_v + \sigma_u \dot{\sigma}_v) + G(\sigma_u \dot{\sigma}_u)$$

$$= \beta_v EF + \gamma_v EG - \alpha EN + \frac{1}{2}\beta EG_u + \frac{1}{2}\gamma EG_u - \beta_u F^2 - \gamma_u FG$$
$$+\alpha FM - \beta FF_u + \beta \frac{1}{2} FE_v - \frac{1}{2}\gamma FG_u$$
$$-\beta_v EF - \gamma_v F^2 + \alpha FM - \frac{1}{2}\beta FE_v$$
$$-\gamma FF_v + \gamma \frac{1}{2} FG_u + \beta_u EG + \gamma_u FG - \alpha LG + \frac{1}{2}\beta E_u G + \frac{1}{2}\gamma E_v G$$
$$= \beta_u (EG - F^2) + \frac{1}{2}\beta(E_u G + EG_u - 2FF_u) + \gamma_v (EG - F^2)$$
$$+\frac{1}{2}\gamma(E_v G + EG_v - 2FF_v) - \alpha(LG - 2MF + NE).$$

If we consider the formula for H, we see that the derivative of the area function is given by

$$\dot{A}(\tau) = \int\int_{int(\pi)} \Big(\beta_u \sqrt{EG - F^2} + \frac{1}{2}\beta \frac{E_u G + EG_u - 2FF_u}{\sqrt{EG - F^2}}$$
$$+\gamma_u \sqrt{EG - F^2} + \frac{1}{2}\gamma \frac{E_v G + EG_v - 2FF_v}{\sqrt{EG - F^2}} - 2\alpha H \sqrt{EG - F^2}\Big) du\ dv$$
$$= \int\int_{int(\pi)} \Big((\beta \sqrt{EG - F^2})_u + (\gamma \sqrt{EG - F^2})_v - 2\alpha H \sqrt{EG - F^2} \Big) du\ dv.$$

By Green's Theorem this integral is equal to

$$\int_{\pi} \sqrt{EG - F^2}(\beta dv - \gamma du) - 2\int\int_{int(\pi)} \alpha H \sqrt{EG - F^2} du dv.$$

We obtain the derivative of the area functional in terms of H

$$\dot{A}(\tau) = \int_{\pi} \sqrt{E^\tau G^\tau - (F^\tau)^2}(\beta^\tau dv - \gamma^\tau du)$$
$$-2\int\int_{int(\pi)} \alpha^\tau H^\tau \sqrt{E^\tau G^\tau - (F^\tau)^2} du dv.$$

If by hypothesis $\sigma = \sigma^0$ has the minimal area among all surfaces with the same boundary curve and if we set $E^0 = E, F^0 = F, G^0 = G, H^0 = H$ and $\alpha^0 = \alpha$, then

$$\dot{A}(0) = \int_{\pi} \sqrt{EG - F^2}(\beta^0 dv - \gamma^0 du)$$
$$-2\int\int_{int(\pi)} \alpha H \sqrt{EG - F^2} du\ dv = 0.$$

Since on the boundary curve the variation is zero, i.e. $\varphi^\tau(u,v) = 0$ for $(u,v) \in \pi$, we have $\beta = \gamma = 0$ along the boundary curve π and

$$\dot{A}(0) = -2\int\int_{int(\pi)} \alpha H \sqrt{EG - F^2} du\ dv = 0$$

for all smooth families of surfaces as above. This means that the integral must vanish for all smooth functions $\alpha : U \longrightarrow \mathbb{R}$. This can happen only if the term that multiplies α in the integrand vanishes, that is if $H = 0$. $\square$

Remark: From the proof we see that the relevant part of the variation is the part in the normal direction i.e. the part in the direction of the normal at the surface of a point. It is intuitively clear that the variation in the tangent direction changes the parameterization but not the area. With this in mind we can rewrite the above calculation in a simpler form. First we will define the normal variation and then make the simpler calculation.

Definition 3.3. *Let $r : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a regular parameterized surface. Let $D \subset U$ be a bounded domain. Let $h : \overline{D} \longrightarrow \mathbb{R}$ be a differentiable function, where $\overline{D}$ is the union of the domain D with its boundary. We define the normal variation of $r(\overline{D})$ determined by h to be the map, given by*

$$\varphi : \overline{D} \times (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^3$$

$$\varphi(u, v, t) = r(u, v,) + th(u, v,)N(u, v).$$

Here $(u, v) \in \overline{D}$, $t \in (-\epsilon, \epsilon)$ and N is the normal vector to the surface.

Remark: For each fixed $t \in (-\epsilon, \epsilon)$, the map

$$r^t : D \longrightarrow \mathbb{R}^3$$

$$r^t(u, v) = \varphi(u, v, t)$$

is a parameterized surface.

Theorem 3.4. *Let $r : U \longrightarrow \mathbb{R}^3$ be a regular parameterized surface and let $D \subset U$ be a bounded domain in U. Then the area function $A(t)$ of any normal variation of r which associates to the parameter $t \in (-\epsilon, \epsilon)$ the area of the surface $r^t(\overline{D})$, has a critical point in $t = 0$ if and only if the mean curvature of the surface r is zero everywhere.*

Proof. Let us calculate the area of the surface $r^t(\overline{D})$:

$$A(t) = \int_{\overline{D}} \sqrt{E^t G^t - (F^t)^2} du\ dv,$$

where E^t, F^t, G^t are the coefficients of the first fundamental form of r^t. For calculating them we need the partial derivatives of the parameterization:

$$\frac{\partial r^t}{\partial u} = r_u + thN_u + th_u N$$

$$\frac{\partial r^t}{\partial v} = r_v + thN_v + th_v N.$$

So the coefficients of the first fundamental form are

$$\begin{aligned} E^t &= E + 2th_u\langle \vec{N}, r_u\rangle + 2th\langle r_u, \vec{N}_u\rangle \\ &+t^2h_u^2\langle \vec{N}, \vec{N}\rangle + 2ht^2\langle \vec{N}, \vec{N}_u\rangle + t^2h^2\langle \vec{N}_u, \vec{N}_u\rangle \\ &= E + 2th\langle r_u, \vec{N}_u\rangle + t^2h^2\langle \vec{N}_u, \vec{N}_u\rangle + t^2h_u^2, \\ F^t &= F + th(\langle r_u, \vec{N}_v\rangle + \langle r_v, \vec{N}_u\rangle) + t^2h^2\langle \vec{N}_u, \vec{N}_v\rangle + t^2h_uh_v, \\ G^t &= G + 2th\langle r_v, \vec{N}_v\rangle + t^2h^2\langle \vec{N}_v, \vec{N}_v\rangle + t^2h_v^2. \end{aligned}$$

Now we use the fact that the coefficients of the second fundamental form of r satisfy the following equations:

$$\langle r_u, \vec{N}_u\rangle = -L, \qquad \langle r_u, \vec{N}_v\rangle + \langle r_v, \vec{N}_u\rangle = -2M, \qquad \langle r_v, \vec{N}_v\rangle = -N.$$

We find the area of r^t

$$A(t) = \int_{\overline{D}} \sqrt{EG - F^2 - 2th(EN - 2FM + GL) + R} \quad du\ dv,$$

where R is a polynomial in t with all the powers of t greater than 2 so

$$\lim_{t \to 0} \frac{R}{t} = 0.$$

Since we need a relation between the area and the mean curvature, we insert H in the area expression:

$$\begin{aligned} A(t) &= \int_{\overline{D}} \sqrt{EG - F^2 - 4thH(EG - F^2) + R} \;\; du\ dv \\ &= \int_{\overline{D}} \sqrt{EG - F^2}\sqrt{1 - 4thH^2 + \overline{R}} \;\; du\ dv, \end{aligned}$$

where $\overline{R} = \frac{R}{EG - F^2}$.

We notice that if ϵ is sufficiently small, the map r^t is a regular parameterized surface and A is a differentiable function. Its derivative at $t = 0$ is

$$A'(0) = -\int_{\overline{D}} 2hH\sqrt{EG - F^2} du\ dv.$$

We have to prove that $A'(0) = 0$ if and only if $H = 0$. If $H = 0$ then $A'(0) = 0$. Conversely assume that $A'(0) = 0$ and that $H(q) \neq 0$ for some $q \in D$. If we choose h that is identically zero outside a small neighbourhood of q and $h(q) = H(q)$, then $A'(0) < 0$, which is a contradiction. Thus we get $H = 0$. □

Remark: In the proof above we did not use any conditions on the boundary of the surface so it holds not only for the Plateau's problem, but also for more general minimal surfaces.

Remark: A critical point of the area function of the precedent theorem may not be a minimum of the area. However, we still define all the surfaces that are extremals of the area function as 'minimal surfaces'. This is the standard terminology introduced by Lagrange in 1760. Lagrange was the first to define minimal surfaces. Such terminology can be justified if we take into account that the definition of 'minimal surface' refers to the surface locally. Globally, a minimal surface can be a surface which does not have the minimal area. But the surface, whose area is minimal, is a minimal surface. If we take the two examples in the figures we notice that different surfaces may not be surfaces of minimal area but they are still part (or made of parts) of a 'minimal surface' (in our examples the catenoid and the plane).

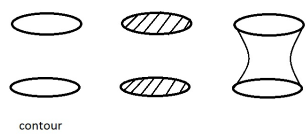

Figure 1. Different surfaces with same boundary curve

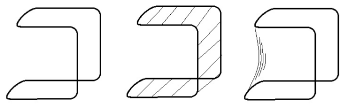

FIGURE 2. Different surfaces with same boundary curve

4. EXAMPLES

4.1. **The catenoid.** The catenoid is a surface obtained by rotating the curve

$$x = \frac{1}{a}\cosh az$$

in the xz-plane around the z-axis, where a is a non-zero constant. The catenoid can be parameterized by

$$\sigma(u, v) = (a\cosh u\cos v, a\cosh u\sin v, au),$$

where $v \in (0, 2\pi), u \in (-\infty, \infty)$.

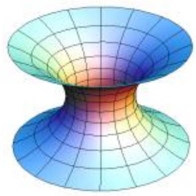

FIGURE 3. Catenoid

Let us check that the helicoid is a minimal surface for $a = 1$:

$$\sigma_u = (\cos v\sinh u, \sin v\sinh u, 1),$$
$$\sigma_v = (\cos v\sinh u, \sin v\sinh u, 1),$$
$$E = \cosh^2 u, \qquad F = 0, \qquad G = \cosh^2 u,$$
$$\sigma_{uu} = (\cos v\cosh u, \sin v\cosh u, 0),$$
$$\sigma_{uv} = (-\sinh u\sin v, \sinh u\cos v, 0),$$
$$\sigma_{vv} = (-\cosh u\cos v, -\cosh u\sin v, 0),$$

$$\vec{n} = \frac{(-\cosh u\cos v, \cosh u\sin v, \cosh u\sinh u)}{\cosh^2 u},$$

$$L = -1, \qquad M = 0, \qquad N = -1.$$

So the mean curvature is

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = -\cosh^2 u + \cosh^2 u = 0.$$

Remark: The catenoid does not necessarily have the minimal area. In the following example we will show with calculationas that the catenoid in some cases does not have the minimal area. Let us fix $a > 0$ and take as boundary curve two circles of radius $b = \cosh a$ located in the planes $z = \pm a$ and with centers on the z-axis. There are two surfaces with mean curvature equal to zero with such boundary curve: one is the surface consisting of two disks of radius b given by $x^2 + y^2 \leq b^2$, the other is a part of the catenoid with $|z| < a$. Let us calculate when the area of the part of the catenoid is not minimal. The area of the part of the catenoid is

$$A_1 = \int_0^{2\pi} \int_{-a}^{a} (EG - F^2)^{1/2} du\, dv = \int_0^{2\pi} \int_{-a}^{a} \cosh^2 u du\, dv$$
$$= 2\pi(a + \sinh a \cosh a).$$

The area of the two disks is

$$A_2 = 2\pi b^2 = 2\pi \cosh^2 a.$$

The area of the catenoid is not minimal if $A_1 > A_2$ i.e. $2\pi \cosh^2 a > 2\pi(a + \sinh a \cosh a)$ i.e. $\cosh^2 a > (a + \sinh a \cosh a)$ i.e.

$$\frac{(e^a + e^{-a})^2}{4} < a + \frac{(e^a - e^{-a})(e^a + e^{-a})}{4}$$

i.e.

$$1 + e^{-2a} < 2a.$$

The graphs of $1 + e^{-2a}$ and $2a$ intersect in exactly one point $a = a_0$ so the area of the part of the catenoid is not minimal if $a > a_0$.

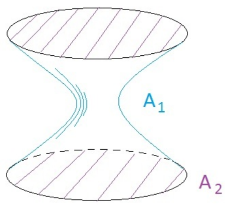

FIGURE 4. Two surfaces with same boundary curve

Proposition 4.1. *Any minimal surface of revolution is either part of a plane or can be obtained by applying a rigid motion (i.e. a transformation consisting of rotations and translations) to part of a catenoid.*

Proof. We take a minimal surface of revolution parameterized by

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)),$$

where the profile curve $u \mapsto (f(u), 0, g(u))$ is of unit speed, $f > 0$, and f, g are continuous functions. It is non restrictive to take as the axis of the surface the z-axis and the curve in the xz-plane because we can apply a rigid motion to the surface and make it look like that. Let us calculate the mean curvature of the surface of revolution. We have

$$\sigma_u = (\dot{f} \cos v, \dot{f} \sin v, \dot{g}), \qquad \sigma_v = (-f \sin v, f \cos v, 0),$$

$$\sigma_{uu} = (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}), \qquad \sigma_{uv} = (-\dot{f}\sin v, \dot{f}\cos v, 0),$$
$$\sigma_{vv} = (-f\cos v, -f\sin v, 0).$$

The normal to the surface is

$$\vec{N} = (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f}).$$

So the coefficients of the first and second fundamental form are

$$E = \dot{f}^2 + \dot{g}^2 = 1, \qquad F = 0, \qquad G = f^2,$$
$$L = -\dot{g}\ddot{f} + \dot{f}\ddot{g}, \qquad M = 0, \qquad N = \dot{g}f.$$

The mean curvature is

$$H = \frac{1}{2}(\dot{f}\ddot{g} - \dot{g}\ddot{f} + \frac{\dot{g}}{f}) = 0.$$

The function f cannot be zero. If it were, S would not be a surface any more but would be a curve.

If $\dot{g}(u) = 0$ for every u in the definition domain of g, then g is a constant so the parameterization of the surface is

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, d)$$

and therefore S is part of the plane $z = d$.

If there exists u_0 such that $\dot{g}(u_0) \neq 0$, then due the continuity of $\dot{g}$ there exists an interval (α, β) such that for every $u \in (\alpha, \beta)$ we have $\dot{g}(u) \neq 0$. Let (α, β) be the largest such interval. Suppose $u \in (\alpha, \beta)$. If we multiply the first two elements of the sum in the expression of the mean curvature by $\frac{\dot{f}}{\dot{f}}$. We get

$$H = \frac{1}{2}(\frac{\dot{g}\dot{f}\ddot{g} - \dot{g}^2\ddot{f}}{\dot{g}} + \frac{\dot{g}}{f}) = 0$$

Now we will use the fact that the profile curve is unit-speed i.e. $\dot{f}^2 + \dot{g}^2 = 1$. Differentiating this expression we get $2\dot{f}\ddot{f} + 2\dot{g}\ddot{g} = 0$. If we use $\dot{g}\ddot{g} = -\dot{f}\ddot{f}$ in the formula of the mean curvature we obtain

$$H = \frac{1}{2}(\frac{-\dot{f}^2\ddot{f} - \dot{g}^2\ddot{f}}{\dot{g}} + \frac{\dot{g}}{f})$$
$$= \frac{1}{2}(\frac{-\ddot{f}}{\dot{g}} + \frac{\dot{g}}{f}) = 0$$

i.e.

$$\frac{\ddot{f}}{\dot{g}} = \frac{\dot{g}}{f}$$

i.e.

$$\ddot{f}f = \dot{g}^2.$$

Using again the fact that the profile curve is unit-speed we get $\ddot{f}f = 1 - \dot{f}^2$ i.e.

$$\frac{d^2f}{dt^2}\frac{df}{dt} = 1 - \left(\frac{df}{dt}\right)^2.$$

We will solve this differential equation by defining $h := \frac{df}{dt}$ and so $\ddot{f} = \frac{dh}{dt} = \frac{dh}{df}h$. The equation becomes

$$\left(\frac{dh}{df}\right) hf = 1 - h^2$$
$$\int \frac{h}{1-h^2} dh = \int \frac{1}{f} df$$
$$\ln(1-h^2)^{-\frac{1}{2}} = \ln f + k,$$

where k is a constant. We obtain the following expression of h:

$$h = \pm\frac{\sqrt{a^2f^2-1}}{af},$$

where a is a non zero constant. We omit the $\pm$ because we can change the sign by replacing u by $-u$ if necessary. Now we can find f by integrating h:

$$f = \frac{\sqrt{1+a^2(u+b)^2}}{a},$$

where b is a constant that we can assume to be equal to zero (because if necessary we can do a change of parameter $u \mapsto u + b$). We can find g by the formula that connects f and g i.e. the unit-speed assumption:

$$\dot{g} = \pm\sqrt{1-(\dot{f})^2} = \pm\sqrt{\frac{1}{1+a^2u^2}}.$$

By integrating we find $g = \pm\frac{1}{a}(\sinh au)^{-1} + c$, where c is a constant that we can assume equal to zero because if necessary we can apply a translation along the z-axis. So we find a surface of revolution parameterized by

$$\sigma(u,v) = (\frac{\sqrt{1+a^2u^2}}{a}\cos v, \frac{\sqrt{1+a^2u^2}}{a}\sin v, \pm\frac{1}{a}(\sinh au)^{-1}).$$

We want to prove that this surface is a catenoid of the form

$$\sigma(x,y) = (a\cosh x\cos y, a\cosh x\sin y, ax).$$

We put $x = \pm\frac{1}{a}(\sinh au)^{-1}$ and we get $au = \pm\sinh(ax)$. We insert au in the term of f:

$$f = \frac{\sqrt{1+(\sinh ax)^2}}{a} = \frac{1}{a}\cosh(ax).$$

We obtain a catenoid parameterized by

$$\sigma(x,y) = (\frac{1}{a}\cosh(ax)\cos y, \frac{1}{a}\cosh(ax)\sin y, x).$$

To complete the proof we argue as follows. Let $\beta < 0$. Then, if the profile curve is defined for values of $u \geq \beta$, we must have $\dot{g}(\beta) = 0$, for otherwise $\dot{g}$ would be non-zero on an open interval containing β, which would contradict our assumption that (α, β) is the largest open interval containing u_0 on which $\dot{g} \neq 0$. But the formulas above show that

$$\dot{g}^2 = \frac{1}{1+a^2u^2} \; if \; u \in (\alpha,\beta),$$

so, since $\dot{g}$ is a continous function of u, $\dot{g}(\beta) = \pm(1+a^2\beta^2)^{-\frac{1}{2}} \neq 0$. This contradiction shows that the profile curve is not defined for values of $u \geq \beta$. Of course, this also holds trivially if $\beta = \infty$. A similar argument applies to α, and shows that (α, β) is the entire domain of definition of the profile curve. Hence, the whole of S is part of a catenoid.

$\square$

4.2. **Helicoid.** The helicoid is a ruled surface swept by a straight line that rotates at a constant speed about an axis orthogonal to the line, while simultaneously moving at a constant speed along the axis. We can take the axis to be the z-axis. Let ω be the angular velocity of the rotating line and α its speed along the z-axis. A parameterization of the helicoid is

$$\sigma(u, v) = (u \cos \omega v, u \sin \omega v, \alpha v),$$

where $\alpha \neq 0$.

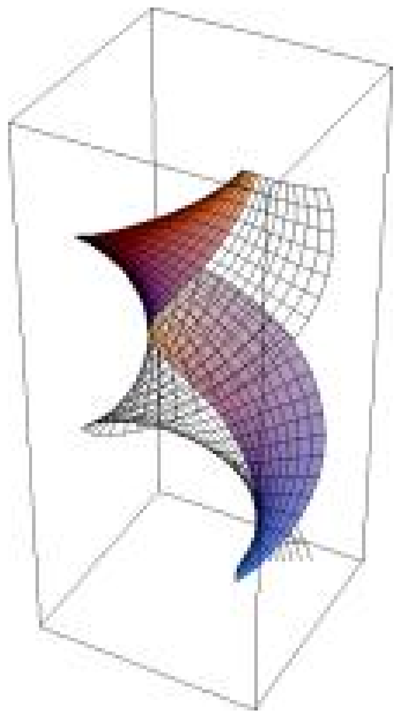

FIGURE 5. Helicoid

Let us check that the helicoid is a minimal surface for $\omega = 1$:

$$\sigma_u = (\cos v, \sin v, 0),$$

$$\sigma_v = (-u \sin v, u \cos v, \lambda),$$

$$E = 1, \qquad F = 0, \qquad G = u^2 + \lambda^2,$$

$$\sigma_{uu} = (0, 0, 0), \quad \sigma_{uv} = (-\sin v, \cos v, 0), \quad \sigma_{vv} = (-u \cos v, -u \sin v, 0),$$

$$\vec{n} = \frac{(\lambda \sin v, -\lambda \cos v, u)}{\sqrt{\lambda^2 + v^2}},$$

$$L = 0, \qquad M = \frac{\lambda}{\sqrt{\lambda^2 + v^2}}, \qquad N = 0.$$

So the mean curvature is $H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0$.

Proposition 4.2. *Any ruled minimal surface is part of a plane or part of a helicoid.*

Proof. We take a ruled surface parameterized by

$$\sigma(u,v) = \gamma(u) + v\delta(u).$$

We can make a simplification by assuming that $\| \delta(u) \| = 1$. This assumption implies that $\delta\dot{\delta} = 0$.

If $\dot{\delta}(u) = 0$ for all values of u, than δ is a constant vector and the surface is parameterized by $\sigma(u,v) = \gamma(u) + vp$, where p is a fixed point. So $\sigma_u = \dot{\gamma}$, $\sigma_v = p$, $\sigma_{uu} = \ddot{\gamma}$, $\sigma_{vv} = 0$, $\sigma_{uv} = 0$ and the coefficients of the first and second fundamental form are

$$E = \dot{\gamma}^2, \quad F = p^2, \quad G = p\dot{\gamma}, \quad L = \frac{\ddot{\gamma}(\dot{\gamma} \times p)}{\| \dot{\gamma} \times p \|}, \quad M = 0, \quad N = 0.$$

The surface is minimal if and only if

$$H = \frac{GL - 2FM - EN}{EG - F^2} = \frac{\dot{\gamma}\ddot{\gamma}(\dot{\gamma} \times p)}{(\dot{\gamma}^3 - p^3)(\| \dot{\gamma} \times p \|)} = 0.$$

We notice that $\dot{\gamma} \neq 0$ otherwise γ is a constant and σ is a line. It follows that $\ddot{\gamma} = 0$ and so γ is a line and σ is part of a plane.

If $\dot{\delta}$ is never zero we can assume that δ is a unit speed curve i.e. $\| \dot{\delta} \| = 1$. This assumption implies that $\dot{\delta}\ddot{\delta} = 0$. In this case we have

$$\sigma_u = \dot{\gamma} + v\dot{\delta}, \quad \sigma_v = \delta, \quad \sigma_{uu} = \ddot{\gamma} + v\ddot{\delta}, \quad \sigma_{vv} = 0, \quad \sigma_{uv} = \dot{\delta}$$

and the normal vector to the surface is

$$\vec{N} = \frac{(\dot{\gamma} + v\dot{\delta}) \times \delta}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|}.$$

The coefficients of the first fundamental form are $E = \| \dot{\gamma} + v\dot{\delta} \|^2$, $F = (\dot{\gamma} + v\dot{\delta})\delta$. Using the fact that δ is a unit speed curve we have $F = \dot{\gamma}\delta$. If we use the assumption $\| \delta \| = 1$, than we have $G = 1$. Let us calculate the coefficients of the second fundamental form. We get

$$L = \frac{((\dot{\gamma} + v\dot{\delta}) \times \delta)(\ddot{\gamma} + v\ddot{\delta})}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|},$$

$$M = \frac{((\dot{\gamma} + v\dot{\delta}) \times \delta)\dot{\delta}}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|} = \frac{\dot{\delta}((\dot{\gamma} \times \delta) + v(\dot{\delta} \times \delta))}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|} = \frac{\dot{\delta}(\dot{\gamma} \times \delta}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|}$$

because $| \dot{\delta} \times \delta | = \dot{\delta}\delta \sin\frac{\pi}{2} = \dot{\delta}\delta = 0$,

$$N = 0.$$

The surface is minimal if and only if

$$H = \frac{GL - 2FM - EN}{EG - F^2} = 0$$

i.e.

$$\frac{((\dot{\gamma} + v\dot{\delta}) \times \delta)(\ddot{\gamma} + v\ddot{\delta})}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|} - 2(\delta\dot{\gamma})(\frac{\dot{\delta}(\dot{\gamma} \times \delta)}{\| (\dot{\gamma} + v\dot{\delta}) \times \delta \|}) = 0$$

i.e.

$$((\dot{\gamma} + v\dot{\delta}) \times \delta)(\ddot{\gamma} + v\ddot{\delta}) = 2\delta\dot{\gamma}\dot{\delta}(\dot{\gamma} \times \delta)$$

i.e.

$$(\ddot\gamma(\dot\gamma \times \delta) + \ddot\gamma v(\dot\delta \times \delta) + v\ddot\delta(\dot\gamma \times \delta) + v^2\ddot\delta(\dot\delta \times \delta)) = 2\delta\dot\gamma\dot\delta(\dot\gamma \times \delta).$$

This equation must hold for all values of (u, v). Equating coefficients of powers of v gives

$$\ddot\gamma(\dot\gamma \times \delta) = 2\delta\dot\gamma\dot\delta(\dot\gamma \times \delta), \tag{1}$$

$$\ddot\gamma(\dot\delta \times \delta) + \ddot\delta(\dot\gamma \times \delta) = 0, \tag{2}$$

$$\ddot\delta(\dot\delta \times \delta) = 0. \tag{3}$$

These equations show us three important facts. The equation (3) shows us that the vector $\ddot\delta$ is orthogonal to the vector $(\dot\delta \times \delta)$ i.e. $\ddot\delta$ lies in the plane given by δ and $\dot\delta$. So there exist smooth functions $\alpha(u)$ and $\beta(u)$ such that

$$\ddot\delta = \alpha\delta + \beta\dot\delta.$$

Since δ is a unit speed curve, we have $\dot\delta\ddot\delta = 0$ and therefore $\beta = 0$. Since $\| \delta \| = 1$, we have $\delta\dot\delta = 0$. Differentiation gives $\delta\ddot\delta + \dot\delta^2 = 0$. We obtain $\delta\ddot\delta = -\dot\delta^2 = -1$ and so $\alpha = -1$.

It follows that equation (3) shows us that the curve δ is such that $\ddot\delta = -\delta$. So this curve has curvature $k = \| \ddot\delta \| = \| -\delta \| = 1$. Let us calculate it's torsion τ through the Frenet's formula $\dot b = -\tau n$, where b is the binormal vector and n is the normal vector. We have

$$\dot b = \frac{d}{du}(\dot\delta \times n) = \frac{d}{du}(\dot\delta \times \frac{\ddot\delta}{\| \ddot\delta \|}) = \frac{d}{du}(\dot\delta \times -\delta)$$
$$= -(\ddot\delta \times \delta + \dot\delta \times \dot\delta) = -(-\delta \times \delta) = 0.$$

It follows that the torsion of δ is zero so the curve is located in a plane and since its curvature is 1, it is a circle. We have supposed that $\| \delta \| = 1$ so δ parameterizes a circle of radius 1. By applying a rigid motion, we can assume that δ is the circle with the center at the origin of the xy-plane, so that

$$\delta(u) = (\cos u, \sin u, 0).$$

If we consider the equation (2) and the above result $\ddot\delta = -\delta$, we have

$$\ddot\gamma(\dot\delta \times \delta) + \ddot\delta(\dot\gamma \times \delta) = \ddot\gamma(\dot\delta \times \delta) - \delta(\dot\gamma \times \delta) = \ddot\gamma(\dot\delta \times \delta) = 0,$$

because the vector $(\dot\gamma \times \delta)$ is orthogonal to δ and $\dot\gamma$. So $\ddot\gamma$ is orthogonal to the vector $(\dot\delta \times \delta)$, that is orthogonal to the xy-plane because δ and $\dot\delta$ lie in the xy-plane. It follows, that $\ddot\gamma$ is parallel to the xy-plane so the vector $\ddot\gamma$ has the third component equal to zero and the parameterization will have the following form:

$$\gamma(u) = (f(u), g(u), au + b),$$

where f and g are smooth functions and a, b are constants. If $a = 0$ the surface is part of the plane $z = b$. We suppose $a \neq 0$. Up to now our surface parameterization has the following form:

$$\sigma(u, v) = (f(u) + v\cos u, g(u) + \sin u, au + b).$$

We would like to prove that f, g and b are zero. To prove this we will use the equation (1) and the following assumption on the curves γ and δ.

Remark: we can assume that $\dot{\gamma}\dot{\delta} = 0$ without loosing generality because we can reparameterize the surface with

$$\sigma(u, \tilde{v}) = \tilde{\gamma} + \tilde{v}\delta(u),$$

where $\tilde{v} = v + \dot{\gamma}\dot{\delta}$ and $\tilde{\gamma}(u) = \gamma(u) - (\dot{\gamma}\dot{\delta})\delta(u)$. With this new parameterization we can write

$$\dot{\tilde{\gamma}}\dot{\delta} = \left(\dot{\delta} - \frac{d}{du}(\dot{\gamma}\dot{\delta})\delta - (\dot{\gamma}\dot{\delta})\dot{\delta}\right)\dot{\delta} = \dot{\gamma}\dot{\delta} - \ddot{\gamma}\dot{\delta}\dot{\delta}\delta - \dot{\gamma}\ddot{\delta}\dot{\delta}\delta - \dot{\gamma}\dot{\delta}\dot{\gamma}^2 = 0$$

because $\delta\dot{\delta} = 0$ and $\dot{\delta}^2 = 1$. This means that we could have assumed at the beginning that $\dot{\gamma}\dot{\delta} = 0$.

Now we will use equation (1) and the assumption $\dot{\gamma}\dot{\delta} = 0$ to prove that f and g are zero. If we insert the components of the parameterization of the curves $\delta(u) = (\cos u, \sin u, 0)$ and $\gamma(u) = (f(u), g(u), au{+}b)$, their derivatives $\dot{\delta} = (-\sin u, \cos u, 0)$, $\dot{\gamma} = (\dot{f}(u), \dot{g}(u), a)$ and the acceleration $\ddot{\gamma} = (\ddot{f}, \ddot{g}, 0)$ into the equation (1), we get

$$(\ddot{f}, \ddot{g}, 0)(-a\sin u, a\cos u, \dot{f}(u)\sin u - \dot{g}\cos u)$$
$$= 2(\dot{f}\cos u + \dot{g}\sin u)(a\sin^2 u, a\cos^2 u, 0)$$

i.e.

$$\ddot{g}\cos u - \ddot{f}\sin u = 2(\dot{f}\cos u - \dot{g}\sin u). \tag{4}$$

The differentiation of the condition $\dot{\gamma}\dot{\delta} = 0$ gives

$$\frac{d}{du}(-\dot{f}\sin u + \dot{g}\cos u) = 0$$

i.e.

$$-\ddot{f}\sin u - \dot{f}\cos u + \ddot{g}\cos u - \dot{g}\sin u = 0. \tag{5}$$

If we put together (4) and (5), we get

$$\dot{f}\cos u + \dot{g}\sin u = 0.$$

This equation, together with the condition $\dot{\gamma}\dot{\delta} = 0$, gives $\dot{f} = \dot{g} = 0$. So f and g are constants. By a translation of the surface, we can assume that the constants f, g and b are zero so that the parameterization of the surface is

$$\sigma(u, v) = (v\cos u, v\sin u, au),$$

that is a helicoid.

The proof is now completed by an argument similar to that used at the end of the previous proposition, which shows that the whole surface is either part of a plane or part of a helicoid.

□

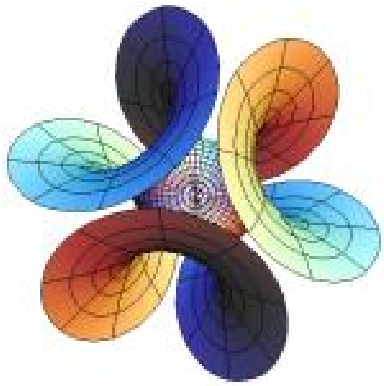

FIGURE 6. Enneper surface

4.3. **Enneper's minimal surface.** Enneper's minimal surface is parameterized by

$$\sigma(u,v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2\right).$$

Let us check that the Enneper surface is indeed minimal:

$$\sigma_u = (1 - u^2 + v^2, 2uv, 2u),$$
$$\sigma_v = (2uv, 1 - v^2 + u^2, -2v),$$
$$E = 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4 = G, \qquad F = 0$$
$$\sigma_{uu} = (-2u, 2v, 2), \qquad \sigma_{uv} = (2v, 2u, 0), \qquad \sigma_{vv} = (2u, -2v, -2),$$
$$\vec{n} = \frac{(-2v^2u - 2u - 2u^3, 2u^2v + 2v + 2v^3, 1 - 2u^2v^2 - u^4 - v^4)}{\| \sigma_u \times \sigma_v \|},$$
$$L = \frac{2 + 4u^2 + 4v^2 + 2u^4 + 2v^4 + 4u^2v^2}{\| \sigma_u \times \sigma_v \|} = -N, \qquad M = 0.$$

So the mean curvature is $H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{LE - LE}{2(EG - F^2)} = 0$.

Strictly speaking, this is not a surface patch, as it is not injective and it has self-intersections. However, if we restrict (u, v) to lie in sufficiently small open sets, σ will be injective by the inverse function theorem. We can show that the Enneper surface has self-intersections by setting $u = \varrho\cos\theta$, $v = \varrho\sin\theta$.

Similarly, it can be shown that the intersection of the surface with the plane $x = 0$ is also a curve of self-intersection. It is easily seen that they are the only self-intersections of Enneper surface.

4.4. **Scherk's minimal surface.** Scherk's minimal surface is the surface with Cartesian equation

$$z = \ln\left(\frac{\cos y}{\cos x}\right).$$

Note, that the surface exists only when $\cos x$ and $\cos y$ are both > 0 or < 0.

Let us check that the Scherk's surface is indeed minimal:

$$\sigma = \left(u, v, \ln\frac{\cos v}{\cos u}\right),$$
$$\sigma_u = (1, 0, \tan u), \qquad \sigma_v = (0, 1, -\tan v),$$

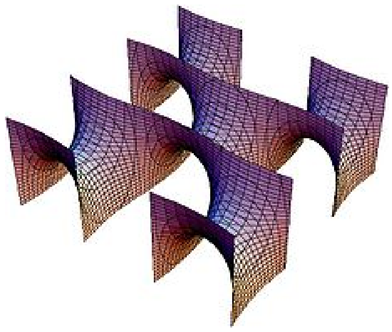

FIGURE 7. Scherk's surface

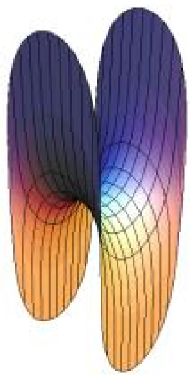

FIGURE 8. A part of the Scherk's surface

$$E = 1 + \tan^2 u, \qquad F = -\tan u \tan v, \qquad G = 1 + \tan^2 v,$$

$$\sigma_{uu} = \left(0, 0, \frac{1}{\cos^2 u}\right), \qquad \sigma_{uv} = (0, 0, 0), \qquad \sigma_{vv} = \left(0, 0, -\frac{1}{\cos^2 v}\right),$$

$$\vec{n} = \frac{(-\tan u, \tan v, 1)}{\sqrt{\tan^2 u + \tan^2 v + 1}},$$

$$L = \frac{1}{\cos^2 u \sqrt{\tan^2 u + \tan^2 v + 1}},$$

$$M = 0,$$

$$N = -\frac{1}{\cos^2 v \sqrt{\tan^2 u + \tan^2 v + 1}}.$$

So the mean curvature is

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{\frac{1+\tan^2 v}{\cos^2 u} - \frac{1+\tan^2 u}{\cos^2 v}}{2(EG - F^2)\sqrt{\tan^2 u + \tan^2 v + 1}}$$

$$= \frac{\frac{\cos^2 v + \sin^2 v - \cos^2 u - \sin^2 u}{\cos^2 u \cos^2 v}}{2(EG - F^2)\sqrt{\tan^2 u + \tan^2 v + 1}} = 0.$$

4.5. **Henneberg's minimal surface.** Henneberg's minimal surface is given by the following parametric equations:

$$x(u,v) = 2\sinh u \cos v - \frac{2}{3}\sinh(3u)\cos(3v)$$
$$y(u,v) = 2\sinh u \sin v + \frac{2}{3}\sinh(3u)\sin(3v)$$
$$z(u,v) = 2\cosh u \cos(2v).$$

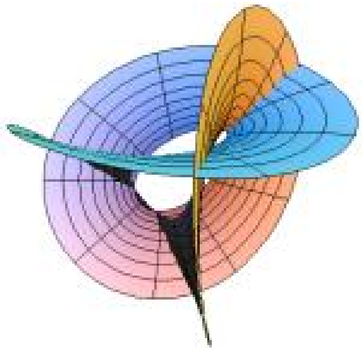

FIGURE 9. Henneberg's surface

Let us check that it is indeed a minimal surface. The coefficients of the first and second fundamental form are

$$E = 8\cosh^2 u(\cosh(4u) - \cos(4v)),$$
$$F = 0,$$
$$G = 8\cosh^2 u(\cosh(4u) - \cos(4v)),$$
$$L = -4\cos(2v)\sinh(2u),$$
$$M = 4\cosh(2u)\sin(2v),$$
$$N = 4\sinh(2u)\cos(2v).$$

So the mean curvature is

$$H = \frac{LE - LE}{2E^2} = 0.$$

4.6. **Other examples.** Others examples of minimal surfaces are shown in the following pictures:

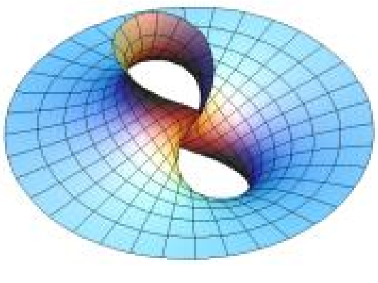

FIGURE 10. Plannar Enneper

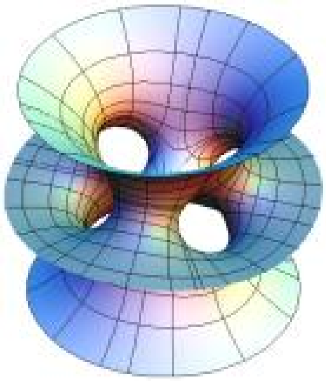

Figure 11. Costa-Hoffmann-Meeks surface

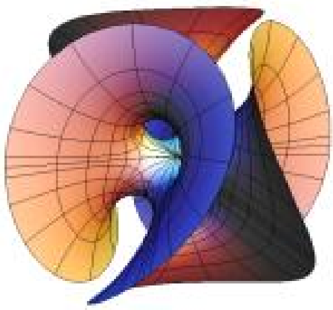

Figure 12. Chen Gackstatter

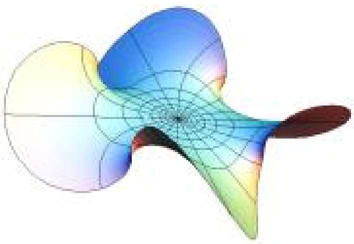

Figure 13. Kusner (Dihedral Symmetric)

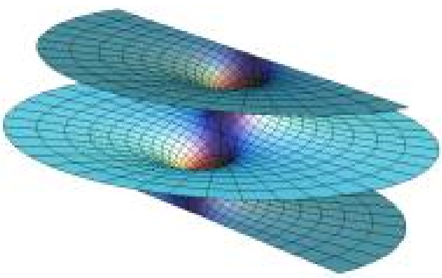

Figure 14. Riemann surface

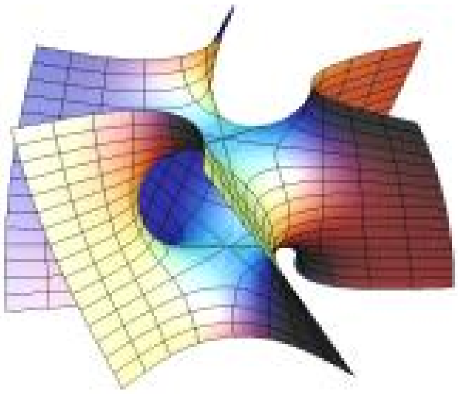

FIGURE 15. Twisted Scherk

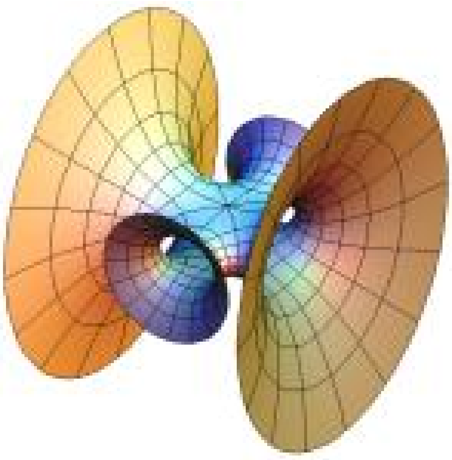

FIGURE 16. Symmetric 4-noid

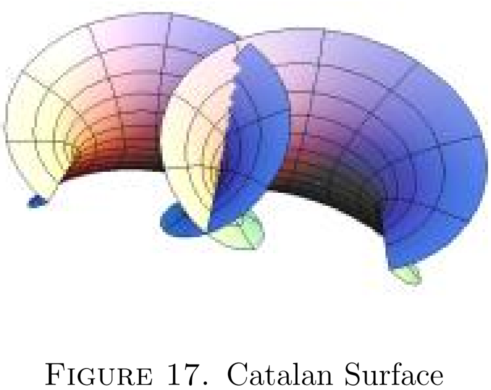

FIGURE 17. Catalan Surface

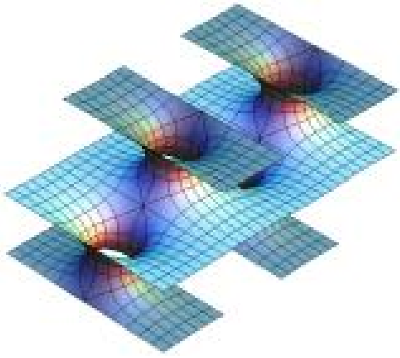

Figure 18. Karcher JE Saddle Tower

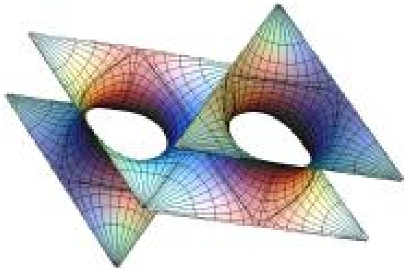

Figure 19. Schwarz H Family Surface

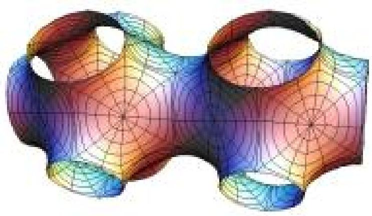

Figure 20. Schearz PD Family Surface

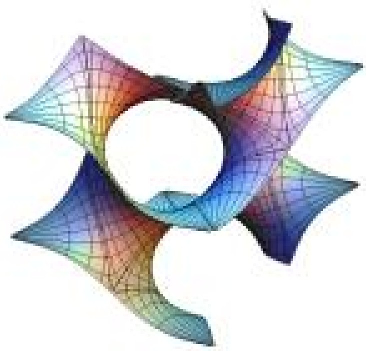

Figure 21. Lidinoid

5. Gauss map of a minimal surface

In this section we will start with the definitions of the Gauss map and of the conformal map. Then we will recall some important results like the existence of a conformal parameterization for every surface and the relation between the first fundamental forms of two surfaces related by a conformal map. At the end of the section, we will prove the basic result, that the Gauss map of a minimal surface is a conformal map.

Definition 5.1. *The Gauss map of a surface patch is the map $\sigma : U \to \mathbb{R}^3$ that associates to a point $\sigma(u, v)$ of the surface the standard unit normal N(u,v) regarded as a point of the unit sphere S^2.*

Definition 5.2. *A map $f\colon U \to V$ is called conformal at u_0 if it preserves oriented angles between curves through u_0 with respect to their orientation (i.e. not just the acute angle).*

The definition of conformal parameterization of a surface or of a conformal patch is the following.

Definition 5.3. *A surface patch $\sigma : U \to \mathbb{R}^3$ is called conformal if the coefficients of the first fundamental form are such that $F = 0$ and $E = G$ is a positive smooth function on U.*

Proposition 5.4. *A conformal patch $\sigma\colon U \to S \subset \mathbb{R}^3$ is a conformal map.*

Proof. Let $(u(t), v(t))$ and $(\tilde{u}(t), \tilde{v}(t))$ be two curves in $\mathbb{R}^3$. Let σ be a surface patch that preserves oriented angles between curves through u_0 with respect to their orientation. So the angle between $(u(t), v(t))$ and $(\widetilde{u(t)}, \widetilde{v(t)})$ is the same as the angle between $\sigma(u(t), v(t))$ and $\sigma(\tilde{u}(t), \tilde{v}(t))$. To simplify the notation, we drop the t. The angle between $(u(t), v(t))$ and $(\widetilde{u(t)}, \widetilde{v(t)})$ is

$$\frac{(\dot{u}, \dot{v})(\dot{\tilde{u}}, \dot{\tilde{v}})}{\sqrt{\dot{u}^2 + \dot{v}^2}\sqrt{\dot{\tilde{u}}^2 + \dot{\tilde{v}}^2}} = \frac{\dot{u}\dot{\tilde{u}} + \dot{v}\dot{\tilde{v}}}{\sqrt{\dot{u}^2 + \dot{v}^2}\sqrt{\dot{\tilde{u}}^2 + \dot{\tilde{v}}^2}}.$$

The angle between $\sigma(u(t), v(t))$ and $\sigma(\tilde{u}(t), \tilde{v}(t))$ is

$$\begin{aligned} &\frac{(\sigma_u\dot{u} + \sigma_v\dot{v})(\sigma_u\dot{\tilde{u}} + \sigma_v\dot{\tilde{v}})}{\sqrt{(\sigma_u\dot{u} + \sigma_v\dot{v})^2}\sqrt{(\sigma_u\dot{\tilde{u}} + \sigma_v\dot{\tilde{v}})^2}} \\ =\ &\frac{E\dot{u}\dot{\tilde{u}} + F\dot{u}\dot{\tilde{v}} + F\dot{\tilde{u}}\dot{v} + G\dot{v}\dot{\tilde{v}}}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2}}. \end{aligned}$$

Since these two angles are the same, it follows that $E = G$ and $F = 0$. □

Proposition 5.5. *Suppose that we have in Euclidean space $\mathbb{R}^3$ a 2-dimensional surface with the parameterization*

$$\sigma(p, q) = (x(p, q), y(p, q), z(p, q)),$$

where (p,q) ranges over some region of $\mathbb{R}^2$. Suppose that the coefficients of the first fundamental form of the surface E, F, G are real valued analytic functions of the real variables p, q (i.e. are representable as power series in p, q). Then, there exist new (real) local co-ordinates u, v for the surface in terms of which the coefficients of the first fundamental form are $\tilde{F} = 0$ and $\tilde{E} = \tilde{G}$.

Proof. [10] □

Theorem 5.6. *Let S_1 and S_2 be two surfaces. A map $f : S_1 \longrightarrow S_2$ is conformal if and only if for every parameterization σ of S_1 its fundamental form is non-zero and a multiple of the first fundamental form of $f(\sigma)$.*

Proof. [17] □

Lemma 5.7. *The expressions of N_u and N_v in terms of σ_u and σ_v are*

$$N_u = a\sigma_u + b\sigma_v$$
$$N_v = c\sigma_u + d\sigma_v,$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\mathcal{W}$, $\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II}$ is the Weingarten matrix and $\mathcal{F}_I$, $\mathcal{F}_{II}$ are the first and the second fundamental form.

Proof. [17] □

Remark: in the proposition 2.8 we have proved that the principal curvatures k_1 and k_2 are the eigenvalues of the Weingarten matrix.

Theorem 5.8. *Let $\sigma(u, v)$ be a minimal surface patch with a nowhere vanishing Gaussian curvature. Then, the Gauss map is a conformal map from σ to a part of the unit sphere.*

Proof. We have to show that $N(u, v)$ is conformal. By the previous theorem this means that the first fundamental form of the sphere corresponding to N is proportional to the first fundamental form E, F, G of σ. Explicitly

$$\begin{pmatrix} \langle N_u, N_u\rangle & \langle N_u, N_v\rangle \\ \langle N_u, N_v\rangle & \langle N_v, N_v\rangle \end{pmatrix} = \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{6}$$

for some scalar λ. Let us denote the first fundamental form of σ by $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \mathcal{F}_I$. By the above lemma $N_u = a\sigma_u + b\sigma_v$ and $N_v = c\sigma_u + d\sigma_v$, where the coefficients a, b, c, d are the coefficients of the Weingarten matrix. It follows that

$$N_u N_u = a^2E + 2abF + b^2G,$$
$$N_u N_v = acE + (ad + bc)F + bdG,$$
$$N_v N_v = c^2E + 2cdF + d^2G.$$

So we are looking for a λ such that

$$\begin{pmatrix} a^2E + 2abF + b^2G & acE + (ad + bc)F + bdG \\ acE + (ad + bc)F + bdG & c^2E + 2cdF + d^2G \end{pmatrix} = \lambda\mathcal{F}_I. \tag{7}$$

The above can be rewritten as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \lambda\mathcal{F}_I, \tag{8}$$

which, in turn, becomes

$(-\mathcal{W})^t\mathcal{F}_I(-\mathcal{W}) = \lambda\mathcal{F}_I$

$(-\mathcal{F}_I^{-1}\mathcal{F}_{II})^t\mathcal{F}_I(-\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \lambda\mathcal{F}_I.$

$\mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_I(-\mathcal{F}_I^{-1}\mathcal{F}_{II}) = \lambda\mathcal{F}_I$

$\mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_{II} = \lambda\mathcal{F}_I$

$\mathcal{F}_I^{-1}\mathcal{F}_{II}\mathcal{F}_I^{-1}\mathcal{F}_{II} = \lambda\mathcal{I}$

$(-\mathcal{W})^2 = \lambda\mathcal{I}$
$\begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix} = \lambda\mathcal{I}.$

Now we will use four facts to show that $a+d=0$. From the definition of minimal surface and the definition of the mean curvature we get $H = \frac{1}{2}(k_1+k_2) = 0$; from the fact that the principal curvature k_1 and k_2 are the eigenvalues of the Weingarten matrix $\mathcal{W} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ and from the fact that the sum of eigenvalues of a matrix is equal to the sum of its diagonal entries we get $k_1+k_2 = -(a+d)$. So we obtain the equation $a+d=0$ i.e. $a=-d$. We are searching for the λ such that

$$\begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix} = \lambda\mathcal{I}.$$

It follows that $\lambda = a^2+bc$. □

6. Harmonic parameterization

In this section we will see that every minimal surface has a conformal parameterization, which is harmonic. Harmonicity of conformal parameterizations is characteristic for minimal surfaces. Every surface has a conformal parameterization, but only the conformal parameterizations of minimal surfaces are harmonic. In this section, we will first see the definition of a harmonic function and then the theorem which states the above fact.

Definition 6.1. *A harmonic function is a twice continuously differentiable function $f : U \to \mathbb{R}$ (where U is an open subset of $\mathbb{R}^n$) which satisfies the Laplace equation*

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + ... + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. This is usually written as $\Delta f = 0$.

Proposition 6.2. *In terms of conformal coordinates, the parameterization of a minimal surface is harmonic.*

Proof. Let $r(u,v)$ be a locally conformal parameterization of a minimal 2-dimensional surface V^2 in Euclidean $\mathbb{R}^3$. Since our parameterization is conformal, we have $F=0$, $E=G$, and since the surface is minimal, we have $H=0$. So

$$H = \frac{GL-2FM-EN}{EG-F^2} = \frac{L-N}{E} = \frac{\langle r_{uu}+r_{vv}, n\rangle}{E} = \frac{\langle \Delta r, n\rangle}{E} = 0,$$

where n is the normal vector to the surface and Δ is the Laplace operator. This holds if and only if $\langle \Delta r, n\rangle = 0$.

Let us show that the equation $\langle \Delta r, n\rangle = 0$ is equivalent to $\Delta r = 0$ i.e. to the parameterization being harmonic. It is trivial that $\Delta r = 0$ implies $\langle \Delta r, n\rangle = 0$. For the reverse, we will use the fact that whenever $\langle \Delta r, n\rangle = 0$, $\langle \Delta r, r_u\rangle = 0$ and $\langle \Delta r, r_v\rangle = 0$ we have $\Delta r = 0$. This is so because $r_u,\ r_v,\ n$ are linearly independent at a non-singular point of V^2. So we have to show that $\langle \Delta r, n\rangle = 0$ and $\langle \Delta r, r_u\rangle = 0$. By virtue of the conformality of the parameterization we have $F=0$ and $E=G$

i.e. $\langle r_u, r_u\rangle = \langle r_v, r_v\rangle$ and $\langle r_u, r_v\rangle = 0$. If we differentiate with respect to u and v we obtain

$$2\langle r_{uu}, r_u\rangle = 2\langle r_{vu}, r_v\rangle$$

$$2\langle r_{uv}, r_u\rangle = 2\langle r_{vv}, r_v\rangle$$

$$\langle r_{uu}, r_v\rangle + \langle r_u, r_{uv}\rangle = 0$$

$$\langle r_{uv}, r_v\rangle + \langle r_{vv}, r_v\rangle = 0.$$

By joining together, we get

$$\langle r_{uu}, r_u\rangle + \langle r_{vv}, r_u\rangle = 0$$

$$\langle r_{uu}, r_v\rangle + \langle r_{vv}, r_v\rangle = 0$$

i.e. $\langle \Delta r, n\rangle = 0$ and $\langle \Delta r, r_u\rangle = 0$. □

7. Holomorphic functions

In the following section we will present an interesting relation between minimal surfaces and holomorphic functions which is called Weierstrass representation.

Definition 7.1. *A map $\varphi : D \subset \mathbb{C} \longrightarrow D_1 \subset \mathbb{C}$ is holomorphic if it is differentiable in a complex way. For every $z_0 \in D$ there exists a complex number $\varphi'(z_0) \in \mathbb{C}$ such that for every $h \in \mathbb{C}$ the following equation holds:*

$$\varphi(z_0 + h) = \varphi(z_0) + \varphi'(z_0)h + o(h),$$

where $\lim_{|h|\to 0} \frac{|o(h)|}{|h|} = 0$.

Theorem 7.2. *Let $\sigma : U \longrightarrow \mathbb{R}^3$ be a conformal surface patch. We introduce complex coordinate $\zeta = u + iv$ in the ambient plane of U. We define*

$$\varphi : U \subset \mathbb{C} \longrightarrow \mathbb{C}^3$$

$$\varphi(\zeta) = \sigma_u - i\sigma_v$$

$$\varphi = (\varphi_1, \varphi_2, \varphi_3).$$

Then σ is minimal if and only if the function φ is holomorphic on U.

Proof. The function φ is holomorphic if and only if it satisfies the Cauchy-Riemann equations:

$$\sigma_{uu} = -\sigma_{vv}$$

$$\sigma_{uv} = \sigma_{vu}.$$

(These are the necessary and sufficient condition for φ to be holomorphic.) The second equation imposes no condition on σ so φ is holomorphic already if

$$\sigma_{uu} = -\sigma_{vv}$$

or $\Delta\sigma = 0$. This is true since σ is a conformal minimal parameterization and the proposition in the previous section says that conformal parameterization are harmonic precisely when they parameterize minimal surfaces.

□

Theorem 7.3. *Let $\sigma : U \longrightarrow \mathbb{R}^3$ be a conformally parameterized minimal surface. We define (as in the previous theorem)*

$$\varphi : U \subset \mathbb{C} \longrightarrow \mathbb{C}^3$$

$$\varphi(\zeta) = \sigma_u - i\sigma_v$$

$$\varphi = (\varphi_1, \varphi_2, \varphi_3).$$

(The function φ is holomorphic because of the previous theorem.) Than the following conditions hold:

$$(i)\ \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0;$$

$$(ii)\ \varphi \textit{ is nowhere zero.}$$

Conversely, if U is simply-connected and $\varphi_1,\ \varphi_2,\ \varphi_3 : U \longrightarrow \mathbb{C}$ are holomorphic functions satisfying the conditions (i) and (ii) above, then there exists a conformaly parameterized minimal surface $\sigma : U \longrightarrow \mathbb{R}^3$ such that $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\varphi(\zeta) = \sigma_u - i\sigma_v$. Moreover, σ is uniquely determined by $\varphi_1,\ \varphi_2$ and φ_3 up to a translation.

Proof. Let $\sigma : U \longrightarrow \mathbb{R}^3$ be a conformally parameterized minimal surface. We denote the components of σ_u and σ_v as $\sigma_u = (\sigma_{u1}, \sigma_{u2}, \sigma_{u3})$ and $\sigma_v = (\sigma_{v1}, \sigma_{v2}, \sigma_{v3})$. Let φ be such that $\varphi : U \subset \mathbb{C} \longrightarrow \mathbb{C}^3$, $\varphi(\zeta) = \sigma_u - i\sigma_v$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

We have to prove that the conditions (i) and (ii) hold. Let us start with the first condition:

$$\begin{aligned}\varphi_1^2 + \varphi_2^2 + \varphi_3^2 &= \sigma_{u1}^2 - 2i\sigma_{u1}\sigma_{v1} - \sigma_{v1}^2 + \sigma_{u2}^2 - 2i\sigma_{u2}\sigma_{v2} - \sigma_{v2}^2 + \sigma_{u3}^2 - 2i\sigma_{u3}\sigma_{v3} - \sigma_{v3}^2 \\ &= \parallel \sigma_u \parallel^2 - \parallel \sigma_v \parallel^2 - 2i\sigma_u\sigma_v.\end{aligned}$$

Since σ is conformal, $F = \langle \sigma_u, \sigma_v \rangle = 0$ and $E = F = \parallel \sigma_u \parallel^2 = \parallel \sigma_v \parallel^2$, so we obtain $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$. The second condition (ii) also holds because $\varphi = 0$ if and only if $\sigma_u = \sigma_v = 0$, which is impossible since σ is regular.

For the converse we suppose that U is simply-connected and $\varphi_1,\ \varphi_2,\ \varphi_3 : U \longrightarrow \mathbb{C}$ are holomorphic functions satisfying conditions (i) and (ii). We want to prove that there exists a conformal parameterized minimal surface $\sigma : U \longrightarrow \mathbb{R}^3$ such that $\varphi(\zeta) = \sigma_u - i\sigma_v$, where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. Let us fix $(u_0, v_0) \in U$ and define

$$\sigma(u, v) = \mathfrak{Re} \int_\pi \varphi(\xi) d\xi,$$

where $\mathfrak{Re}$ indicates the real part of the complex line integral and π is any curve in U from (u_0, v_0) to $(u, v) \in U$.

We have to show the following facts: σ is well defined (i.e. independent from π), σ is a regular surface, σ is conformal, the equation $\varphi(\zeta) = \sigma_u - i\sigma_v$ holds and σ is uniquely determined up to a translation.

The parameterization σ is independent of π because of the Cauchy's theorem, which essentially says that if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same (for the proof see [1]).

The parameterization σ is a regular surface because of the condition (ii).

The parameterization σ is conformal i.e. $F = \langle \sigma_u, \sigma_v \rangle = 0$ and $E = F =$- $\| \sigma_u \|^2 - \| \sigma_v \|^2$ because from the condition (i) we obtain

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \| \sigma_u \|^2 - \| \sigma_v \|^2 - 2i\sigma_u\sigma_v = 0.$$

So $\sigma_u\sigma_v = 0$ and $\| \sigma_u \|^2 = \| \sigma_v \|^2$.

The parameterization σ is minimal because it is conformal and φ is holomorphic.

Let us prove that the equation $\varphi(\zeta) = \sigma_u - i\sigma_v$ holds. We have

$$\varphi(\zeta) = \mathfrak{Re}(\varphi(\zeta)) + i\mathfrak{Im}(\varphi(\zeta)),$$

where $\mathfrak{Re}(\varphi(\zeta))$ and $\mathfrak{In}(\varphi(\zeta))$ are the real and imaginary part of φ. We define

$$\phi(\zeta) := \int_\pi \varphi(\xi) d\xi$$

so that $\frac{d}{d\zeta}\phi(\zeta) = \phi'(\zeta) = \varphi(\zeta)$ and $\sigma = \mathfrak{Re}(\phi)$. We can write

$$\varphi(\zeta) = \mathfrak{Re}(\phi'(\zeta)) + i\mathfrak{In}(\phi'(\zeta)).$$

Since $\mathfrak{Im}(x) = -\mathfrak{Re}(ix)$ for every complex number x, we have

$$\varphi = \mathfrak{Re}(\phi'(\zeta)) - i\mathfrak{Re}(i\phi'(\zeta)).$$

Since if F is a holomorphic function of $\zeta = u + iv$, then

$$F_u = F', \quad F_v = iF', \quad (\overline{F})_u = \overline{F'}, \quad (\overline{F})_v = -i\overline{F'},$$

where the bar denotes the complex-conjugate. It follows

$$\varphi = \mathfrak{Re}(\phi_u) - i\mathfrak{Re}(\phi_v) = (\mathfrak{Re}(\phi))_u - i(\mathfrak{Re}(\phi))_v = \sigma_u - i\sigma_v.$$

It remains to show that σ is uniquely determined up to a translation: let $\tilde{\sigma}$ be another conformal minimal surface corresponding to the same holomorphic function φ. Then both following equations hold:

$$\varphi = \sigma_u - i\sigma_v,$$

$$\varphi = \tilde{\sigma}_u - i\tilde{\sigma}_v.$$

It follows that $\sigma_u = \tilde{\sigma}_u$ and $\sigma_v = \tilde{\sigma}_v$ everywhere on U i.e. $(\sigma - \tilde{\sigma})_u = 0$ and $(\sigma - \tilde{\sigma})_v = 0$

i.e. $\sigma - \tilde{\sigma} = a$, where a is a constant

i.e. $\tilde{\sigma} = \sigma + a$

i.e. $\tilde{\sigma}$ is obtained from σ by translating by the vector a. □

Definition 7.4. *A meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all D except on a set of isolated points, which are poles for the function.*

Remark: Every meromorphic function on D can be expressed as the ratio between two holomorphic functions (with the denominator not identically zero) defined on D. The poles then occur at the zeroes of the denominator. At every pole the principal part (the negative powers) of the Laurent series is finite. Thus, poles are singularities of "finite order".

Definition 7.5. *The Laurent series for a complex function $f(z)$ about a point c is given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n,$$

where the a_n are constants defined by a line integral which is a generalization of Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-c)^{n+1}} dz.$$

The path of integration γ is a closed, rectifiable path containing no self-intersections, enclosing c and lying in an annulus A in which f(z) is holomorphic (analytic). The expansion for $f(z)$ will then be valid anywhere inside the annulus.

Definition 7.6. *An analytic function $f(x)$ is an infinitely differentiable function such that the Taylor series at any point x_0 in its domain*

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

converges to $f(x)$ for x in a neighbourhood of x_0.

Theorem 7.7. *Holomorphic functions are analytic.*

Proof. [1] □

Theorem 7.8. *Let φ be a holomorphic function such that*

$$(i) \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0,$$
$$(ii)\ \varphi \text{ is nowhere zero.}$$

Than φ has the following form:

$$\varphi = \left(\frac{1}{2} f(1-g^2), \frac{i}{2} f(1+g^2), fg \right),$$

where $f : U \longrightarrow \mathbb{C}$ is a holomorphic function not identically zero, U is an open set, $g : U \longrightarrow \mathbb{C}$ is a meromorphic function on U, such that if $\zeta_0 \in U$ is a pole of g of order $m \geq 1$ then ζ_0 is also a zero of f of order $\geq 2m$.

Conversely let U be an open set, let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function not identically zero, let $g : U \longrightarrow \mathbb{C}$ be a meromorphic function on U, such that if $\zeta_0 \in U$ is a pole of g of order $m \geq 1$ then ζ_0 is also a zero of f of order $\geq 2m$. Then the function

$$\varphi = (\frac{1}{2} f(1-g^2), \frac{i}{2} f(1+g^2), fg)$$

is holomorphic and it satisfies the conditions (i) and (ii).

Proof. Let φ be a holomorphic function satisfying conditions (i) and (ii). We have to find the functions f, g such that they will have the desired characteristics, and that φ will be expressible as

$$\varphi = \left(\frac{1}{2} f(1-g^2), \frac{i}{2} f(1+g^2), fg \right).$$

We define

$$f := \varphi_1 - i\varphi_2, \quad g := \frac{\varphi_3}{\varphi_1 - i\varphi_2},$$

where we can suppose $\varphi_1 - i\varphi_2 \neq 0$. Indeed, in the case $\varphi_1 - i\varphi_2 = 0$ we can always replace $\varphi_1 \pm i\varphi_2$ by $\varphi_1 \mp i\varphi_2$ because $\varphi_1 - i\varphi_2$ and $\varphi_1 + i\varphi_2$ cannot both be zero. If they were, we would have $\varphi_1 = \varphi_2 = 0$, hence $\varphi_3 = 0$ by condition (i) and this would violate the condition (ii).

Let us show that φ has the form in the statement using the following tree equations

$$f := \varphi_1 - i\varphi_2,$$
$$g := \frac{\varphi_3}{\varphi_1 - i\varphi_2},$$
$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2.$$

By inserting the first equation into the second, we find the expression of the third component: $\varphi_3 = fg$. By inserting the first equation in the third we get

$$(f + i\varphi_2)^2 + \varphi - 2^2 + g^2 f^2 = 0,$$

and from which we get $\varphi_2 = \frac{i}{2} f(1 - g^2)$. Inserting this last result in the first equation we obtain $\varphi_1 = \frac{1}{2} f(1 - g^2)$.

We still have to show that the functions f and g have the wanted characteristics. The function f is holomorphic because φ is holomorphic and the function g is meromorphic because it is a ratio between two holomorphic functions. If $\zeta_0 \in U$ is a pole of g of order $m \geq 1$ then by the Laurent expansions of f about ζ_0, we get

$$g(\zeta) = \frac{b}{(\zeta - \zeta_0)^m} + ...,$$

where b is a non-zero complex number and the three points indicate terms involving higher powers of $\zeta - \zeta_0$. Since the relation between g and f is $g(\zeta) = \frac{\varphi_3(\zeta)}{f(\zeta)}$ we get

$$f(\zeta) = \frac{\varphi_3(\zeta)}{g(\zeta)} = \frac{\varphi_3(\zeta)(\zeta - \zeta_0)^m}{b} + ...$$

So ζ_0 is a zero of f.

Conversely, let U be an open set, let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function not identically zero, let $g : U \longrightarrow \mathbb{C}$ be a meromorphic function on U such that if $\zeta_0 \in U$ is a pole of g of order $m \geq 1$ then ζ_0 is also a zero of f of order $\geq 2m$. We want to prove that the function $\varphi = (\frac{1}{2} f(1 - g^2), \frac{i}{2} f(1 + g^2), fg)$ is holomorphic and that it satisfies the conditions (i) and (ii). First, we have to show that the components of φ are holomorphic i.e. analytic due to the above theorem. If $\zeta_0 \in U$ is a pole of g of order $m \geq 1$, then ζ_0 is also a zero of f of order $\geq 2m$. Thus, the Laurent expansions of f and g about ζ_0 have the following form:

$$f(\zeta) = a(\zeta - \zeta_0)^n + ...$$
$$g(\zeta) = \frac{b}{(\zeta - \zeta_0)^m} + ...,$$

where a and b are non-zero complex numbers and the three points indicate terms involving higher powers of $\zeta - \zeta_0$. Thus, the components od φ are

$$\frac{1}{2} f(1 - g^2) = -\frac{1}{2} ab^2 (\zeta - \zeta_0)^{n-2m} + ... \ ,$$

$$\frac{i}{2}f(1+g^2) = \frac{i}{2}ab^2(\zeta-\zeta_0)^{n-2m} + \dots \,,$$
$$fg = ab(\zeta-\zeta_0)^{n-m} + \dots \,.$$

Since $n \geq 2m$, the components of φ involve only non-negative powers of $\zeta - \zeta_0$, so that φ is holomorphic near ζ_0. It follows that φ is holomorphic near the poles of g. The function φ is holomorphic wherever g is holomorphic, therefore it is holomorphic everywhere on U.

Now let us prove that φ satisfies the conditions (i) and (ii):

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \frac{1}{4}f^2(1-2g^2+g^4) - \frac{1}{4}f^2(1-2g^2+g^4) + f^2g^2 = 0$$

The function φ is identically zero only if f is identically zero. But we supposed f not to be identically zero. Thus, the second condition is also satisfied. $\square$

Definition 7.9. *The correspondence given by the last two theorems between pairs of functions f, g and minimal surfaces is called Weierstrass representation.*

<u>WEIERSTRASS'S REPRESENTATION IN A SCHEMATIC WAY</u>:

Minimal surface with a conformal parameterization σ.

$$\Updownarrow$$

Holomorphic function $\varphi(\zeta) = \sigma_u - i\sigma_v$ with three components, which satisfies some conditions $[(i)\varphi_1^2+\varphi_2^2+\varphi_3^2 = 0$, (ii) φ *is nowhere zero*$]$ and is at the same time of the form $\varphi = (\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg)$.

$$\Updownarrow$$

Scalar holomorphic function f, scalar meromorphic function g, which satisfy a certain relation regarding their poles and zeros.

Theorem 7.10. *The Gaussian curvature of the minimal surface corresponding to the functions f and g in Weierstrass representation is given by*

$$K = \frac{-16 \mid dg/d\zeta \mid^2}{\mid f \mid^2 (1+ \mid g \mid^2)^4}.$$

Proof. From the previous theorems we know that

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) = \sigma_u - i\sigma_v = \left(\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg\right).$$

Let us define $\bar{\varphi}$ by taking the complex-conjugate of each component of φ i.e.

$$\bar{\varphi} = (\bar{\varphi_1}, \bar{\varphi_2}, \bar{\varphi_3}) = \sigma_u + i\sigma_v = \left(\frac{1}{2}\bar{f}(1-\bar{g}^2), \frac{i}{2}\bar{f}(1+\bar{g}^2), \bar{f}\bar{g}\right).$$

If we compute σ_u and σ_v in terms of φ and $\bar{\varphi}$ we get

$$\sigma_u = \frac{1}{2}(\bar{\varphi} + \varphi),$$

$$\sigma_v = \frac{1}{2i}(\bar{\varphi} - \varphi).$$

The coefficients of the first fundamental form are

$$E = \langle \frac{1}{2}(\bar{\varphi} + \varphi), \frac{1}{2}(\bar{\varphi} + \varphi) \rangle = \frac{1}{4}\bar{\varphi}^2 + \frac{1}{4}\varphi^2 + \frac{1}{2}\bar{\varphi}\varphi,$$

$$F = \langle \frac{1}{2}(\bar{\varphi} + \varphi), \frac{1}{2}(\bar{\varphi} - \varphi) \rangle = \frac{1}{4i}(\bar{\varphi}^2 - \varphi^2) = 0,$$

$$G = \langle \frac{1}{2i}(\bar{\varphi} - \varphi), \frac{1}{2i}(\bar{\varphi} - \varphi) \rangle = \frac{1}{2}\bar{\varphi}\varphi.$$

Since σ is conformal, we have $\varphi^2 = \sigma_u^2 - 2i\sigma_u\sigma_v - \sigma_v^2 = 0$ and $\bar{\varphi}^2 = \sigma_u^2 + 2i\sigma_u\sigma_v - \sigma_v^2 = 0$ and so $E = \frac{1}{2}\bar{\varphi}\varphi = G$. Furthermore

$$E = G = \frac{1}{2}\bar{\varphi}\varphi = \frac{1}{2}(\frac{1}{4}|f|^2(1 - |g|^2)^2 + \frac{1}{4}|f|^2(1 + |g|^2)^2 + |f|^2|g|^2)$$

$$= \frac{1}{4}|f|^2(1 + |g|^2).$$

The normal to the surface is given by

$$N = \frac{\sigma_u \times \sigma_v}{\parallel \sigma_u \times \sigma_v \parallel} = \frac{\frac{1}{4i}(\varphi + \bar{\varphi}) \times (\bar{\varphi} - \varphi)}{\sqrt{-\frac{1}{4}(\varphi \times \bar{\varphi}).(\varphi \times \bar{\varphi})}}$$

$$= \frac{\frac{1}{2i}\varphi \times \bar{\varphi}}{\sqrt{-\frac{1}{4}((\varphi.\varphi)(\bar{\varphi}.\bar{\varphi}) - (\varphi\bar{\varphi})^2)}} = i\frac{\varphi \bar{\times} \varphi}{\varphi.\bar{\varphi}}$$

$$= \frac{1}{1 + |g|^2}(g + \bar{g}, -i(g - \bar{g}), |g|^2 - 1).$$

Since

$$L = -\sigma_u.N_u, \qquad M = -\sigma_u.N_v, \qquad N = -\sigma_v.N_v,$$

(this follows from differentiating $\sigma_u.N = \sigma_v.N = 0$), we find the second fundamental form:

$$L = N = -\frac{1}{2}(fg' + \bar{f}\bar{g'}), \qquad M = 2i(fg' - \bar{f}\bar{g'}).$$

By inserting the first fundamental form and the second fundamental form in the formula of the mean curvature, expressed in terms of the two fundamental forms, we obtain the wanted result. □

Corollary 7.11. *Let S be a minimal surface that is not part of a plane. Then, the zeros of the Gaussian curvature of S are isolated.*

This means that if a point P lies in a surface patch σ of S, say $P = \sigma(u_0, v_0)$, there is a number $\epsilon > 0$ such that the Gaussian curvature K does not vanish at the point $\sigma(u, v)$ of S if $0 < (u - u_0)^2 + (v - v_0)^2 < \epsilon^2$.

Proof. If the surface S is not a plane then K is not identically zero. It is a standard result of complex analysis that the zeros of a non-zero meromorphic function are isolated, so if K is not identically zero its zeros must be isolated. □

8. Examples of Weierstrass representation

In this section we will see some examples of the correspondence between minimal surfaces and pairs of functions f and g.

8.1. **Helicoid.** Let $\mathbb{C}$ be the domain of the functions f and g. We define $g = -ie^z$ and $f = e^{-z}$. Observe that neither g has poles nor does f have zeros in $\mathbb{C}$, so the condition regarding poles and zeros is satisfied. The components of the function φ are

$$\varphi_1 = \frac{1}{2}f(1-g^2) = \frac{1}{2}(1+e^{2z})e^{-z} = \cosh(z),$$
$$\varphi_2 = \frac{i}{2}f(1+g^2) = \frac{i}{2}(1-e^{2z})e^{-z} = -i\sinh(z),$$
$$\varphi_3 = fg = (-e^z)e^{-z} = -i.$$

Since $\cosh(z)$, $\sinh(z)$ and multiplication by a constant are holomorphic functions in $\mathbb{C}$ the vector function φ is also holomorphic. We observe that the condition $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ also holds. Let us find a conformal parameterization σ using the formula $\sigma(u,v) = \mathfrak{Re}\int_\pi \varphi(\xi)d\xi$. Fore $z = u - iv$ we find that

$$\sigma_1 = \mathfrak{Re}\int_0^z \cosh(z)dz = \mathfrak{Re}(\sinh(z)) = \mathfrak{Re}\left(\frac{e^{u+iv} - e^{-u-iv}}{2}\right)$$
$$= \mathfrak{Re}\left(\frac{1}{2}\left(e^u(\cos(v) + i\sin(v)) - e^{-u}(\cos(-v) + i\sin(v))\right)\right)$$
$$= \cos(v)\sinh(u),$$
$$\sigma_2 = \mathfrak{Re}\int_0^z -i\sinh(z)dz = \mathfrak{Re}(-i\cosh(z) + i) = \sin(v)\sinh(u),$$
$$\sigma_3 = \mathfrak{Re}\int_0^z -idz = (\,-iz) = v.$$

If we take $t = \sinh(u)$, the parameterization $\sigma(t,v) = (\cos(v)t, \cos(v)t, v)$ exactly describes the helicoid.

8.2. **Catenoid.** Let us take $\mathbb{C}$ as domain of the functions f and g. We define $g = -e^z$ and $f = -e^{-z}$. Observe that neither g has poles nor f has zeros in $\mathbb{C}$ so the condition regarding poles and zeros is satisfied. The components of the function φ are

$$\varphi_1 = \frac{1}{2}f(1-g^2) = \sinh(z),$$
$$\varphi_2 = \frac{i}{2}f(1+g^2) = -i\cosh(z),$$
$$\varphi_3 = fg = 1.$$

Since $\cosh(z)$, $\sinh(z)$ and multiplication by a constant are holomorphic functions in $\mathbb{C}$, the function φ is also a holomorphic function. We observe that the condition $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ holds. Let us find a conformal parameterization σ by using the formula $\sigma(u,v) = \mathfrak{Re}\int_\pi \varphi(\xi)d\xi$. For $z = u - iv$ we find

$$\sigma_1 = \mathfrak{Re}\int_0^z \sinh(z)dz = \mathfrak{Re}(\cosh(z) - 1) = \cos(v)\cos(u) - 1,$$
$$\sigma_2 = \mathfrak{Re}\int_0^z -i\cosh(z)dz = \mathfrak{Re}(-i\sinh(z)) = \sin(v)\cosh(u),$$
$$\sigma_3 = \mathfrak{Re}\int_0^z 1dz = (z) = u.$$

We obtain $\sigma(u,v) = (\cos(v)\cosh(u), \sin(v)\cosh(u), u) - (1,0,0)$ which is, up to a translation, the parameterization of the catenoid.

Remark: Notice, that in terms of Weierstrass representation, the catenoid and the helicoid are closely related. Taking $g = -ie^z$ and $f = e^{-z}$ we obtain the helicoid, while taking $g = -e^z$ and $f = -e^{-z}$ yields the catenoid.

Another way of obtaining the catenoid is the following: take $\mathbb{C} - \{0\}$ as the domain of the functions f, g and define $g(z) = z$ and $f = \frac{1}{z^2}$. We obtain

$$\varphi_1 = \frac{1}{2} f(1 - g^2) = \frac{1}{2}(\frac{1}{z^2} - 1),$$

$$\varphi_2 = \frac{i}{2} f(1 + g^2) = \frac{i}{2}(\frac{1}{z^2} + 1),$$

$$\varphi_3 = fg = \frac{1}{z}.$$

For $z = u - iv$ we have

$$\sigma_1 = -\frac{u}{2}(1 + \frac{1}{u^2 + v^2}) + 1,$$

$$\sigma_2 = -\frac{v}{2}(1 + \frac{1}{u^2 + v^2}),$$

$$\sigma_3 = \frac{1}{2} \log(u^2 + v^2).$$

If we set $\rho = \frac{1}{2}\log(u^2 + v^2)$ and $\theta = (\arctan \frac{v}{u}) - \pi$, we can see that the equations describe the catenoid.

8.3. **Scherk surface.** Let us take $\{z \in \mathbb{C}; |z| < 1\}$ as domain of the functions f and g. If we define $g = z$ and $f = \frac{4}{(1-z^4)}$, then the components of the function φ are

$$\varphi_1 = \frac{1}{2} f(1 - g^2) = \frac{2}{1 + z^2} = (\frac{i}{z + i} - \frac{i}{z - i}),$$

$$\varphi_2 = \frac{i}{2} f(1 + g^2) = \frac{2}{1 - z^2} = (\frac{i}{z + i} - \frac{i}{z - i}),$$

$$\varphi_3 = fg = \frac{4z}{1 - z^4} = (\frac{2z}{z^2 + 1} - \frac{2z}{z^2 - 1}).$$

For $z = u - iv$, we have

$$\sigma_1 = \mathfrak{Re}(i \log \frac{z + i}{z - i}) = -\arg(\frac{z + i}{z - i}),$$

$$\sigma_2 = \mathfrak{Re}(i \log \frac{z + 1}{z - 1}) = -\arg \frac{z + 1}{z - 1},$$

$$\sigma_3 = \mathfrak{Re}(\log \frac{z^2 + 1}{z^2 - 1}) = log|\frac{z^2 + 1}{z^2 - 1}|,$$

where the argument φ of a complex number w is such that $w = re^{i\varphi}$, where $r \in \mathbb{R}$. We observe that $(\sigma_1, \sigma_2, \sigma_3)$ satisfy the equation $z = \ln \frac{\cos y}{\cos x}$ for $(\sigma_1, \sigma_2) \in (-\frac{3\pi}{2}, -\frac{\pi}{2}) \times (-\frac{3\pi}{2}, -\frac{\pi}{2})$ so $(\sigma_1, \sigma_2, \sigma_3)$ describe a piece of the Scherk's minimal surface.

8.4. **Enneper surface.** If we take $\{z \in \mathbb{C}; |z| < 1\}$ as domain of the functions f, g and define $g = z$ and $f = 1$. The components of the function φ are

$$\varphi_1 = \frac{1}{2}(1 - z^2),$$
$$\varphi_2 = \frac{i}{2}(1 + z^2),$$
$$\varphi_3 = z.$$

We obtain the following parameterization:

$$\sigma(u, v) = \frac{1}{2}(u - \frac{u^3}{3} + uv^2, -v + \frac{v^3}{3} - u^2 v, u^2 - v^2),$$

which describes the Enneper surface.

8.5. **Henneberg surface.** If we take $\mathbb{C} - \{0\}$ as domain of the functions f, g and define $g = z$ and $f = 2 - \frac{2}{z^4}$. The components of the function φ are

$$\varphi_1 = (-\frac{1}{z^4} + \frac{1}{z^2} + 1 - z^2),$$
$$\varphi_2 = i(-\frac{1}{z^4} - \frac{1}{z^2} + 1 - z^2),$$
$$\varphi_3 = 2(z - \frac{1}{z^3}).$$

We obtain the following parameterization:

$$\sigma_1(u, v) = \frac{u^3(1 - u^2 - v^2)^3 - 3uv^2(1 - u^2 - v^2)(1 + u^2 + v^2)^2}{3(u^2 + v^2)^3}),$$
$$\sigma_2(u, v) = \frac{3u^2 v(1 + u^2 + v^2)^2(1 - u^2 - v^2) - v^3(1 - u^2 - v^2)^3}{3(u^2 + v^2)^3},$$
$$\sigma_3(u, v) = \frac{(1 - u^2 - v^2)^2 u^2 - (1 + u^2 + v^2)v^2}{(u^2 + v^2)^2},$$

which describes the Henneberg surface.

9. Higher dimensional minimal surfaces and the regularity problem

The present section will deal with the problem of regularity of a minimal surface in higher dimensions. In the early 50s Renato Caccioppoli introduced a generalization of the idea of area, and precisely the concept of perimeter, which can be applied to a larger class of hypersurfaces.

In 1960s Reifenberg, De Giorgi, Federer and Fleming proved that minimal hypersurfaces are regular outside a set (called the singular set) of small dimension. After that De Giorgi, Fleming, Almgren and Simons proved a more precise result: an $(n-1)$-dimensional minimal hypersurface in $\mathbb{R}^n$ is regular outside a singular set whose dimension is at most $(n-8)$. It follows that all minimal hypersurfaces in $\mathbb{R}^n$ with $n \leq 7$ are regular.

The mathematician De Giorgi studied the Bernstein's problem considering the Caccioppoli's idea of perimeter and together with Bombieri and Giusti in 1968 [5] proved the minimality of a cone in $\mathbb{R}^8$. They also proved that, as a nontrivial consequence, there exist minimal hypersurfaces in $\mathbb{R}^9$ represented as a graph of a function f, defined for all $(x_1, ..., x_8) \in \mathbb{R}^8$, which have continuous first and second partial derivatives, and they are not planes.

The example of a minimal hypersurface in $\mathbb{R}^8$ with a singular point is the Simons cone, given by

$$S = \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

It has a singular point in the origin.

The original proof of Bombieri, De Giorgi and Giusti of the minimality of the Simons cone was very technical and complicated. Successively, several different proofs were exhibited (see [13], [18], [15], [14], [12], [6], [16], [4], [7]).

In this thesis we shall explain the proof given by G. De Philippis and E. Paolini in ‘A short proof of the minimality of Simons cone’ in 2009 [8]. The original proof in [5] and the following simplifications use the tool of calibrations. A calibration is a divergence free unit vector field which extends the normal field of the surface to the whole ambient space. Using the divergence theorem one finds that if such a field can be found, then the surface is minimal. In the simplified proof [8] the authors consider sub-minimal surfaces, i.e. oriented surfaces which are minimal with respect to the internal deformations, and they prove a sub-calibration result for sub-minimal surfaces. Hence, they explicitly find a sub-calibration for the Simons cone obtaining its sub-minimality. Passing to the complementary set (i.e. changing the orientation of the cone), they find that the complementary subset is also sub-minimal, hence, the Simons cone is indeed minimal.

9.1. **Perimeters, sub-minimal sets and sub-calibrations.**

Definition 9.1. *Let Ω be an open set in $\mathbb{R}^n$. The perimeter of a measurable set $E \subseteq \mathbb{R}^n$ in Ω is*

$$P(E,\Omega) = sup\{\int_E div\ g\ dx : g \in C_c^1(\Omega,\mathbb{R}^n), |g| \leq 1\},$$

where $|g|$ denotes the euclidean norm.

Remark: if E has a regular compact boundary, then $P(E,\Omega)$ is equal to the $(n-1)$-dimensional surface area of the boundary $\partial E \cap \Omega$. To show this, we will see that both of the following inequalities hold: $P(E,\Omega) \leq \mathcal{H}^{n-1}(\partial E \cap \Omega)$ and $P(E,\Omega) \geq \mathcal{H}^{n-1}(\partial E \cap \Omega)$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, which is defined as follows:

Definition 9.2. *For any subset $U \subset \mathbb{R}^n$, let*

$$diam(U) = \sup\{d(x,y)|x,y \in U\},$$

$diam(\varnothing) := 0$, where d is the euclidean distance. Let S be any subset of X and $\delta > 0$ a real number. Define

$$\mathcal{H}^d_\delta(S) = \frac{\omega_d}{2^d} \inf\left\{\sum_{i=1}^{\infty} diam(U_i)^d\ : \bigcup_{i=1}^{\infty} U_i \supset S, diam(U_i) < \delta\right\},$$

where ω_d is the Lebesgue measure of the d-dimensional unit ball of $\mathbb{R}^d$. We define the d-dimensional Hausdorff measure of S as

$$\mathcal{H}^d(S) := \sup_{\delta>0} \mathcal{H}^d_\delta(S) = \lim_{\delta\to 0} \mathcal{H}^d_\delta(S).$$

Now we can prove the above inequalities. Using the Gauss Divergence Theorem (see [11]) we get

$$P(E,\Omega) = sup\{\int_E div\ g\ dx : g \in C^1_c(\Omega,\mathbb{R}^n), |g| \leq 1\}$$
$$= sup\{\int_{\partial E} \langle g, \nu_E\rangle d\mathcal{H}^{n-1} : g \in C^1_c(\Omega,\mathbb{R}^n), |g| \leq 1\}$$
$$\leq sup\{\int_{\partial E} |g||\nu_E| d\mathcal{H}^{n-1} : g \in C^1_c(\Omega,\mathbb{R}^n), |g| \leq 1\}$$
$$\leq sup\{\int_{\partial E} 1 d\mathcal{H}^{n-1} : g \in C^1_c(\Omega,\mathbb{R}^n), |g| \leq 1\}$$
$$= \mathcal{H}^{n-1}(\partial E \cap \Omega),$$

where ν_E is the exterior unit normal vector to ∂E.

To prove the converse inequality, it suffices to find a vector field $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is C^1_c, which gives the exterior normal to ∂E and with norm ≤ 1 in every point. If we find such a function, we will have

$$P(E,\Omega) = sup\{\int_E div\ g\ dx : g \in C^1_c(\Omega,\mathbb{R}^n), |g| \leq 1\} \geq \int_E div\ f\ dx$$
$$= \int_{\partial E} \langle f, \nu_E\rangle d\mathcal{H}^{n-1}$$
$$= \int_{\partial E} |\nu_E|^2 d\mathcal{H}^{n-1} = \int_{\partial E} 1 d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial E \cap \Omega).$$

A function with such properties as f is the gradient of the signed distance function, extended to $\mathbb{R}^n$ by a convolution. We define the signed distance function as

$$\bar{d}(x,E) := d(x,E) - d(x,\mathbb{R}^n \setminus E).$$

We notice that, by definition, $\bar{d}(x,E)$ is equal to $-d(x,\partial E)$ in E, and equal to $d(x,\partial E)$ in E^c, where d is the distance function defined by

$$d(x,E) := \inf_{y\in E} |x-y|.$$

Notice also that if $x \in \partial E$, we have $\bar{d}(x,E) = 0$.

Let us see some particular examples to understand how this function $\bar{d}(x,E)$ looks like.

If $n = 1$ and $E = [0,1]$, $\partial E = \{0,1\}$, the values of the distance function are the following: for $x \in (-\infty, 0]$ we have $d(x,E) = -x$, for $x \in [0, \frac{1}{2}]$ we have $d(x,E) = x$, for $x \in [\frac{1}{2}, 1]$ we have $d(x,E) = 1-x$ and for $x \in [1,+\infty)$ we have $d(x,E) = x-1$. So for $x \in (-\infty, \frac{1}{2}]$ we have $\bar{d}(x,E) = -x$ and for $x \in [\frac{1}{2}, +\infty]$ we have $\bar{d}(x,E) = x-1$.

If $n = 2$ and E is the unit disk centered at the origin, then ∂E is the circle of radius 1 centered at $(0,0)$, and the graph of $\bar{d}$ is a vertical cone in $\mathbb{R}^3$ with the apex in (0,0,-1).

The next results concern the differentiability of the signed distance function and the properties of its gradient.

Theorem 9.3 (Property of the gradient of the signed distance function). $|\nabla d| = 1$ *and also* $|\nabla \bar{d}| = 1$ *at any point, in which they are differentiable.*

Proof. [2] $\square$

Theorem 9.4 (Properties of the gradient of the signed distance function). *Let* $E \subset \mathbb{R}^n$ *be a bounded open set with smooth boundary. Then* $\bar{d}(x, E)$ *is smooth in a tubular neighbourhood* U *of* ∂E *and so it is* $\nabla \bar{d}$. *Moreover, the outer normal of* E *is given by* $\nabla \bar{d}$.

Proof. We will prove that $\bar{d}$ is smooth by showing that it coincides with a function, which will be smooth by construction.

Set $\Gamma = \partial E$ and let $x_0 \in \Gamma$. Since Γ is smooth, it can be considered locally as a graph $x^n = f(x^1, ..., x^{n-1})$ and so the normal vector is given by

$$\nu_E(x_0) = \frac{(-f_{x^1}, ..., -f_{x^{n-1}}, 1)}{\sqrt{1 + \sum_{i=1}^{n-1}(f_{x^i})^2}}.$$

So there exists a constant $s > 0$ and a smooth orthonormal vector field

$$\nu_E : B_s(x_0) \cap \Gamma \longrightarrow \mathbb{R}^n,$$

where $B_s(x_0)$ denotes the open ball of radius s centered at x_0. We define for $x \in B_s(x_0) \cap \Gamma$ and $\alpha \in \mathbb{R}$

$$\Phi(x, \alpha) := x + \alpha \nu_E(x).$$

Since the Jacobian of Φ in $(x_0, 0)$ is equal to 1, we can use the Inverse Function Theorem: there exists $r \in (0, s)$ such that

(1) in $(B_r(x_0) \cap \Gamma) \times (-r, r)$, Φ is one to one;

(2) $V = \phi((B_r(x_0) \cap \Gamma) \times (-r, r))$ is an open set containing x_0. For $y \in V$ let

$$\Psi(y) = (x(y), \alpha(y)) \in (B_r(x_0) \cap \Gamma) \times (-r, r)$$

be the smooth inverse of Φ.

Choose $\sigma \in (0, r/2)$ such that $B_\sigma(x_0) \subset V$ (in the codomain of the function Φ). Then choose $y \in B_\sigma(x_0)$ and let $x \in \Gamma$ be any minimizer of the distance i.e. $d(y) = |x - y|$. It follows, that $x \in B_\sigma(x_0) \cap \Gamma$ (in the codomain of the function Φ), hence $x = x(y)$ and $d(y) = d(x, y) = d(x, x + \alpha(y)\nu_E(x)) = |\alpha(y)|$. So if

$$y(x) \in E, \quad d(y) = d(x, y) = d(x, x - \alpha(y)\nu_E(x)) = -\alpha(y), \quad \alpha(y) > 0,$$

and if

$$y(x) \in E^c \quad d(y) = d(x, y) = d(x, x - \alpha(y)\nu_E(x)) = \alpha(y), \quad \alpha(y) > 0.$$

It follows, that $\bar{d}(y, E) = \alpha(y)$ for every $y \in B_\sigma(x_0)$. So $\bar{d}$ is smooth in $B_\sigma(x_0)$. Since Γ is compact, we use a covering argument to extend the smoothness to a tubular neighbourhood $\{d^2 < \sigma^2\}$.

Since Γ is a level set of $\bar{d}$ i.e.

$$\Gamma = \{x \in \mathbb{R}^n | \bar{d}(x) = 0\},$$

it follows that the gradient of $\bar{d}$ is orthogonal to $\Gamma = \partial E$, so is parallel to the outer normal to E. We know that $|\nabla \bar{d}| = 1$ hence, $\nabla \bar{d}$ is the outer normal to E.

$\square$

We have proved that the signed distance function is C^1_c in a tubular neighbourhood U of ∂E, but we need a function that is C^1_c in all $\mathbb{R}^n$, so we will extended $\nabla \bar{d}$ to $\mathbb{R}^n$ by performing a convolution. Consider the function

$$\bar{d} * \rho_\lambda(x) := \int_{\mathbb{R}^n} \bar{d}(y)\rho_\lambda(x-y)\, dy,$$

where $\rho_\lambda(x) = \lambda^{-n}\rho(\frac{x}{\lambda})$ and the function $\rho : \mathbb{R}^n \to [0, +\infty)$ has the following properties: $\rho(x) = \rho(-x)$, $\rho \in C^\infty$, $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$. Such a function is smooth on all $\mathbb{R}^n$; on the boundary of E its gradient gives the exterior normal to ∂E and its norm is in every point ≤ 1.

Definition 9.5. *Let $\Omega \subseteq \mathbb{R}^n$ and let E_k and E be measurable sets in Ω. We say that $E_k \to E$ in $L^1_{loc}(\Omega)$ if for every bounded set $A \subset \Omega$ we have $|(E_k \triangle E) \cap A| \to 0$, where $|X|$ is the Lebesgue measure of the set X.*

Proposition 9.6 (Properties of the perimeter)**.** *The perimeter has the following properties:*

(1) the perimeter is lower-semicontinuous i.e.

$$P(E, A) \leq \liminf_k P(E_k, A);$$

(2) $P(E, A) = P(E^c, A)$;

(3) $P(E \cap F, A) + P(E \cup F, A) \leq P(E, A) + P(F, A)$.

Proof. For the proof of (1) see [12]; for (2) and (3) see [3]. The proof of the second property of the perimeter is nontrivial, and another equivalent definition of perimeter has to be considered.

□

Definition 9.7. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A measurable set E is a minimum of the perimeter (or minimal in Ω) if in all bounded open sets $A \subseteq \Omega$ one has*

$$P(E, A) \leq P(F, A) \qquad \textit{for all } F \textit{ such that } E \triangle F \Subset A,$$

where with $E \triangle F$ we denote the symmetric difference $(E \setminus F) \cup (F \setminus E)$ and with $A \Subset B$ we mean that the closure of A is a compact subset of B.

Definition 9.8. *A measurable set E is sub-minimal in Ω if in all bounded open sets $A \subseteq \Omega$ one has*

$$P(E, A) \leq P(F, A) \qquad \textit{for all } F \subseteq E \textit{ such that } E \setminus F \Subset A.$$

Proposition 9.9. *If E and $E^c = \Omega \setminus E$ are sub-minimal in Ω, then E is minimal in Ω.*

Proof. Let E and E^c be sub-minimal. So, if A is a bounded open set in Ω, then $P(E, A) \leq P(F, A)$ for all $F \subseteq E$ such that $E \setminus F \Subset A$ and $P(E^c, A) \leq P(F, A)$ for all $F \subseteq E^c$ such that $E^c \setminus F \Subset A$.

We want to prove that $P(E, A) \leq P(F, A)$ for all F such that $E \triangle F \Subset A$. Define

$$F' = E \cap F,$$

so $F' \subseteq E$ and $E \setminus F' = E \setminus (E \cap F) = E \setminus E \cup E \setminus F = E \setminus F \Subset A$, and define

$$F'' = (E \cup F)^c,$$

so $F'' \subseteq E^c$ and $E^c \setminus F'' = E^c \setminus (E \cup F)^c = E^c \setminus (E^c \cap F^c) = (E^c \setminus E^c) \cup (E^c \setminus F^c) = E^c \setminus F^c = E^c \cap F \subseteq F \setminus E \Subset A$. From the sub-minimality of E and E^c we have

$$P(E, A) \leq P(F', A) \tag{9}$$

$$P(E^c, A) \leq P(F'', A). \tag{10}$$

By the second property of the perimeter the inequality (10) can be written as

$$P(E, A) \leq P((F'')^c, A) = P(E \cup F, A).$$

If we sum the two inequalities (9) and (10), we obtain

$$2P(E, A) \leq P(E \cap F, A) + P(E \cup F, A).$$

Finally, by the third property of the perimeter we obtain

$$P(E \cap F, A) + P(E \cup F, A) \leq P(E, A) + P(F, A).$$

We have $2P(E, A) \leq P(E \cap F, A) + P(E \cup F, A) \leq P(E, A) + P(F, A)$, hence $P(E, A) \leq P(F, A)$. □

Proposition 9.10 (Convergence of subminimal sets to a subminimal set)**.** *Let E_k and E be measurable sets in Ω with $E_k \subseteq E$ and $E_k \to E$ in $L^1_{loc}(\Omega)$. If E_k is sub-minimal in Ω for every k, then also E is sub-minimal in Ω.*

Proof. Let E_k and E be measurable sets in Ω such that $E_k \subseteq E$, $E_k \to E$ in $L^1_{loc}(\Omega)$ and let A be an open bounded subset of Ω. By the sub-minimality of E_k we have $P(E_k, A) \leq P(H, A)$ for every $H \subseteq E_k$ such that $E_k \setminus H \Subset A$. Let F be a measurable set such that $F \subseteq E$, $E \setminus F \Subset A$. We have to prove that $P(E, A) \leq P(F, A)$. If we define

$$F'_k = F \cap E_k,$$

we have that $F'_k \subseteq E_k$ and $E_k \setminus F'_k = E_k \setminus (F \cap E_k) = E_k \setminus F \cup E_k \setminus E_k = E_k \setminus F \subseteq E \setminus F \Subset A$. So by sub-minimality of E_k we have

$$P(E_k, A) \leq P(F'_k, A).$$

By the lower-semicontinuity of the perimeter we have

$$P(E, A) \leq \liminf_k P(E_k \cup F, A).$$

Using the third property of the perimeter we get

$$\begin{aligned} P(E, A) &\leq \liminf_k P(E_k \cup F, A) \\ &\leq \liminf_k [P(E_k, A) + P(F, A) - P(E_k \cap F, A)] \\ &\leq \liminf_k [P(F'_k, A) + P(F, A) - P(E_k \cap F, A)] \\ &= \liminf_k [P(E_k \cap F, A) + P(F, A) - P(E_k \cap F, A)] = P(F, A) \end{aligned}$$

□

Definition 9.11. *Let $E \subset \Omega$ be a measurable set such that the boundary $\partial E \cap \Omega$ has C^2 regularity. A vector field $\xi \in C^1(\Omega, \mathbb{R}^n)$ is a sub-calibration of E in Ω if it satisfies the following properties:*

(i) $\xi(x) = \nu_E(x)$ is the exterior normal vector to ∂E for all $x \in \partial E \cap \Omega$;

(ii) div $\xi(x) \leq 0$ for all $x \in E \cap \Omega$;

(iii) $|\xi(x)| \leq 1$ for all $x \in \Omega$.

Theorem 9.12 (Sub-calibrated sets are sub-minimal). *If ξ is a sub-calibration of a set E with boundary of class C^2 in Ω, then E is sub-minimal in Ω.*

Proof. Let $A \subseteq \Omega$ be an open bounded set and let $F \subseteq E$, with $E \setminus F \Subset A$. Let us define

$$\xi_j = \eta_j \xi,$$

where the $\eta_j \in C_c^1(A, \mathbb{R})$ are such that

$$\eta_j(x) = 1 \qquad for\ x \in E \setminus F$$

$$0 \leq \eta_j(x) \leq 1 \qquad for\ all\ \ x \in A$$

and the sequence of sets $A_j = \{x \in A : \eta_j(x) = 1\}$ is increasing with $\bigcup A_j = A$. We want to prove that $P(F, A) \geq P(E, A)$. Since $\xi_j \in C_c^1(A, \mathbb{R}^n)$ for every j, we have

$$P(F,A) = \sup_h \{ \int_{F \cap A} div\ h\ dx, h \in C_c^1(A, \mathbb{R}^n), |h| \leq 1\}$$

$$\geq \int_{F \cap A} div\ \xi_j\ dx.$$

Since $E \triangle F = E \setminus F \subset A$, we have $(E \cap A) \setminus (F \cap A) = E \setminus F$; so

$$\int_{E \cap A} div\ \xi_j\ dx - \int_{F \cap A} div\ \xi_j\ dx = \int_{E \setminus F} div\ \xi_j\ dx$$

$$= \int_{E \setminus F} \xi \frac{\partial \eta_j}{\partial x_1} + \eta_j \frac{\partial \xi}{\partial x_1} + ... + \xi \frac{\partial \eta_j}{\partial x_n} + \eta_j \frac{\partial \xi}{\partial x_n} dx$$

$$= \int_{E \setminus F} \langle \eta_j, div\ \xi \rangle\ dx + \int_{E \setminus F} \langle \xi, div\ \eta_j \rangle\ dx = \int_{E \setminus F} div\ \xi\ dx \leq 0.$$

It follows that for every j we have

$$P(F, A) \geq \int_{F \cap A} div\ \xi_j\ dx \geq \int_{E \cap A} div\ \xi_j\ dx.$$

Applying the Gauss-Green theorem for every j we get

$$\int_{E \cap A} div\ \xi_j\ dx = \int_{\partial(E \cap A)} \langle \xi_j, \nu_E \rangle\ d\mathcal{H}^{n-1}$$

$$= \int_{\partial E \cap A} \langle \xi_j, \nu_E \rangle\ d\mathcal{H}^{n-1} + \int_{E \cap \partial A} \langle \xi_j, \nu_E \rangle\ d\mathcal{H}^{n-1}$$

$$= \int_{\partial E \cap A} \langle \xi_j, \nu_E \rangle\ d\mathcal{H}^{n-1} = \int_{\partial E \cap A} \eta_j\ d\mathcal{H}^{n-1} \geq \int_{\partial E \cap A_j} \eta_j\ d\mathcal{H}^{n-1}$$

$$= \int_{\partial E \cap A_j} d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial E \cap A_j).$$

We observe that since the A_j are increasing and $UA_j = A$, we have $A_j \uparrow A$. It follows that if we take the infimum for $j \to \infty$, we obtain $P(F, A) \geq \mathcal{H}^{n-1}(\partial E \cap A)$. Since E has regular boundary, we have $\mathcal{H}^{n-1}(\partial E \cap A) = P(E, A)$ and so we have proved the sub-minimality of E.

□

9.2. **Simons cone.**

Definition 9.13. *The Simons cone is*

$$C = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : |x| \leq |y|\} \subseteq \mathbb{R}^{2m}.$$

Remark: such a cone is the zero sub-level $C = \{f \leq 0\}$ of the function

$$f : \mathbb{R}^{2m} \to \mathbb{R}$$

$$f(x, y) = \frac{|x|^4 - |y|^4}{4}, \qquad (x, y) \in \mathbb{R}^m \times \mathbb{R}^m.$$

Moreover, the following sequences of sets

$$E_k = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : f(x, y) \leq -\frac{1}{k}\} \subseteq C$$

$$F_k = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : f(x, y) \leq \frac{1}{k}\} \supseteq C$$

both converge to C in $L^1_{loc}(\mathbb{R}^{2m})$.

The crucial part of the construction is in the following:

Proposition 9.14. *If $m \geq 4$, then the vector field*

$$\xi = \frac{Df}{|Df|},$$

where $f(x, y) = \frac{|x|^4 - |y|^4}{4}$ for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, is a sub-calibration of the sets E_k in $\Omega = \mathbb{R}^n \setminus \{0\}$, and the vector field $-\xi$ is a sub-calibration of the sets F_k^c in Ω.

Proof. Let us prove that ξ is a sub-calibration of E_k in Ω, i.e.

(1) $\xi(x, y) = \nu_{E_k}(x, y)$ is the exterior normal vector to ∂E_k for all $(x, y) \in \partial E_k \cap \Omega$;

(2) *div* $\xi(x, y) \leq 0$ for all $(x, y) \in E_k \cap \Omega$;

(3) $|\xi(x, y)| \leq 1$ for all $(x, y) \in \Omega$.

The point (3) is satisfied because in Ω we have $|\xi| = \frac{|Df|}{|Df|} = 1$ and $Df \neq 0$ in Ω.

Since the level set given by $\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : f(x, y) = -\frac{1}{k}\}$ is orthogonal to Df we have that ξ is orthogonal to ∂E_k in $\partial E_k \cap \Omega$. Considering that ξ is also a unit vector it follows that ξ is the exterior normal to E_k in ∂E_k.

To prove the point (2) we have to compute the divergence:

$$div\left(\frac{Df}{|Df|}\right) = \sum_{i=1}^{m}\left(\frac{f_{x_i}}{|Df|}\right)_{xi} + \sum_{i=1}^{m}\left(\frac{f_{y_i}}{|Df|}\right)_{y_i}.$$

We recall that $f = \frac{1}{4}(|x|^4 - |y|^4)$, so

$$f_{x_i} = \frac{1}{4}4|x|^3\frac{x_i}{|x|} = |x|^2 x_i,$$

$$f_{y_i} = -|y|^2 y_i,$$

$$|Df| = \sqrt{|x|^4 x_1^2 + ... + |x|^4 x_m^2 + |y|^4 y_1^2 + .. + |y|^4 y_m^2} = \sqrt{|x|^6 + |y|^6},$$

$$f_{x_i x_j} = 2|x|\frac{x_j}{|x|}x_i + \delta_{ij}|x|^2 = 2x_j x_i + \delta_{ij}|x|^2,$$

$$f_{y_i y_j} = 2y_j y_i - \delta_{ij}|y|^2,$$
$$f_{x_i y_j} = 0,$$
$$(|Df|)_{x_i} = \frac{1}{2|Df|}(2f_{x_1}f_{x_1x_i} + ... + 2f_{x_m}f_{x_mx_i} + 2f_{y_1}f_{y_1x_i} + ... + 2f_{y_m}f_{y_mx_i})$$
$$= \frac{1}{|Df|}\sum_{j=1}^{m} f_{x_j}f_{x_jx_i} = \frac{1}{|Df|}\sum_{j=1}^{m}|x|^2x_j(2x_ix_j + \delta_{ij}|x|^2).$$

We obtain

$$\left(\frac{f_{x_i}}{|Df|}\right)_{x_i} = \frac{(2x_i^2+|x|^2)\left(|x|^6+|y|^6\right)^{1/2} - |x|^2x_i\frac{1}{|Df|}\sum_{j=1}^{m}|x|^2x_j(2x_ix_j+\delta_{ij}|x|^2)}{|Df|^6}$$
$$= \frac{(2x_i^2+|x|^2)(|x|^6+|y|^6) - |x|^2x_i\sum_{j=1}^{m}|x|^2x_j(2x_ix_j+\delta_{ij}|x|^2)}{|Df|^3}.$$

If we sum all these terms we have

$$|Df|^3\sum_{i=1}^{m}\left(\frac{f_{x_i}}{|Df|}\right)_{x_i} = \sum_{i=1}^{m}2x_i^2|x|^6+|x|^8+2x_i^2|y|^6+|x|^2|y|^6-2|x|^6x_i^2-|x|^6x_i^2$$
$$= m|x|^8 + 2|y|^6|x|^2 + m|y|^6|x|^2 - |x|^8$$
$$= (m-1)|x|^8 + (m+2)|x|^2|y|^6.$$

The derivatives with respect to y_i are the same, with signs changed, so

$$|Df|^3 div\left(\frac{Df}{|Df|}\right)$$
$$= (m-1)|x|^8 + (m+2)|x|^2|y|^6 - (m-1)|y|^8 - (m+2)|y|^2|x|^6$$
$$= (|x|^4 - |y|^4)[(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4].$$

We want to prove that the divergence is negative for $(x,y) \in E_k \cap \Omega$.

If $(x,y) \in E_k \cap \Omega$ then $|x|^4 - |y|^4$ is negative so we have to prove that $[(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4]$ is non-negative. We make the substitution $t = \frac{|x|^2}{|y|^2}$ and we obtain

$$[(m-1)t^2 - (m+2)t + (m-1)] \geq 0.$$

Hence the discriminant must be non-positive we have

$$(m+2)^2 - 4(m-1)^2 = 3m(4-m) \leq 0,$$

which holds for $m \geq 4$.

Finally, to prove that $-\xi$ is a sub-calibration of F_k^c in Ω we have to show that

(1) $-\xi(x,y) = \nu_{F_k^c}(x,y)$ is the exterior normal vector to ∂F_k^c for all $(x,y) \in \partial F_k^c \cap \Omega$;

(2) $div(-\xi(x,y)) \leq 0$ for all $(x,y) \in F_k^c \cap \Omega$;

(3) $|\xi(x,y)| \leq 1$ for all $(x,y) \in \Omega$.

The point (1) holds because Df is orthogonal to ∂F_k. The point (2) can be checked by computing the divergence which has the opposite sign as above i.e.

$$|Df|^3 div\left(-\frac{Df}{|Df|}\right)$$
$$= -(|x|^4 - |y|^4)[(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4].$$

Since $(x, y) \in F_k^c \cap \Omega$ then $|x|^4 - |y|^4$ is positive so we have to prove that $[(m-1)|x|^4 - (m+2)|x|^2|y|^2 + (m-1)|y|^4]$ is non-negative and we proceed as before.

The last point can be proved as above.

□

Theorem 9.15 (Minimality of the cone). *For $m \geq 4$, the Simons cone C is a minimal set in $\mathbb{R}^{2m}$.*

Proof. The vector field $\xi = \frac{Df}{|Df|}$, where $f(x, y) = \frac{|x|^4 - |y|^4}{4}$, $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, is a sub-calibration of the sets E_k in $\Omega = \mathbb{R}^n \setminus \{0\}$ and the vector field $-\xi$ is a sub-calibration of the sets F_k^c in Ω. So E_k and F_k^c are subminimal in Ω. E_k is sub-minimal also in the whole space $\mathbb{R}^n$ because, if A is any open subset of $\mathbb{R}^n$, then $P(E, A) = P(E, A \setminus \{0\})$ for every measurable set E. For the same reason, F_k^c is sub-minimal in $\mathbb{R}^n$. Then E and E^c are sub-minimal and so E is minimal. □

Remark: The Simons cone is a particular example of a Lawson cone, that is the cone

$$C_{k,h} = \{(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^h : |x|^2 = \frac{k-1}{h-1}|y|^2\}.$$

The main result regarding this cone is the following theorem [7]:

Theorem 9.16. *Let $C_{k,h}$ be the Lawson's cone with $k,\ h \geq 2$.*

(i) If $n > 8$ then $C_{k,h}$ is of minimal area;

(ii) If $n = 8$ then $C_{k,h}$ has mean curvature zero at every point except at the origin, which is singular, and it is of minimal area if and only if $|k - h| \leq 2$.

Proof. [7] □

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[20] The pictures are taken from $http://xahlee.org/surface/gallery\_m.html$